COALGEBRAS

Enric Cosme

Contents

CHAPTER 1

Coalgebras

1. Basic concepts

We present in this section the most fundamental definitions of category theory; the definition of category and the definition of functor between categories.

Definition 1.1. **Category**
A *category* $\mathbf{X}$ is given by a collection $\mathbf{X}_0$ of *objects* and a collection $\mathbf{X}_1$ of *arrows* which have the following structure:

- Each arrow has a *domain* and a *codomain* which are objects. One writes $f : X \to Y$ if X is the domain of the arrow f and Y its codomain. One also writes $X = dom(f)$ and $Y = cod(f)$.
- Given two arrows f and g such that $cod(f) = dom(g)$ the *composition* of f and g written gf, is defined and has domain $dom(f)$ and codomain $cod(g)$, i.e.:

$$\left.\begin{array}{l} f : X \to Y \\ g : Y \to Z \end{array}\right| \quad \longmapsto \quad gf : X \to Z$$

- Composition is associative, that is: given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ it holds that $h(gf) = (hg)f$.
- For every object X there is an identity arrow $id_X : X \to X$, satisfying $id_X g = g$ for every $g : Y \to X$ and $f id_X = f$ for every $f : X \to Y$.

Example 1.2. We present here the category of sets, **Set**. **Set** is given by the collection $\mathbf{Set}_0$ consisting in sets and $\mathbf{Set}_1$ of usual mappings between sets.

Definition 1.3. **Functor**
Given two categories $\mathbf{X}$ and $\mathbf{X}'$, a functor $F : \mathbf{X} \to \mathbf{X}'$ consists of operations $F_0 : \mathbf{X}_0 \to \mathbf{X}'_0$ and $F_1 : \mathbf{X}_1 \to \mathbf{X}'_1$ such that:

- For each $f : X \to Y$, $F_1(f) : F_0(X) \to F_0(Y)$.
- For each $f : X \to Y$, $g : Y \to Z$ it holds that $F_1(gf) = F_1(g)F_1(f)$.
- For each $X \in \mathbf{X}_0$, $F_1(id_X) = id_{F_0(X)}$.

When $\mathbf{X}$ and $\mathbf{X}'$ are the same category, we say that F is an *endofunctor*.

We introduce here a useful proposition on functors.

Proposition 1.4. *Let F be an arbitrary functor over* **Set**. *Let X and Y be arbitrary sets with $X \neq \emptyset$. If $f : X \to Y$ is mono, then $Ff : FX \to FY$ is mono as well.*

Proof. Let $x_0 \in X$ and define $g : Y \to X$ as:

$$g(y) = \begin{cases} x & \text{if there is a unique } x \in X \\ & \text{such that } y = f(x) \\ x_0 & \text{otherwise} \end{cases}$$

Clearly $gf = Id_X$, therefore

$$FgFf = F(gf) = F(Id_X) = Id_{F(X)}$$

Thus, Ff is injective, that is, mono. □

2. Coalgebras for an endofunctor

Definition 1.5. **Coalgebra**

Given a category $\mathbf{X}$, called the *base category*, and an endofunctor $F : \mathbf{X} \to \mathbf{X}$, a F-*coalgebra* (or F-*system*) consists of a pair (X, α), where X is an object of $\mathbf{X}$ and $\alpha : X \to FX$ an arrow in $\mathbf{X}$. We call X the *base* and α the *structure map* of the coalgebra. When the endofunctor is clear we will refer to the pair simply as a coalgebra (or system).

One can see coalgebras as structures that allow us to focus our attention on the decomposition of elements.

Definition 1.6. **Coalgebra Homomorphism**

Let $\mathbf{X}$ be any category. Let F be an endofunctor over $\mathbf{X}$. Let (X, α) and (Y, β) be two F-coalgebras. A F-*coalgebra homomorphism*,

$$f : (X, \alpha) \to (Y, \beta)$$

is an arrow $f : X \to Y$ in $\mathbf{X}$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & FX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[\beta]{} & FY \end{array}$$

The F-coalgebras and their homomorphisms form a category, denoted by $\mathbf{CoAlg}(F)$.

