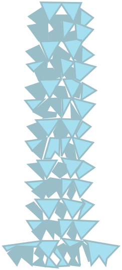
CALCULUS OF RELATIONS

Axiomatisation and algorithms

Primer Congrés Predoc

Burjassot, 2016



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THE CALCULUS OF RELATIONS

GROUND NOTIONS

A relation R on a set X is a set of pairs from X.

The set of relations $\mathcal{P}(X \times X)$ is equipped with set-theoretic inclusion $(\subseteq)$ as partial order and it contains different distinguished elements and binary operations.

We highlight the following ones.

Empty relation	$0 = \emptyset$
Identity relation	$1 = \{(x, x) \mid x \in X\} = \Delta_X$
Total relation	$\top = X \times X = \nabla_X$

GROUND NOTIONS

For relations R, S on X, we consider the following operations.

Union $R + S$ Intersection $R \cap S$

Relational composition
$R{\cdot}S = \{(x, z) \mid \exists y \in X\ ((x, y) \in R \text{ and } (y, z) \in S)\}$

Converse $R^\circ = \{(y, x) \mid (x, y) \in R\}$

Reflexive-transitive closure
$R^\star = \{(x_0, x_{n-1}) \mid \exists n \in \mathbb{N}\ \exists (x_i)_{i \in n} \in X^n\ ((x_i, x_{i+1}) \in R)\}$

GROUND NOTIONS

In this way we can state many properties in a concise way.

Example

Reflexivity	$1 \subseteq R$
Symmetric	$R^\circ \subseteq R$
Transitive	$R \cdot R \subseteq R$
Acyclicity	$R \cdot R^\star \cap 1 = 0$

Some of these properties do not always hold.

GROUND NOTIONS

Others do always hold. These (in)equations are called universally valid.

Example

$$R{\cdot}(S{\cdot}T) = (R{\cdot}S){\cdot}T$$
$$1 \cap R \subseteq R^2 \cap R^3$$
$$(R{\cdot}S) \cap T \subseteq R{\cdot}(S \cap (R^\circ{\cdot}T))$$

Contemporary theory of relations are to be found in the writings of A. De Morgan, C. S. Peirce, and E. Schröder pursuing this kind of universally valid (in)equations.

GROUND NOTIONS

Two questions arise naturally:

1. Decidability

   Is it possible to decide whether a law is universally valid ?

2. Finite axiomatisability

   Is it possible to give a finite system that axiomatise the set of universally valid laws?

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   Is it possible to decide whether a law is universally valid ?
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   Is it possible to give a finite system that axiomatise the set of universally valid laws?

1. NO, [Tarski, 1941] 2. NO, [Monk, 1964]

THE WORK OF TARSKI

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For a set of variables Σ, consider the following grammar Λ

$$e, f ::= e + f \mid e \cap f \mid e {\cdot} f \mid e^{\circ} \mid e^{\star} \mid 0 \mid 1 \mid \top \mid a \qquad (a \in \Sigma).$$

These algebraic structures will be called relation algebras.

Tarski also provided an algebraic interpretation of universally valid equations.

THE WORK OF TARSKI

Other structures admit the same algebraic type.

Among others, languages over a set Σ. Consider the free monoid $(\Sigma^\star, \cdot, \varepsilon)$. Operations and constants in the grammar admit natural interpretations on languages.

$$0 = \emptyset \qquad 1 = \{\varepsilon\} \qquad \top = \Sigma^\star \qquad L + K \qquad L \cap K$$

$$LK = \{vw \mid v \in L,\ w \in K\} \qquad L^\star = \bigcup_{n\in\mathbb{N}} L^n$$

$$L^\circ = \{v^\circ \mid v \in L\}$$

THE WORK OF TARSKI

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   Are all the relation algebras isomorphic to an algebra of relations over a set X?

3. NO

$$\mathsf{Rel}(\Lambda) \models e \leq e{\cdot}e^{\circ}{\cdot}e \qquad \text{but} \qquad \mathsf{Lang}(\Lambda) \not\models e \leq e{\cdot}e^{\circ}{\cdot}e$$

THE WORK OF MONK

Tarski also appealed to a geometric interpretation of relations. This is the basis for Monk's results on finite axiomatisability.

