Universal Coalgebra

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Barcelona, February 2012

Contents

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Universal Coalgebra is a theory of Systems

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Universal Coalgebra is a theory of Systems

It suffices to:

- Model systems.
- Construct morphisms between systems.
- Detect behavioural equivalent states.
- Simplify systems.
- Define new concepts and operators via coinduction.

1. Coalgebra

Definition

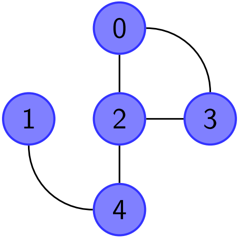
Given a category $\mathbf{X}$, called the *base category*, and an endofunctor $F : \mathbf{X} \to \mathbf{X}$, a *$F$-coalgebra* (or *$F$-system*) consists of a pair (X, α), where X is an object of $\mathbf{X}$ and $\alpha : X \to FX$ an arrow in $\mathbf{X}$. We call X the *base* and α the *structure map* of the coalgebra.

1. Example: Graphs

Definition

A *graph* is an ordered pair $G = (V, E)$ comprising a set V of *vertices* together with a set $E \subseteq [V]^2$ called *edges*.

Example



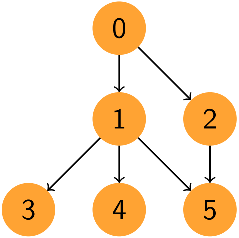
$$V = 5$$
$$E = \{\{0,2\},\{0,3\},\{1,4\},\{2,3\},\{2,4\}\}$$

1. Example: Posets

Definition

A *partially ordered set* (or *poset*) is a pair $\mathbb{P} = (P, \leq)$ where P is a set and $\leq$ is an order over P.

Example



To each $p \in P$ we can associate two subsets of P:

Its *downset* $\downarrow p = \{q \in P : \; q \leq p\}$

Its *upset* $\uparrow p = \{q \in P : \; p \leq q\}$

2. Coalgebra Homomorphisms

Definition

Let $\mathbf{X}$ be any category. Let F be an endofunctor over $\mathbf{X}$. Let (X, α) and (Y, β) be two F-coalgebras. A *F-coalgebra homomorphism*, $f : (X, \alpha) \to (Y, \beta)$ is an arrow $f : X \to Y$ in $\mathbf{X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\
FX & \xrightarrow[\quad Ff \quad]{} & FY
\end{array}
$$

2. Example: $\mathcal{P}$-Coalgebra Homomorphisms

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Remark

Given a $\mathcal{P}$-coalgebra (X, α), we can write it as (X, R_α) with $R_\alpha \subseteq X \times X$ and $x_1 R_\alpha x_2 \Leftrightarrow x_2 \in \alpha(x_1)$. Notice that R_α can also play the role of α by setting $R_\alpha x_1 = \{x_2 \in X : \; x_1 R_\alpha x_2\} = \alpha(x_1)$.

Proposition

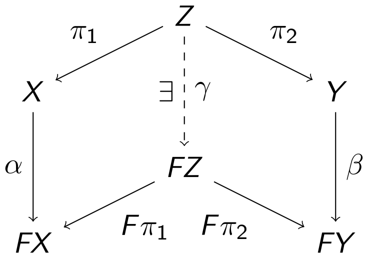
Let (X, R_α) and (Y, R_β) be two $\mathcal{P}$-coalgebras. A function $f : X \to Y$ is a $\mathcal{P}$-coalgebra homomorphism if and only if:

- $x_1 R_\alpha x_2 \implies f(x_1) R_\beta f(x_2)$
- $f(x_1) R_\beta y \implies \exists x_2 \in X \; (x_1 R_\alpha x_2 \text{ and } f(x_2) = y)$

3. Bisimulation

Definition

Let F be any endofunctor over **Set**. Let (X, α), (Y, β) be two F-coalgebras. A subset $Z \subseteq X \times Y$ of the cartesian product of X and Y is called a *F-bisimulation* if there exists a structure map $\gamma : Z \to FZ$ such that the projections from Z to X and Y are F-coalgebra homomorphisms.