We introduce here some useful results on coalgebra homomorphisms.

Proposition 1.7. *Consider any category $\mathbf{X}$ and any endofunctor F over it. Let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be a F-coalgebra homomorphism over them. If $g : Y \to X$ is an inverse of f as arrow, then g is also a F-coalgebra homomorphism*

Proof. Since g is an inverse of f and F is an endofunctor follows that $FgFf = Id_{FX}$. Therefore holds that:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & FX \\ {\scriptstyle g}\uparrow\downarrow{\scriptstyle f} & & {\scriptstyle Fg}\uparrow\downarrow{\scriptstyle Ff} \\ Y & \xrightarrow[\beta]{} & FY \end{array}
\qquad
\begin{array}{rcl} \alpha g & = & (FgFf)\alpha g \\ & = & Fg(Ff\alpha)g \\ & = & Fg(\beta f)g \\ & = & Fg\beta(fg) \\ & = & Fg\beta \end{array}$$

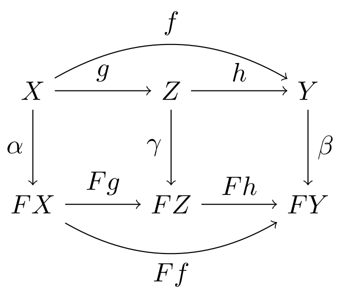
Therefore g makes the diagram commute. □

Proposition 1.8. *Consider any category $\mathbf{X}$ and any endofunctor F over it. Let (X, α), (Y, β) and (Z, γ) be three F-coalgebras and let $f : X \to Y$, $g : X \to Z$ and $h : Z \to Y$ be arbitrary functions. Then holds:*

- *If $f = hg$, f and g are F-coalgebra homomorphisms and g is surjective then h is a F-coalgebra homomorphism.*
- *If $f = hg$, f and h are F-coalgebra homomorphisms and h is injective then g is a F-coalgebra homomorphism.*

PROOF. We will do the proof for the first statement. The other is quite similar.

Consider any $z \in Z$. Since g is surjective there exists some $x \in X$ with $g(x) = z$. Since $f = hg$ and F is an endofunctor we obtain that $Ff = FhFg$. Therefore holds that:



$$\begin{aligned} \beta h(z) &= \beta hg(x) \\ &= \beta f(x) \\ &= Ff\alpha(x) \\ &= FhFg\alpha(x) \\ &= Fh\gamma g(x) \\ &= Fh\gamma(z) \end{aligned}$$

Therefore h makes the diagram commute. □

In the rest of this work we will focus our attention only on the category $\mathbf{X} = \mathbf{Set}$.

3. Coproducts

DEFINITION 1.9. **Coproduct of sets**

Let $\{X_j : j \in J\}$ be a family of sets. We define the *coproduct*, $\coprod_{j\in J} X_j$, of the family to be the set:

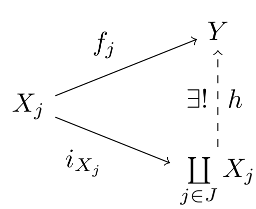
$$\coprod_{j\in J} X_j = \bigcup_{j\in J} (X_j \times \{j\})$$

We can associate to each $j \in J$ the *injection mapping*, $i_{X_j} : X_j \to \coprod_{j\in J} X_j$, defined as:

$$\begin{array}{rccl} i_{X_j} : & X_j & \longrightarrow & \coprod_{j\in J} X_j \\ & x & \longmapsto & \langle x, j \rangle \end{array}$$

PROPOSITION 1.10. **Universal Property of Coproduct**

Let $\{X_j : j \in J\}$ be a family of sets and let Y be an arbitrary set. Assume that for each $j \in J$ there exists some mapping $f_j : X_j \to Y$. Then there exists a unique mapping $h : \coprod_{j\in J} X_j \to Y$ such that for each $j \in J$, it makes the following diagram commute:



PROOF. Notice that each $z \in \coprod_{j\in J}$ has the form $\langle x, j \rangle$ for some $j \in J$ and $x \in X_j$. Therefore we define the mapping h as:

$$\begin{array}{rccc} h: & \coprod_{j\in J} X_j & \longrightarrow & Y \\ & z & \longmapsto & f_j(x) \end{array}$$

where $z = \langle x, j \rangle$.

