A COMPLETE AXIOMATISATION OF ISOMORPHISM
of graphs of treewidth 2

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Algebras of relations appear naturally in many contexts in computer science as they constitute a framework well suited to the semantics of imperative programs.

Many objects of interest either are relations or can be seen as relations. A major benefit of a relational approach in computer science is the surprisingly small number of relations needed to express complex notions.

ALLEGORIES

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$$u, v ::= u \cdot v \mid u \cap v \mid u^\circ \mid 1 \mid \top \mid a \qquad\qquad (a \in \Sigma).$$

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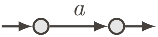
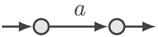
One model for this algebra is the set of relations on a given set with the usual interpretation of the operators.

TERMS AS GRAPHS

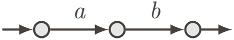
$\mathsf{g}(a)$ 

TERMS AS GRAPHS

$\mathsf{g}(a)$



$\mathsf{g}(a \cdot b)$



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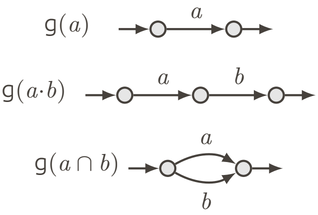

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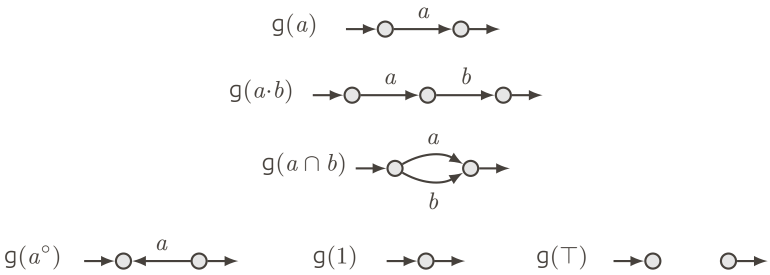

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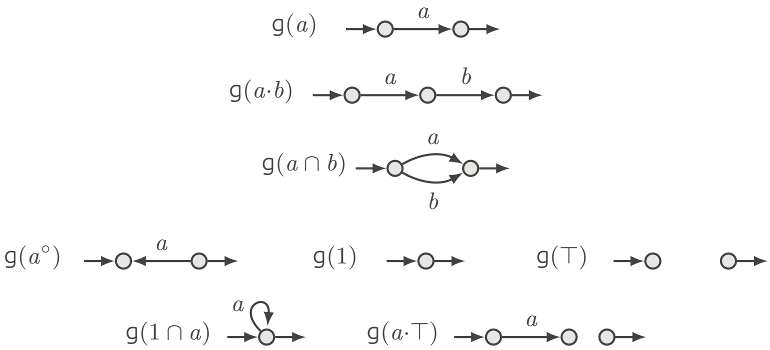

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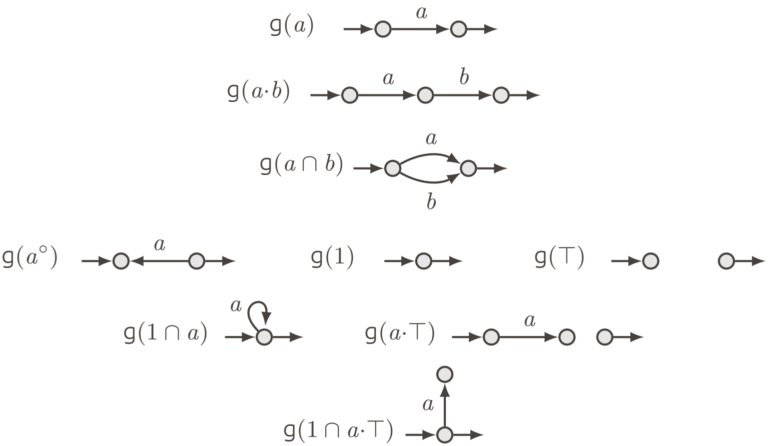
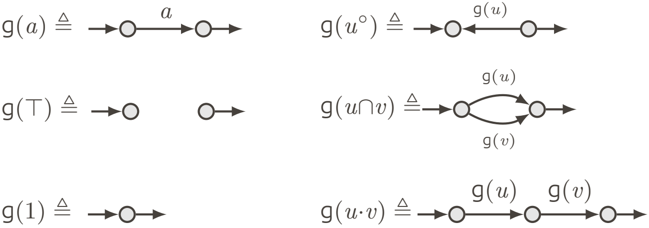


ASSOCIATED GRAPH



$$\text{Term} \xrightarrow{\;\mathsf{g}\;} \text{Graphs}$$

Theorem [FS90]

For any terms u, v, we have

$$\mathsf{Rel} \models u \subseteq v \qquad \Leftrightarrow \qquad \mathsf{g}(u) \blacktriangleleft \mathsf{g}(v).$$

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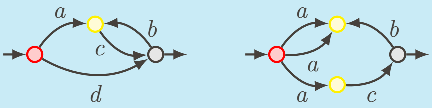
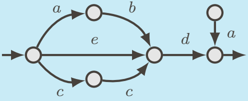
Example

$$\mathsf{Rel} \models a{\cdot}(b^\circ \cap c) \cap d \subseteq (a \cap a){\cdot}b^\circ \cap a{\cdot}c$$