Definition

- We will denote by $B(X, Y)$ to the set of all bisimulations between X and Y.
- If $(X, \alpha) = (Y, \beta)$, then (Z, γ) is called a bisimulation on (X, α). We will write $B(X)$ instead of $B(X, X)$. A *bisimulation equivalence* is a bisimulation that is also an equivalence relation.
- Two states $x \in X$, $y \in Y$ are called *bisimilar* if there exists a bisimulation Z with $\langle x, y \rangle \in Z$.

Example

The empty set, $\emptyset \subseteq X \times Y$, is always a bisimulation. $\emptyset \in B(X, Y)$.

3. Basic Properties

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Properties

Let (X, α), (Y, β) and (W, δ) be three F-coalgebras.

- $f : X \to Y$ is a F-coalgebra homomorphism iff $G(f) \in B(X, Y)$.

$$G(f) = \{\langle x, f(x)\rangle : x \in X\}$$

- If $Z \in B(X, Y)$ then $Z^{-1} \in B(Y, X)$.

$$Z^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in Z\}$$

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Theorem

Let (X, α) and (Y, β) be two F-coalgebras and let $\{Z_j : \ j \in J\}$ be a family of $B(X, Y)$. Then the union of the family is also a bisimulation between X and Y.

Corollary

$B(X, Y)$ is a complete lattice for the inclusion order, with least upper bound and greatest lower bound given by:

$$\bigvee_{j \in J} Z_j = \bigcup_{j \in J} Z_j$$

$$\bigwedge_{j \in J} Z_j = \bigcup \{Z : \ Z \in B(X, Y) \text{ and } Z \subseteq \bigcap_{j \in J} Z_j\}$$

3. Example: Kripke Models

Definition

Given a set of atomic propositions Prop and an arbitrary set A, the set of all *multimodal formulas* $\mathcal{ML}$ is defined inductively by:

$$\begin{aligned} p \in \text{Prop} \quad &\Rightarrow \quad p \in \mathcal{ML} \\ \bot \in \mathcal{ML} & \\ \varphi, \psi \in \mathcal{ML} \quad &\Rightarrow \quad \varphi \to \psi \in \mathcal{ML} \\ \varphi \in \mathcal{ML},\ a \in A \quad &\Rightarrow \quad \Box_a \varphi \in \mathcal{ML} \end{aligned}$$

As usual, $\top$, $\neg$, $\wedge$, $\vee$, can be defined from $\bot$, $\to$. The modal operator $\Diamond_a$ for each $a \in A$ is defined as $\neg \Box_a \neg$.

3. Example: Kripke Models

Definition

A *Kripke Model* is a triple $\mathbb{X} = (X, (R_a)_{a \in A}, V)$ consisting on a set X, a relation $R_a \subseteq X \times X$ for each $a \in A$ and a valuation $V : X \to \mathcal{P}(\text{Prop})$.

Elements of X are called *states*. R_a is called the *accesibility relation* according to a. As usual we think of V as a mapping assigning to each possible state the set of atomics propositions holding in x.

Remark

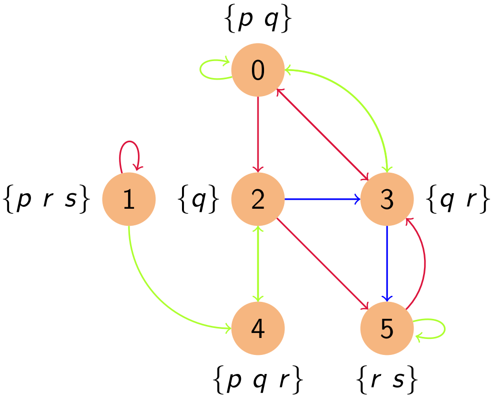
For $A = 1$ we reduce that construction for the case of the usual modal logic.

We think of A as a set of agents and of $\Box_a \varphi$ as 'agent a knows φ'.

Atomic propositions describe the facts agents can know.