This mapping makes the preceding diagram commute. For each $j \in J$ and for each $x \in X_j$ it holds:

$$hi_{X_j}(x) = h(\langle x, j \rangle) = f_j(x)$$

We only need to prove unicity. Assume there exists another $g : \coprod_{j\in J} X_j \to Y$ such that for each $j \in J$ it holds that $gi_{X_j} = f_j$. Take any $z \in \coprod_{j\in J} X_j$, as we have seen there exists some $j \in J$ and some $x \in X_j$ such that $z = \langle x, j \rangle = i_{X_j}(x)$. Then, $g(z) = g(i_{X_j}(x)) = f_j(x) = h(z)$. It finally holds that $g = h$. □

Remark 1.11. Notice that in Proposition 1.10, the resulting mapping is uniquely determined by the given mappings f_j, therefore we will use $\coprod_{j\in J} f_j$ instead of h.

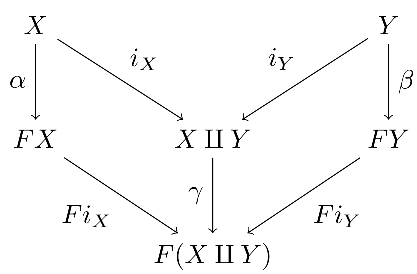
Definition 1.12. **Coproduct of Coalgebras**

Let F be any endofunctor over **Set**.

Assume (X, α) and (Y, β) are F-coalgebras.

The *coproduct* of this two systems is defined as the pair $(X \amalg Y, \gamma)$, where $X \amalg Y$ is the coproduct of X and Y and γ is an structure map $\gamma : X \amalg Y \to F(X \amalg Y)$ that turns the *injections* $i_X : X \hookrightarrow X \amalg Y$ and $i_Y : Y \hookrightarrow X \amalg Y$ into F-coalgebra homomorphisms.

That means that γ that the following diagram is commutative:



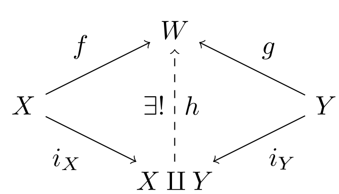
Remark 1.13. Notice that γ exists and it is unique. It is due to the Universal Property of the Coproduct 1.10. In fact, $\gamma = Fi_X\alpha \amalg Fi_Y\beta$.

Proposition 1.14. **Universal Property of Coproduct of Coalgebras**

Let F be an endofunctor over **Set**.

Let (X, α) and (Y, β) be two F-coalgebras.

For any F-coalgebra (W, δ) and any F-coalgebra homomorphisms $f : X \to W$ and $g : Y \to W$, there exists a unique F-coalgebra homomorphism $h : X \amalg Y \to W$ making the following diagram commute:



Proof. Notice that h exists by the Universal Property of Coproduct 1.10. In fact, $h = f \amalg g$. Let us see it is a F-coalgebra homomorphism. Hence take any

$z \in X \amalg Y$, then it must happen that $z = \langle x, 1\rangle$ for some $x \in X$ or $z = \langle y, 2\rangle$ for some $y \in Y$. In both cases hold that $\delta h = Fh\gamma$:

$z = \langle x, 1\rangle$, $x \in X$	$z = \langle y, 2\rangle$, $y \in Y$
$\begin{aligned} \delta h(z) &= \delta f(x) \\ &= Ff\alpha(x) \end{aligned}$	$\begin{aligned} \delta h(z) &= \delta g(y) \\ &= Fg\beta(y) \end{aligned}$
$\begin{aligned} Fh\gamma(z) &= FhFi_X\alpha(x) \\ &= F(h\ i_X)\alpha(x) \\ &= Ff\alpha(x) \end{aligned}$	$\begin{aligned} Fh\gamma(z) &= FhFi_Y\beta(y) \\ &= F(h\ i_Y)\beta(y) \\ &= Fg\beta(y) \end{aligned}$