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2. The theorem of Bruck and Ryser on the nonexistence of projective planes of certain finite orders (1949);
3. The fundamental theorem of Łoś on ultraproducts (1961).

THE WORK OF ANDRÉKA

The proof of Monk has proved to be very successful.

Theorem (Andréka and Mikulás, 2011)

Let $\{+, \cdot\} \subseteq \Lambda \subseteq \{+, \cap, \cdot, ^\circ, 0, 1, \top\}$, then the class Rel($\Lambda$) is not finitely axiomatizable.

The idea behind: Construct a family of relation algebras not being representable. Take an ultraproduct of this family and check that it is representable. If there is an axiomatization of this kind of algebras, it cannot be finite.

KLEENE ALGEBRAS

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To each regular expression e we can define inductively a language $[e]$ as follows

$$[e+f] \triangleq [e] + [f] \qquad [e{\cdot}f] \triangleq [e][f] \qquad [e^\star] \triangleq [e]^\star$$
$$[0] \triangleq \emptyset \qquad [1] \triangleq \{\epsilon\} \qquad [a] \triangleq \{a\}$$

DECIDABILITY OF KLEENE ALGEBRAS

Kleene algebras are decidable.

Theorem

For any regular expressions, e, f, we have

$$\mathsf{Rel}(\Lambda) \models e = f \quad \text{iff} \quad [e] = [f]$$

Language equivalence can be checked coinductively by establishing a bisimulation on suitable deterministic automata.

AXIOMATISABILITY OF KLEENE ALGEBRAS

In 1956 Kleene asks for an axiomatisation of the previous theory. If it exists, it cannot be finite according to Andréka's theorem.

In the nineties Krob and Kozen independently show that one can axiomatise this theory in a finite way, but using Horn clauses instead of equations. It is based in the algebraic encoding of two fundamental constructions in finite automata theory.

- Determinization of an automaton via subset construction
- Minimization via a Myhill-Nerode equivalence relation

AXIOMATISABILITY OF KLEENE ALGEBRAS

Theorem (Kozen, 1991)

For any regular expressions e, f, we have $[e] = [f]$ if and only if the equality $e = f$ is derivable from the axioms listed below, where notation $e \leq f$ is a shorthand for $e + f = f$.

$$\left.\begin{array}{rcl} e + (f + g) & = & (e + f) + g \\ e + f & = & f + e \\ e + 0 & = & e \\ e + e & = & e \end{array}\right\} \quad (+, 0) \text{ is a commutative and idempotent monoid}$$

AXIOMATISABILITY OF KLEENE ALGEBRAS

$$\left.\begin{array}{rcl} e\cdot(f\cdot g) & = & (e\cdot f)\cdot g \\ e\cdot 1 & = & e \\ 1\cdot e & = & e \end{array}\right\} \quad (\cdot, 1) \text{ is a monoid}$$

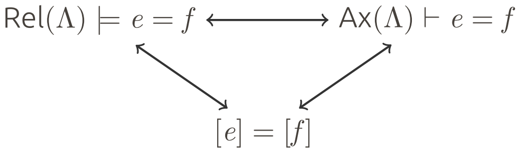
$$\left.\begin{array}{rcl} e\cdot(f+g) & = & e\cdot f + e\cdot g \\ (e+f)\cdot g & = & e\cdot g + f\cdot g \\ e\cdot 0 & = & 0 \\ 0\cdot e & = & 0 \end{array}\right\} \quad \text{distributivity between the two monoids}$$

$$\left.\begin{array}{rcl} 1 + e\cdot e^\star & = & e^\star \\ e\cdot f \leq f & \Rightarrow & e^\star\cdot f \leq f \\ f\cdot e \leq f & \Rightarrow & f\cdot e^\star \leq f \end{array}\right\} \quad \text{laws about Kleene star}$$

It follows that $(+, \cdot, 0, 1)$ is an idempotent semiring.

SUMMARY

For Kleene algebras we have the following summary of results.
No representability results are known for us.



ALLEGORIES

ALLEGORIES

Allegories are algebras of the following type

$$e, f ::= e{\cdot}f \mid e \cap f \mid e^{\circ} \mid 1 \mid \top \mid a \qquad\qquad (a \in \Sigma).$$

This fragment was introduced by Freyd and Scedrov in 1990. Modulo the presence of the constant $\top$, this fragment has been intensively studied by Andréka and Bredikhin.