3. Example: Kripke Models

Example



3. Example: Kripke Models

Definition

Given a Kripke Model $\mathbb{X} = (X, (R_a)_{a\in A}, V)$ and $x \in X$ we define:

$$
\begin{array}{lcllcl}
(\mathbb{X}, x) & \models & p & \Leftrightarrow & p \in V(x) \\
(\mathbb{X}, x) & \not\models & \bot & & \\
(\mathbb{X}, x) & \models & \varphi \to \psi & \Leftrightarrow & \text{if } (\mathbb{X}, x) \models \varphi \text{ then } (\mathbb{X}, x) \models \psi \\
(\mathbb{X}, x) & \models & \Box_a \varphi & \Leftrightarrow & \forall y \in X \text{ such that } xR_a y \text{ then } (\mathbb{X}, y) \models \varphi
\end{array}
$$

When the model $\mathbb{X}$ is clear from the context, we will write $x \models \varphi$ instead of $(\mathbb{X}, x) \models \varphi$. We say that φ *holds* in a model $\mathbb{X}$, written $\mathbb{X} \models \varphi$ if and only if $\forall x \in X \; x \models \varphi$. Finally, φ is *valid*, written $\models \varphi$ if and only if φ holds in all models.

3. Example: Kripke Models

Theorem

Given two Kripke Models $\mathbb{X} = (X, (R_a)_{a \in A}, V)$, $\mathbb{X}' = (X', (R'_a)_{a \in A}, V')$ and $x \in X$ and $x' \in X'$.

$$x,\ x' \text{ are bisimilar} \Rightarrow \text{for all } \varphi \in \mathcal{ML}\ (x \models \varphi \Leftrightarrow x' \models \varphi)$$

3. Example: Kripke Models

Theorem

Given two Kripke Models $\mathbb{X} = (X, (R_a)_{a\in A}, V)$, $\mathbb{X}' = (X', (R'_a)_{a\in A}, V')$ and $x \in X$ and $x' \in X'$.

$$x,\ x' \text{ are bisimilar} \Rightarrow \text{for all } \varphi \in \mathcal{ML}\ (x \models \varphi \Leftrightarrow x' \models \varphi)$$

Theorem (Hennesy and Milner)

Let K be the class of *image-finite Kripke Models*, i.e., for all $\mathbb{X} = (X, (R_a)_{a\in A}, V) \in K$ and each $x \in X$, the set $\{y :\ xR_a y\}$ is finite for each $a \in A$. Then in the class K, the converse hold:

$$\text{For all } \varphi \in \mathcal{ML}\ (x \models \varphi \Leftrightarrow x' \models \varphi) \Rightarrow x,\ x' \text{ are bisimilar}$$

3. Pullbacks

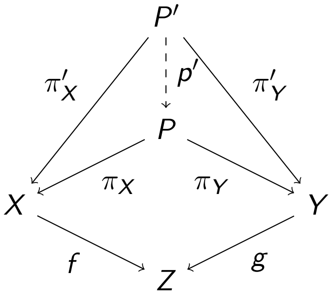
Definition

A *weak pullback* of two mappings $f : X \to Z$ and $g : Y \to Z$ in the category **Set** is a triple (P, π_X, π_Y) such that P is a set, $\pi_X : P \to X$ and $\pi_Y : P \to Y$ are such that:

- $f\ \pi_X = g\ \pi_Y$
- For each triple (P', π'_X, π'_Y) with $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$ and $f\pi'_X = g\pi'_Y$, there is a *mediating mapping* $p' : P' \to P$ such that $\pi_X p' = \pi'_X$ and $\pi_Y p' = \pi'_Y$

Note that the mediating mapping p' need not to be unique; adding this requirement to the definition it would give the more familiar, and stronger, notion of *pullback*.

3. Pullbacks



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Definition

Let F be an endofunctor over **Set**. We say that it *preserves (weak) pullbacks*, written *pwp*, if for any (weak) pullback (P, π_X, π_Y) of (f, g), the triple $(FP, F\pi_X, F\pi_Y)$ is a (weak) pullback of (Ff, Fg).

Definition

Let F be an endofunctor over **Set**. We say that it *preserves (weak) pullbacks*, written *pwp*, if for any (weak) pullback (P, π_X, π_Y) of (f, g), the triple $(FP, F\pi_X, F\pi_Y)$ is a (weak) pullback of (Ff, Fg).