Thus, h is the desired F-coalgebra homomorphism. □

Remark 1.15. One can easily extend the notion of coproduct to an arbitrary set of F-coalgebras, $\{(X_j, \alpha_j)\}_{j\in J}$, by defining its coproduct as the F-coalgebra that turns all the injections i_{X_j} into F-coalgebra homomorphisms. We will denote this F-coalgebra as

$$\left(\coprod_{j\in J} X_j, \gamma\right)$$

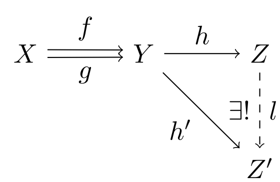
4. Coequalizers

Definition 1.16. **Coequalizer**

Let X and Y be two sets. Let $f : X \to Y$ and $g : X \to Y$ be two mappings. We define the *coequalizer of f and g*, to be the pair (Z, h) where Z is a set and $h : Y \to Z$ with the properties:

1. $hf = hg$
2. For any other set Z' and any other $h' : Y \to Z'$ accomplishing property 1., we can find a unique mapping $l : Z \to Z'$ with $lh = h'$

That is to say that the following diagram is commutative:



Proposition 1.17. *Let X and Y be two sets. Let $f : X \to Y$ and $g : X \to Y$ be two mappings. Then the coequalizer of f and g exists.*

Proof. Let $\sim$ be the minimal equivalence relation on Y for which holds that

$$\forall x \in X \quad f(x) \sim g(x)$$

Take Z to be the quotient of Y under this relation, i.e., $Z = Y/\sim$ and let $h = \pi_\sim$ be the quotient mapping:

$$\begin{array}{rccc} \pi_\sim : & Y & \longrightarrow & Y/\sim \\ & y & \longmapsto & [y]_\sim \end{array}$$

Let us check that all the properties from Definition 1.16 hold for $(Y/\sim, \pi_\sim)$:

1. Take any $x \in X$ it holds that $hf(x) = [f(x)]_\sim$ and $hg(x) = [g(x)]_\sim$. Notice that the classes coincide since $\sim$ contains all the pairs of the form $\langle f(x), g(x)\rangle$ for each $x \in X$.
2. Assume there exists another Z' and another $h' : Y \to Z'$ for which $h'f = h'g$. Define l as the mapping:

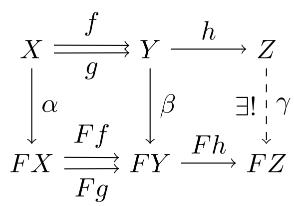
$$\begin{array}{rccc} l: & Y/\sim & \longrightarrow & Z' \\ & [y]_\sim & \longmapsto & h'(y) \end{array}$$

   l is well defined since $h'f = h'g$ and using the condition of miminality upon $\sim$. It also holds that $lh = h'$ and moreover from that follows that l is uniquely determined.

□

We will see that if X and Y have also a coalgebraic structure, one can find an structure map for the coequalizer of f and g with some good properties.

Proposition 1.18. *Let F be an endofunctor over* **Set**. *Let (X, α), (Y, β) be two F-coalgebras. Let $f : X \to Y$ and $g : X \to Y$ be two F-coalgebra homomorphisms. Let (Z, h) be a coequalizer of f and g. Then there exists a unique structure map $\gamma : Z \to F(Z)$ that turns h into a F-coalgebra homomorphism. That is to say that the following diagram commutes:*



Proof. Consider the mapping $Fh\beta : Y \to FZ$
It holds that:

$$\begin{aligned} Fh\beta f &= (Fh)(Ff)\alpha \\ &= F(hf)\alpha \\ &= F(hg)\alpha \\ &= (Fh)(Fg)\alpha = Fh\beta g \end{aligned}$$

So using the 2nd property for h, there exists a unique mapping $\gamma : Z \to F(Z)$ for which it holds that $\gamma h = Fh\beta$ □

5. Pullbacks

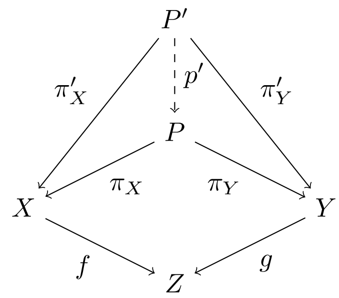
We introduce some notions that will be useful on the next chapters, specially on some results of Chapter 3.