We will see that one can decide the validity of inequations in this fragment but, again, the corresponding theory is not finitely axiomatisable in a purely equational way.

DECIDABILITY OF ALLEGORIES

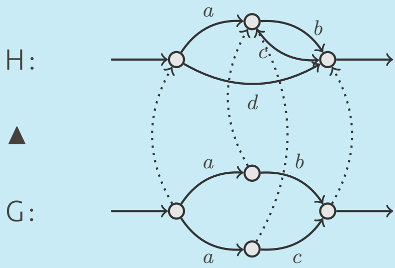
A 2-pointed graph is a tuple (V, E, ι, o), where V is a set of vertices, $E \subseteq V \times \Sigma \times V$ is a set of directed labelled edges, and $\iota, o \in V$ are two distinguished vertices respectivelly called input and output.

A homomorphism from the graph G to the graph H is a function from vertices of G to vertices of H that preserves labelled edges, input, and output.

We write H ◀ G when there is a homomorphism from G to H.

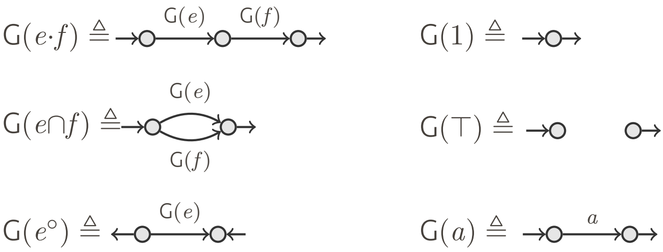
DECIDABILITY OF ALLEGORIES

Example



DECIDABILITY OF ALLEGORIES

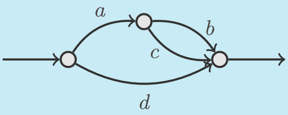
To each regular expression e we can define inductively a 2-pointed graph G(e) as follows



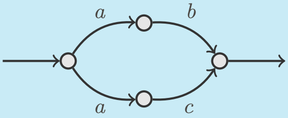
DECIDABILITY OF ALLEGORIES

Example

$\mathsf{G}((a \cdot (b \cap c)) \cap d)$:



$\mathsf{G}((a \cdot b) \cap (a \cdot c))$:



DECIDABILITY OF ALLEGORIES

Theorem (Freyd and Scedrov, 1990)

For any terms e, f, we have

$$\mathsf{Rel}(\Lambda) \models e \subseteq f \qquad\qquad \mathsf{G}(e) \blacktriangleleft \mathsf{G}(f).$$

DECIDABILITY OF ALLEGORIES

Theorem (Freyd and Scedrov, 1990)

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Corollary

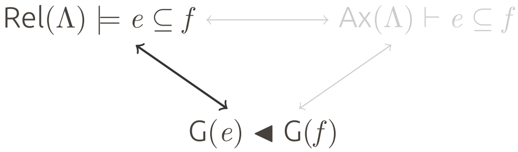
$\mathsf{Rel}(\Lambda) \models a{\cdot}(b \cap c^\circ) \cap d \subseteq a{\cdot}b \cap a{\cdot}c$

Exercise [Dedekind's inequality]

$$\mathsf{Rel}(\Lambda) \models e{\cdot}f \cap g \subseteq (e \cap g{\cdot}f^\circ){\cdot}(f \cap e^\circ{\cdot}g)$$

SUMMARY

For allegories we have the following summary of results. Again, no representability results are known for us.



For Kleene algebras the decision procedure, that is language equivalence, is a very well known procedure whereas, for allegories, 2-pointed graphs associated to regular expressions are not so well understood. The first results we obtained try to characterise these kind of graphs.

COQ

COQ

Coq is a formal proof management system. It provides a formal language to write mathematical definitions, executable algorithms and theorems together with an environment for semi-interactive development of machine-checked proofs.

Typical applications include the certification of properties of programming languages, formalization of mathematics (4-colours theorem, Feit-Thompson's theorem) and teaching.

Our current research will be implemented in a modular library on relation algebras.

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