Proposition

All endofunctors inductively defined upon:

- The Identity Functor $\mathcal{I}$
- The Constant Functor C
- The Coproduct Functor $\amalg$
- The Product Functor $\times$
- The Exponent Functor $(\cdot)^C$
- The Power Set Functor $\mathcal{P}$

preserve weak pullbacks.

3. Pullbacks and Bisimulations

Let F be a *pwp* endofunctor over **Set**.
Let (X, α), (Y, β), (W, δ) be three F-coalgebras.

Theorem

Let Z_1 be a bisimulation between X and Y and let Z_2 be a bisimulation between Y and W. Then the composition $Z_1 \circ Z_2$ is a bisimulation between X and W.

Let F be a *pwp* endofunctor over **Set** and let (X, α) be a F-coalgebra, then:

Corollary

- $X \between X$ is a bisimulation equivalence on X.

Let (Y, β) be another F-coalgebra, let $f : X \to Y$ be a F-coalgebra homomorphism, then:

Corollary

- $Kerf$ is a bisimulation equivalence on X.

$$Kerf = \{\langle x_1, x_2 \rangle : \ f(x_1) = f(x_2)\}$$

3. Idempotent Semirings

Definition

An *idempotent semiring*, or *dioid*, is a 5-tuple, $\mathbb{S} = (S, \oplus, \otimes, \varepsilon, e)$ where:

- $(S, \oplus, \varepsilon)$ is a commutative monoid.
- $(S, \otimes, e)$ is a monoid.
- $\otimes$ distributes over $\oplus$, i.e., $\forall s, t, u \in S$

$$s \otimes (t \oplus u) = (s \otimes t) \oplus (s \otimes u)$$

$$(t \oplus u) \otimes s = (t \otimes s) \oplus (u \otimes s)$$

- $\oplus$ is idempotent, i.e., $\forall s \in S$

$$s \oplus s = s$$

3. Associated Dioid

Let F be a *pwp* endofunctor over **Set** and let (X, α) be a F-coalgebra, then:

Theorem

The set $B(X)$ togheter with the union of bisimulations and the composition of bisimulations forms an idempotent semiring:

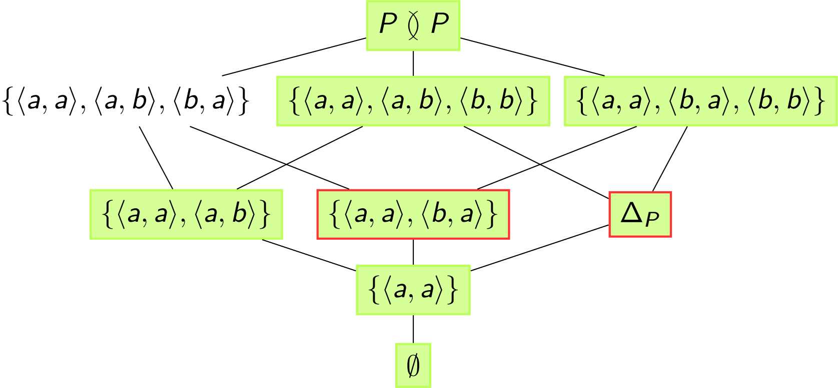
$$(B(X), \cup, \circ, \emptyset, \Delta_X)$$

Definition

We say that $(B(X), \cup, \circ, \emptyset, \Delta_X)$ is the *associated diod* of X, and we denote it by $\pi(X, \alpha)$.