Definition 1.19. **Weak Pullback**

A *weak pullback* of two mappings $f : X \to Z$ and $g : Y \to Z$ in the category **Set** is a triple (P, π_X, π_Y) such that P is a set, $\pi_X : P \to X$ and $\pi_Y : P \to Y$ are such that:

i) $f\ \pi_X = g\ \pi_Y$
ii) For each triple (P', π'_X, π'_Y) with $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$ and $f\pi'_X = g\pi'_Y$, there is a *mediating mapping* $p' : P' \to P$ such that $\pi_X p' = \pi'_X$ and $\pi_Y p' = \pi'_Y$

That is to say that the following diagram commutes:



Note that the mediating mapping p' need not to be unique; adding this requirement to the definition it would give the more familiar, and stronger, notion of *pullback*.

PROPOSITION 1.20. *The category* **Set** *has pullbacks.*

PROOF. Let $f : X \to Z$ and $g : Y \to Z$ be any two mappings in **Set**.
Consider the set

$$P = \{\langle x, y\rangle \in X \times Y : \ f(x) = g(y)\}$$

Let us see that the triple (P, π_1, π_2) consisting on the set P and the usual projections, form a weak pullback for (f, g).

The first property from Definition 1.19 trivially holds.

In order to check the second one, consider any (P', π'_X, π'_Y) with $\pi'_X : P' \to X$ and $\pi'_Y : P' \to Y$ and $f\pi'_X = g\pi'_Y$. Then define the mediating mapping p' as:

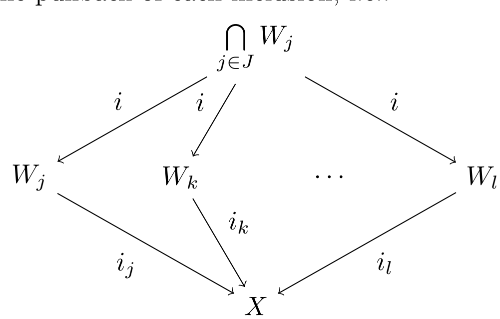
$$\begin{array}{rccl} p' : & P' & \longrightarrow & P \\ & w & \longmapsto & \langle \pi'_X(w), \pi'_Y(w)\rangle \end{array}$$

p' is well defined since $f\ \pi'_X = g\ \pi'_Y$. Furthermore also holds that $\pi_1 p' = \pi'_X$ and $\pi_2 p' = \pi'_Y$.

Therefore, (P, π_1, π_2) is a weak pullback for (f, g).

□

EXAMPLE 1.21. The intersection of a collection $\{W_j : j \in J\}$ of subsets of a set X can be constructed by means of *generalized pullbacks*, which is so to speak a pullback of a whole family of mappings at the same time. The intersection of the family appears as the pullback of each inclusion, i.e.:



where $\{i_j : W_j \to X : j \in J\}$ are the inclusion mappings. Notice that all functions are mono.

Definition 1.22. **Pullback Preservation**

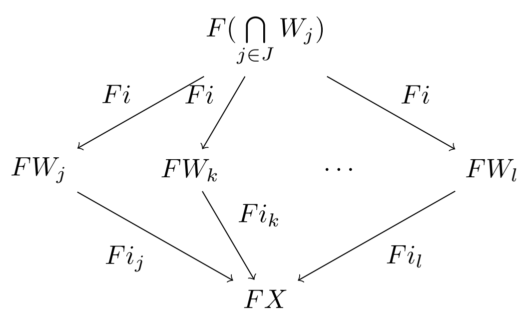
A functor F :**Set**→**Set** *preserves (weak) pullbacks* if for any (weak) pullback (P, π_X, π_Y) of (f, g) holds that the triple $(FP, F\pi_X, F\pi_Y)$ is a (weak) pullback of (Ff, Fg).