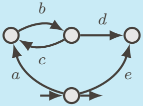
$\mathsf{g}(a{\cdot}(b^\circ \cap c) \cap d) \blacktriangleleft \mathsf{g}((a \cap a){\cdot}b^\circ \cap a{\cdot}c)$



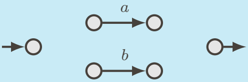
Example



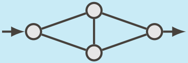
$\mathsf{g}((a\cdot b\cap c^2\cap e)\cdot d\cdot(1\cap a^\circ\cdot\top))$



$\mathsf{g}(1\cap a\cdot(b\cap c^\circ)\cdot d\cdot e^\circ)$



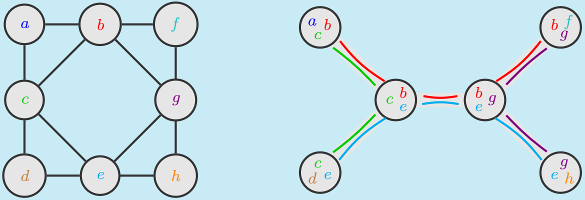
$\mathsf{g}(\top\cdot a\cdot\top\cdot b\cdot\top) \cong \mathsf{g}(\top\cdot b\cdot\top\cdot a\cdot\top)$



It is not a term graph [FS90]

TREE DECOMPOSITION

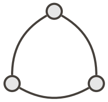
Example



The width of a tree decomposition is the size of the largest set V_t minus one. The treewidth of a graph is the minimal width of a tree decomposition for this graph.

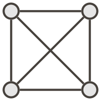
Bounded treewidth can be described by minor exclusion.

TW_1



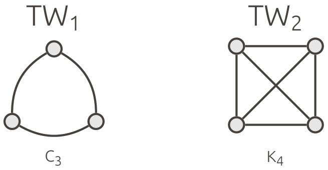
C_3

TW_2



K_4

Bounded treewidth can be described by minor exclusion.

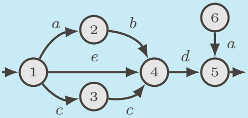


Proposition

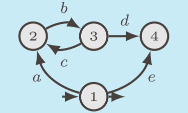
Every term graph has treewidth bounded by 2 with one node containing input and output.

$$\text{Term} \xrightarrow{g} TW_2$$

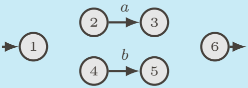
Example



$\mathsf{g}((a{\cdot}b\cap c^2\cap e){\cdot}d{\cdot}(1\cap a^\circ{\cdot}\top))$



$\mathsf{g}(1\cap a{\cdot}(b\cap c^\circ){\cdot}d{\cdot}e^\circ)$



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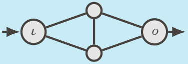

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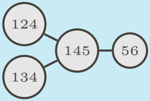
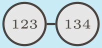
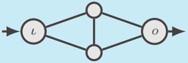
Example

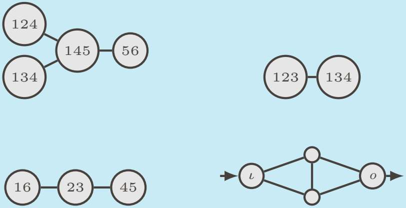






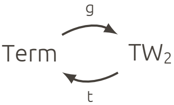
No such tree decomposition

Example



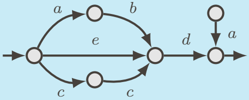
No such tree decomposition

Extract a term from a graph with compatible input and output.



CASE 1: CONNECTED WITH INPUT DIFFERENT FROM OUTPUT

Example

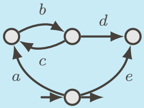


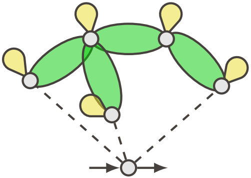
Case 1.1

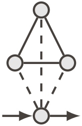
Case 1.2

CASE 2: CONNECTED WITH INPUT EQUALS OUTPUT

Example



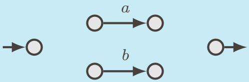




$1 \cap u{\cdot}\top$

CASE 3: DISCONNECTED

Example



$\top \cdot u$	Disconnects the input.
$u \cdot \top$	Disconnects the output.

Theorem

For any 2-pointed graph G with compatible input and output,

$$g(t(G)) \cong G.$$

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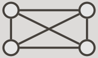
Corollary

Let G be a graph. The following statements are equivalent.

1. G is a term graph.
2. G has treewidth bounded by 2.
3. G is K_4 minor free.



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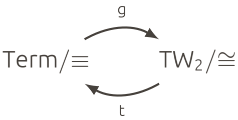

The reduct $(\cap, \top)$ is a commutative monoid, the reduct $(\cdot, 1)$ is a monoid. The converse $^\circ$ is an involution.

$$
\begin{aligned}
1 \cap 1 &\equiv 1 \\
u\cdot(1 \cap v) &\equiv u \cap \top\cdot(1 \cap v) \\
1 \cap u\cdot v &\equiv 1 \cap (u \cap v^\circ)\cdot\top \\
u\cdot\top \cap v &\equiv (1 \cap u\cdot\top)\cdot v
\end{aligned}
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\end{aligned}
$$

Theorem

The axioms listed above give a complete axiomatisation of isomorphism of graphs of treewidth bounded by 2.

$\mathsf{TW}_2/\cong$ is a free algebra.

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