3. Associated Dioid

Example



3. Associated Dioid

Proposition

Let F be a *pwp* endofunctor over **Set**. Let (X, α) and (Y, β) be two F-coalgebras. Let $f : X \to Y$ be a F-coalgebra homomorphism. It holds:

- $Z \in B(X) \;\Rightarrow\; f(Z) \in B(Y)$
- $Z \in B(Y) \;\Rightarrow\; f^{-1}(Z) \in B(Y)$

Remark

- $f(\Delta_X) = \Delta_{f(X)}$
- For each Z bisimulation on X holds that $f(Z^{-1}) = f(Z)^{-1}$
- If f is a F-coalgebra embedding, for each Z_1, Z_2 bisimulations on X holds that $f(Z_1 \circ Z_2) = f(Z_1) \circ f(Z_2)$

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4. Subcoalgebras

Definition

Let F be an arbitrary functor over **Set**. Let (X, α) be a F-coalgebra. Let $W \subseteq X$ be any subset of X.
We say that W is a *subcoalgebra* of X, written $W \leq X$, if there exists an structure map α_W on W such that it turns the inclusion mapping $i : W \to X$ into a F-coalgebra homomorphism.

$$\begin{array}{ccc} W & \xrightarrow{\quad i \quad} & X \\ \exists \; \Big\downarrow \alpha_W & & \Big\downarrow \alpha \\ FW & \xrightarrow[\quad Fi \quad]{} & FX \end{array}$$

4. Basic Properties

Properties for arbitrary functors

- $W \leq X \;\Leftrightarrow\; \Delta_W \in B(X)$
- $B(W) \subseteq B(X)$

Properties for *pwp* functors

- $\Delta_W \circ B(X) \circ \Delta_W = B(W)$

Let $f : X \to Y$ be a coalgebra homomorphism, then:

- $W \leq X \Rightarrow f(W) \leq Y$
- $W \leq Y \Rightarrow f^{-1}(W) \leq X$

Theorem

Let F be a *pwp* endofunctor over **Set** and let (X, α) be a F-coalgebra. The collection of all subcoalgebras of X is a complete lattice in which least upper bounds and greatest lower bounds are given by union and intersection.

Definition

Let $Y \subseteq X$ be any subset of X. We define:
The *subcoalgebra generated by* Y, denoted by $\langle Y \rangle$,

- $\langle Y \rangle = \bigcap \{W : W \leq X \textit{ and } Y \subseteq W\}$

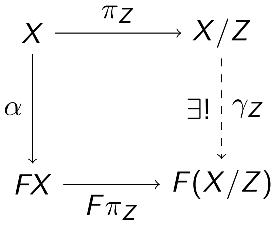
The *greatest subcoalgebra contained in* Y, denoted by $[Y]$,

- $[Y] = \bigcup \{W : W \leq X \textit{ and } W \subseteq Y\}$

5. Quotients

Theorem

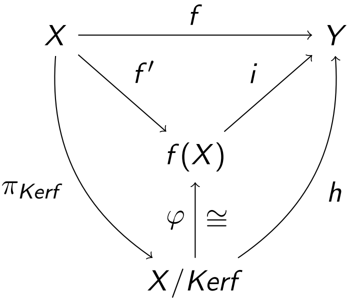
Let F be an arbitrary endofunctor over **Set**. Let (X, α) be a F-coalgebra. Let Z be a bisimulation equivalence on X. Then there exists a unique map structure $\gamma_Z : X/Z \to F(X/Z)$ that turns $\pi_Z : X \to X/Z$ (the quotient mapping), into a F-coalgebra homomorphism.



6. Isomorphism Theorems

1st Isomorphism Theorem

Let F be a *pwp* endofunctor over **Set**, let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be a F-coalgebra homomorphism. Then there is the following factorization of f:



Where:

- i is the inclusion morphism.
- h is a monomorphism.
- f' is a epimorphism with $f(x) = f'(x)$ for each $x \in X$.
- π_{Kerf} is the quotient morphism.

2nd Isomorphism Theorem

Let F be a *pwp* endofunctor over **Set**, let (X, α) be a F-coalgebra, let $W \leq X$ and let Z be a bisimulation equivalence on X.
Let W^Z be defined as

$$W^Z = \{x \in X : \ \exists w \in W(\langle x, w\rangle \in Z)\}$$

The following facts hold:

- $W^Z \leq X$.
- $Z \cap (W \times W)$ is a bisimulation equivalence on W.
- $W/(Z \cap (W \times W)) \cong W^Z/Z$.