Proposition 1.23. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Then F preserves intersections, i.e.:*

$$F(\bigcap_{j\in J} W_j) \cong \bigcap_{j\in J} FW_j$$

for a given family $\{W_j : j \in J\}$ of subsets of a set X.

Proof. Let $\{W_j : j \in J\}$ be a family of subsets of X. Since F preserves weak pullbacks, the diagram of Example 1.21 is transformed by F into a new weak pullback diagram:



Since all the mappings in the diagram of Example 1.21 are mono, we can apply Proposition 1.4 to affirm that each function Fi_j for $j \in J$ is mono as well. This diagram is again a weak pullback in **Set**. Thus, $F(\bigcap_{j\in J} W_j)$ is isomorphic to $\bigcap_{j\in J} FW_j$ since we can find mediating mappings from one to another and viceversa. □

Many but not all endofunctors on **Set** in fact preserve weak pullbacks:

Proposition 1.24. *All endofunctors inductively defined upon:*

- *The Identity Functor* $\mathcal{I}$
- *The Constant Functor* C
- *The Coproduct Functor* $\amalg$
- *The Product Functor* $\times$
- *The Exponent Functor* $(\cdot)^C$
- *The Power Set Functor* $\mathcal{P}$

preserve weak pullbacks.

See [**Ven05**] for more details.

Proposition 1.24 guarantee us that each endofunctor on **Set** used in this work preserves weak pullbacks.

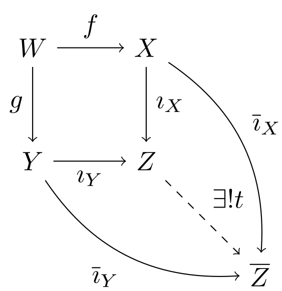
6. Amalgamated sum

Once we have obtained that the coproduct of two arbitrary coalgebras is again a coalgebra and also that the coequalizer of two coalgebra homomorphisms is again a coalgebra, we can derive a new kind of operation among coalgebras: the amalgamated sum.

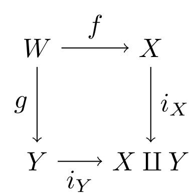
Proposition 1.25. *Let F be any endofunctor over* **Set**. *Assume (X, α) and (Y, β) and (W, λ) are three F-coalgebras, for which there are also two F-coalgebra homomorphisms $f : W \to X$ and $g : W \to Y$. Then there is an F-coalgebra (Z, η) together with two F-coalgebra homomorphisms $\imath_X : X \to Z$ and $\imath_Y : Y \to Z$ with the following properties:*

- S1. $\imath_X f = \imath_Y g$.
- S2. *For any other F-coalgebra $(\overline{Z}, \overline{\eta})$ and any other pair of F-coalgebra homomorpshims $\bar{\imath}_X : X \to \overline{Z}$ and $\bar{\imath}_Y : Y \to \overline{Z}$ accomplishing the equation $\bar{\imath}_X f = \bar{\imath}_Y g$, then there is a unique F-coalgebra homomorphism $t : Z \to \overline{Z}$ such that $t\imath_X = \bar{\imath}_X$ and $t\imath_Y = \bar{\imath}_Y$.*

This can be depicted in the following commutative diagram:



Proof. Let us prove this Proposition step by step. Consider the coproduct of the F-coalgebras (X, α) and (Y, β): $(X \amalg Y, \gamma)$. Thus, we obtain the following diagram:



Notice that the mappings $\mathbf{f} = i_X f$ and $\mathbf{g} = i_Y g$ are F-coalgebra homomorphisms between the coalgebras W and $X \amalg Y$. Let (Z, h) be defined precisely as the coequalizer of this two F-coalgebra homomorphism as seen on Proposition 1.18. Thus, one can find some structure map η for which (Z, η) is a coalgebra and the function h is an F-coalgebra homomorphism for which $h\mathbf{f} = h\mathbf{g}$. It can be depicted as follows:

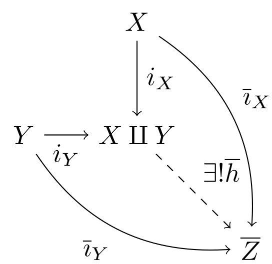
$$W \underset{\mathbf{g}}{\overset{\mathbf{f}}{\rightrightarrows}} X \amalg Y \xrightarrow{\ h\ } Z$$

