A BRIEF INTRODUCTION TO COALGEBRAS

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INTRODUCTION

Universal Coalgebra is a theory of Systems

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It suffices to:

- Model systems.
- Construct morphisms between systems.
- Detect behavioural equivalent states.
- Simplify systems.
- Define new concepts and operators via coinduction.

UNIVERSAL COALGEBRA

COALGEBRA

Given a category $\mathbf{X}$, called the base category, and an endofunctor $F : \mathbf{X} \to \mathbf{X}$.

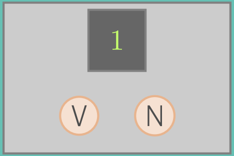
Definition

A F-coalgebra consists of a pair (X, α), where X is an object of $\mathbf{X}$ and $\alpha : X \to FX$ an arrow in $\mathbf{X}$.

We call X the base and α the structure map of the coalgebra.

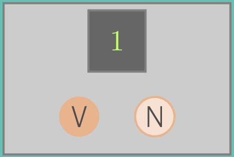
COALGEBRA

Two buttons machine



COALGEBRA

Two buttons machine



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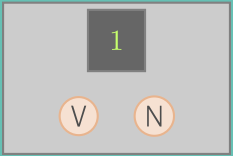

COALGEBRA

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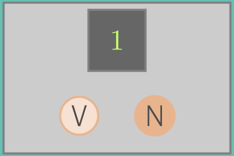

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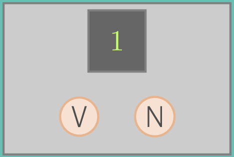

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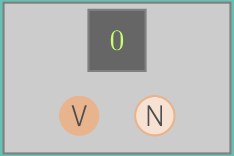

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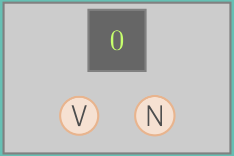

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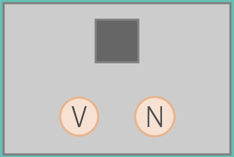


This machine can be described as a coalgebra:

$$(v, n) : X \longrightarrow A \times X$$

COALGEBRA HOMOMORPHISMS

Definition

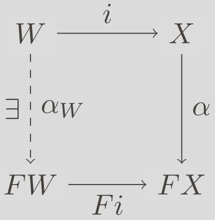
Let (X, α) and (Y, β) be two F-coalgebras. An F-coalgebra homomorphism, $f : (X, \alpha) \to (Y, \beta)$ is an arrow $f : X \to Y$ in $\mathbf{X}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ FX & \xrightarrow[Ff]{} & FY \end{array}$$

SUBCOALGEBRAS

Definition

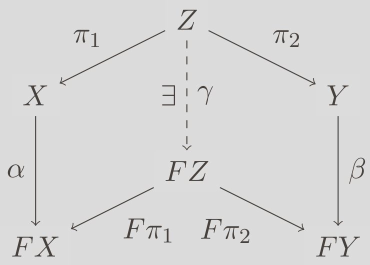
Let (X, α) be an F-coalgebra. Let $W \subseteq X$ be any subset of X. We say that W is a subcoalgebra of X, written $W \leq X$, if there exists an structure map α_W on W such that it turns the inclusion mapping $i : W \to X$ into an F-coalgebra homomorphism.



BISIMULATION

Definition

Let (X, α), (Y, β) be two F-coalgebras. A subset $Z \subseteq X \times Y$ of the cartesian product of X and Y is called a F-bisimulation if there exists a structure map $\gamma : Z \to FZ$ such that the projections from Z to X and Y are F-coalgebra homomorphisms.



BISIMULATION

Definition

If $(X, \alpha) = (Y, \beta)$, then (Z, γ) is called a bisimulation on (X, α). A bisimulation equivalence is a bisimulation that is also an equivalence relation.

Two states $x \in X$, $y \in Y$ are called bisimilar if there exists a bisimulation Z with $\langle x, y \rangle \in Z$.

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Example

The empty set, $\emptyset \subseteq X \times Y$, is always a bisimulation. $\emptyset \in B(X, Y)$.

BISIMULATION

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Two buttons machine

Let $(X, (v, n))$ and $(Y, (v, n))$ be two 2BMs. The elements $x \in X$ and $y \in Y$ are bisimilar if and only if:

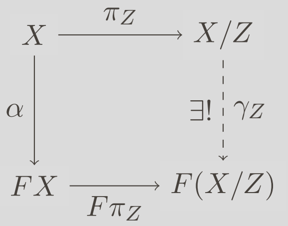
$v(x) = v(y)$

$n(x)$ and $n(y)$ are bisimilar.

QUOTIENTS

Theorem

Let (X, α) be a F-coalgebra. Let Z be a bisimulation equivalence on X. Then there exists a unique map structure $\gamma_Z : X/Z \to F(X/Z)$ that turns $\pi_Z : X \to X/Z$ (the quotient mapping), into a F-coalgebra homomorphism.



FINAL COALGEBRAS

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We say that a F-coalgebra, (X, α), is a final F-coalgebra if for any other F-coalgebra (Y, β) there exists a unique F-coalgebra homomorphism $! : Y \to X$.

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Theorem

Let (X, α) be a final F-coalgebra, then α is a F-coalgebra isomorphism. Final coalgebras if they exist are uniquely determined up to isomorphism.

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They are fixed points of the functor, i.e., $F(X) \cong X$.

FINAL COALGEBRAS

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Consider the following 2BM of infinite streams $(A^{\mathbb{N}}, (h,t))$ with structural map:

$$(h,t): \quad \begin{array}{ccc} A^{\mathbb{N}} & \longrightarrow & A \times A^{\mathbb{N}} \\ (a_i)_{i\in\mathbb{N}} & \longmapsto & (a_0, (a_{i+1})_{i\in\mathbb{N}}) \end{array}$$

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Proposition

$(A^{\mathbb{N}}, (h,t))$ is the final 2BM, with the unique morphism given by:

$$\begin{array}{cccc} !: & (X,(v,n)) & \longrightarrow & (A^{\mathbb{N}}, (h,t)) \\ & x & \longmapsto & (v(n^i(x)))_{i\in\mathbb{N}} \end{array}$$

FINAL COALGEBRAS

The existance of the final coalgebra allow us to define new elements and new operators.

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Let (X, α) and (Y, β) be two coalgebras. Let $x \in X$ and $y \in Y$.

Theorem

$$x \text{ and } y \text{ are bisimilar} \quad \Leftrightarrow \quad !(x) = !(y)$$

This theorem allow us to prove propositions via coinduction.

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