6. Isomorphism Theorems

3rd Isomorphism Theorem

Let F be a *pwp* endofunctor over **Set**. Let (X, α) be a F-coalgebra and let Z_1 and Z_2 be two bisimulation equivalences on X such that $Z_2 \subseteq Z_1$. It holds:

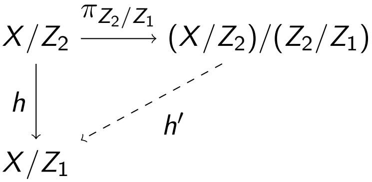
- There is a unique F-coalgebra homomorphism $h : X/Z_2 \to X/Z_1$ such that $h\pi_{Z_2} = \pi_{Z_1}$. That is to say that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_{Z_2}} & X/Z_2 \\
& \searrow^{\pi_{Z_1}} & \Big\downarrow{\scriptstyle \exists!\ h} \\
& & X/Z_1
\end{array}
$$

6. Isomorphism Theorems

3rd Isomorphism Theorem

- Let Z_2/Z_1 denote *Kerh*. It holds that Z_2/Z_1 is a bisimulation equivalence on X/Z_2 and induces a F-coalgebra isomorphism $h' : (X/Z_2)/(Z_2/Z_1) \to X/Z_1$ such that $h = h'\pi_{Z_2/Z_1}$.
  That is to say that the following diagram commutes:



7. Simple Coalgebras

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Definition

Let F be a *pwp* endofunctor on **Set**. We say that a F-coalgebra, (X, α), is *simple* if it has no proper quotients. That is to say, if Z is a bisimulation equivalence on X, then $X/Z \cong X$.

7. Basic Properties

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Theorem

The following statements are equivalent:

- (X, α) is a simple F-coalgebra.

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- Every epimorphism $f : X \to Y$ is an isomorphism.

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- Every epimorphism $f : X \to Y$ is an isomorphism.
- Let Z be a bisimulation on X, then $Z \subseteq \Delta_X$.

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Theorem

The following statements are equivalent:

- (X, α) is a simple F-coalgebra.
- Every epimorphism $f : X \to Y$ is an isomorphism.
- Let Z be a bisimulation on X, then $Z \subseteq \Delta_X$.
- Δ_X is the only bisimulation equivalence on X.

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Theorem

The following statements are equivalent:

- (X, α) is a simple F-coalgebra.
- Every epimorphism $f : X \to Y$ is an isomorphism.
- Let Z be a bisimulation on X, then $Z \subseteq \Delta_X$.
- Δ_X is the only bisimulation equivalence on X.
- Let $f : Y \to X$ and $g : Y \to X$ be two F-coalgebra homomorphisms, then $f = g$.

7. Basic Properties

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The following statements are equivalent:

- (X, α) is a simple F-coalgebra.
- Every epimorphism $f : X \to Y$ is an isomorphism.
- Let Z be a bisimulation on X, then $Z \subseteq \Delta_X$.
- Δ_X is the only bisimulation equivalence on X.
- Let $f : Y \to X$ and $g : Y \to X$ be two F-coalgebra homomorphisms, then $f = g$.
- The quotient homomorphism $\pi_{\between} : X \to X/(X \between X)$ is a F-coalgebra isomorphism.

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Theorem

The following statements are equivalent:

- (X, α) is a simple F-coalgebra.
- Every epimorphism $f : X \to Y$ is an isomorphism.
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- The quotient homomorphism $\pi_{\between} : X \to X/(X \between X)$ is a F-coalgebra isomorphism.
- Any F-coalgebra homomorphism, $f : X \to Y$, is injective.

7. Example: NFAs

Definition

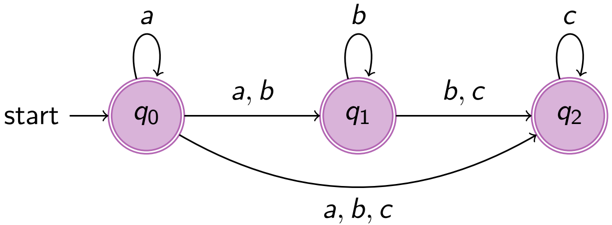
A *nondeterministic finite automaton* or *NFA* is a quintuple $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$, where:

- Q is a finite set of *states*.
- Σ is a finite set of symbols, known as *alphabet*. The elements of Σ are called *letters*.
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is a partial function named *transition function*.
- $q_0 \in Q$ is the *initial* state.
- $F \subseteq Q$ is the set of *final* states.