We, therefore, define $\imath_X = hi_X$ and $\imath_Y = hi_Y$. Let us check that the properties S1. and S2. hold:

S1. Notice the following identities:

$$\imath_X f = (hi_X)f = h(i_X f) = h\mathbf{f} = h\mathbf{g} = h(i_Y g) = (hi_Y)g = \imath_Y g$$

S2. Assume that there is an F-coalgebra $(\overline{Z}, \overline{\eta})$ and some F-coalgebra homomorphisms $\bar{\imath}_X$ and $\bar{\imath}_Y$ with the desired properties. Since $\bar{\imath}_X : X \to \overline{Z}$ and $\bar{\imath}_Y : Y \to \overline{Z}$ are F-coalgebra homomorphisms, using the Universal Property of the Coproduct of F-coalgebras, 1.14, there exists a unique F-coalgebra homomorphism $\overline{h} : X \amalg Y \to \overline{Z}$ that makes the following diagram commute:



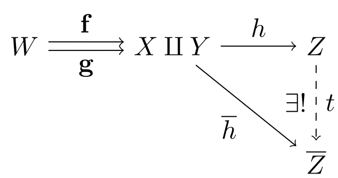
Notice that we get the identities:

$$\bar{\imath}_X = \overline{h} i_X, \qquad \bar{\imath}_Y = \overline{h} i_Y.$$

If we develop the composition $\bar{\imath}_X f$ we obtain that

$$\bar{\imath}_X f = (\overline{h} i_X) f = \overline{h}(i_X f) = \overline{h}\mathbf{f}$$

Analogously, we obtain that $\bar{\imath}_Y g = \overline{h}\mathbf{g}$. By assumption those two compositions are equal, which implies that $\overline{h}\mathbf{f} = \overline{h}\mathbf{g}$. We are now allowed to use the Universal Property of the Coequalizer; There exists a unique F-coalgebra homomorphism $t : Z \to \overline{Z}$ that makes the following diagram commute:



Notice that we get the equation $th = \overline{h}$. Composing both sides of the last equation by i_X we get that

$$(th) i_X = t(h i_X) = t\imath_X = \bar{\imath}_X = \overline{h} i_X$$

therefore $t\imath_X = \bar{\imath}_X$, analogously we can derive the equation $t\imath_Y = \bar{\imath}_Y$.

□

Definition 1.26. **Amalgamated Sum**

Let F be any endofunctor over **Set**. Assume (X, α) and (Y, β) and (W, λ) are three F-coalgebras, for which there are also two F-coalgebra homomorphisms $f : W \to X$ and $g : W \to Y$. The resulting F-coalgebra arising from Proposition 1.25 will be called the *amalgamated sum of X and Y under W relative to f and g* and it will be denoted by $X \amalg_W Y$.

CHAPTER 2

Coalgebraic View of Logical Structures

Our aim in this chapter is to present some logical structures as coalgebras for an specific endofunctor.

In each section we present an endofunctor F and we characterise its F-coalgebra homomorphisms. After this work, one can easily consider a natural structure map for those logical structures and conclude that some well-known mappings or notions are easily described in a coalgebraic sense.

1. $\mathcal{P}$-coalgebras

In this section, we will develop the first important exemple of endofunctor over **Set**, *the power set functor* $\mathcal{P}$:

$$\begin{array}{rccc} \mathcal{P}: & \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ & X & \longmapsto & \mathcal{P}(X) \end{array}$$

For each $f : X \to Y$ in **Set**, $\mathcal{P}f : \mathcal{P}(X) \to \mathcal{P}(Y)$ is a function that map each subset W of X to the subset of the images of the elements of W.

$$\mathcal{P}f(W) = f[W] = \{f(x) : \ x \in W\}$$

As in the example seen before, given two $\mathcal{P}$-coalgebras (X, α), (Y, β), it will important to describe what properties must a mapping $f : X \to Y$ have in order to be a $\mathcal{P}$-coalgebra homomorphism. The next Proposition establishes which are those properties, but first of all we need to fix notation.