7. Example NFAs

Example

Let $\mathbb{M} = (\{q_0, q_1, q_2\}, \{a, b, c\}, \delta, q_0, \{q_0, q_1, q_2\})$ be a NFA with the corresponding transition diagram given by:



7. Example: NFAs

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Definition

In order to define the behaviour of a NFA on a string it is necessary to extend the transition function to a function acting on states and strings. Therefore, we define the *extended transition function* $\hat{\delta} : Q \times \Sigma^{\star} \to \mathcal{P}(Q)$ in the following way:

- $\forall q \in Q,\ x \in \Sigma^{\star},\ a \in \Sigma$:
- $\hat{\delta}(q, \lambda) = \{q\}$
- $\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$

Item 2. means that a NFA can not change its state until it gets a symbol; The 3rd item states the recursive definition of $\hat{\delta}$ on non-empty strings.

7. Example: NFAs

Definition

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ be a NFA, and let $x \in \Sigma^\star$ be a string. We say that x is *accepted* by $\mathbb{M}$ whenever $\delta(q_0, x) \cap F \neq \emptyset$ holds. We define the *accepted language* of the NFA $\mathbb{M}$ as:

$$L(\mathbb{M}) = \{x \in \Sigma^\star : \ \delta(q_0, x) \cap F \neq \emptyset\}$$

Definition

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ be a NFA, and let $x \in \Sigma^\star$ be a string. We say that x is *accepted* by $\mathbb{M}$ whenever $\delta(q_0, x) \cap F \neq \emptyset$ holds. We define the *accepted language* of the NFA $\mathbb{M}$ as:

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Definition

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ be a NFA. The mapping $N : Q \to \mathcal{P}(2)$ is defined for each $p \in Q$ as:

- $0 \in N(p)$ if and only if $p = q_0$.
- $1 \in N(p)$ if and only if $p \in F$.

We say that $N(p)$ is the *nature of the state p*.

7. Example: NFAs

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Theorem

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ and $\mathbb{M}' = (Q', \Sigma, \delta', q'_0, F')$ be two NFA. If q_0 and q'_0 are bisimilar, then $L(\mathbb{M}) = L(\mathbb{M}')$.

7. Example: NFAs

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Theorem

Let L be a regular language, there exists a minimal NFA that accepts L. It is unique up to isomorphism.

8. Final Coalgebras

Definition

Let F be a *pwp* endofunctor on **Set**. We say that a F-coalgebra, (X, α), is *final* if for any other F-coalgebra (Y, β) there exists a unique F-coalgebra homomorphism $f_Y : Y \to X$.

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Proposition

Let (X, α) be a final F-coalgebra, then (X, α) is simple.

8. Coinduction

Example

Let A be an arbitrary set, we define the endofunctor G as:

$$\begin{array}{rccc} G: & \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ & X & \longmapsto & A \times X \\ & f & \longmapsto & id_A \times f \end{array}$$

Let (X, α) be an arbitrary G-coalgebra. The structure map on X can be splitted in two functions $X \to A$ and $X \to X$ which we will call $value : X \to A$ and $next : X \to X$. With these operations we can do two things, given an element $x \in X$:

- Produce an element in A, namely $value(x)$.
- Produce a next element in X, namely $next(x)$.

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Example

Now we can repeat this process and therefore form another element in A, namely $value(next(x))$. By preceding in this way we can get for each element $x \in X$ an infinite sequence $(a_0, a_1, a_2, \cdots) \in A^\omega$ of elements $a_n = value(next^{(n)}(x)) \in A$, where $next^{(0)}(x)$ denotes x. This sequence of elements that x gives rise to is what we can observe about x.

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Proposition

$(A^\omega, \langle head, tail \rangle)$ is a final G-coalgebra.

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