Remark 2.1. Given a $\mathcal{P}$-coalgebra (X, α), we can write it as (X, R) with $R \subset X \times X$ and $x_1 R x_2 \Leftrightarrow x_2 \in \alpha(x_1)$. Notice that R can also play the role of α by setting $Rx_1 = \{x_2 \in X : \ x_1 R x_2\} = \alpha(x_1)$.

Proposition 2.2. *Let (X, R) and (X', R') be two $\mathcal{P}$-coalgebras. A function $f : X \to X'$ is a $\mathcal{P}$-coalgebra homomorphism if and only if:*

P1) $x_1 R x_2 \implies f(x_1) R' f(x_2)$
P2) $f(x_1) R' y \implies \exists x_2 \in X \ (x_1 R x_2 \textit{ and } f(x_2) = y)$

Proof. For a $f : X \to X'$ being $\mathcal{P}$-coalgebra homomorphism we need that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{R} & \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}f} \\ X' & \xrightarrow[R']{} & \mathcal{P}X' \end{array}$$

It means that for each $x_1 \in X$, $\mathcal{P}f(Rx_1) = R'f(x_1)$, which lead us to the following equality between sets:

$$A := \{f(x_2) : \ x_1 R x_2\} = \{y : \ f(x_1) R' y\} =: B$$

Hence, any $f : X \to X'$ is a $\mathcal{P}$-coalgebra homomorphism if and only if $A = B$. Notice that $A \subseteq B$ corresponds to property *P1)* and $A \supseteq B$ corresponds to property *P2)*. □

2. Graphs as $\mathcal{P}$-coalgebras

Once we have presented P-coalgebras and characterised the P-coalgebra homomorphisms, let us see how can we use those notions to present graphs. First of all, let us remember some previous definitions.

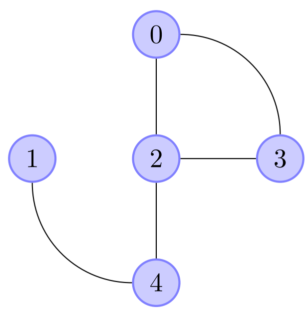
Definition 2.3. **Graph**

A *graph* is an ordered pair $G = (V, E)$ comprising a set V of *vertices* together with a set $E \subset [V]^2$ containing two element subsets of V called *edges*.

Example 2.4. Any graph can be represented by a figure with vertices and edges according to the considered sets, for example:

$$V = 5$$

$$E = \{\{0, 2\}, \{0, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$$



Definition 2.5. **Blockmodels**

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. $f : G \to G'$ is a *complete graph homomorphism* if and only if $f : V \to V'$ is a surjective function such that $\forall x_1, x_2 \in V$ and $\forall y_1, y_2 \in V'$ holds:

BM1) $\{x_1, x_2\} \in E \implies \{f(x_1), f(x_2)\} \in E'$

BM2) $\{y_1, y_2\} \in E' \implies \exists x_1, x_2 \in V$ such that $\{x_1, x_2\} \in E$ and $f(x_1) = y_1$, $f(x_2) = y_2$

G' is called a *Blockmodel* of G by [**WR83**].

Remark 2.6. **Viewing Graphs as $\mathcal{P}$-coalgebras**

Given any graph $G = (V, E)$, we could see G as a $\mathcal{P}$-coalgebra by taking $G = (V, \xi)$, where ξ is defined according to E as follows:

$$\begin{array}{rccl} \xi : & V & \longrightarrow & \mathcal{P}V \\ & x_1 & \longmapsto & \{x_2 : \ \{x_1, x_2\} \in E\} \end{array}$$

Notice that ξ is the *adjacency mapping* for the graph.

Doing this way we can identify some notions presented before.

Proposition 2.7. *Given $G = (V, E)$, $G = (V', E')$ two graphs, and $f : V \to V'$ any mapping, then it holds that f is a complete graph homomorphism if and only if f is a surjective $\mathcal{P}$-coalgebra homomorphism.*

3. Posets as $\mathcal{P}$-coalgebras

We use the notions seen before to present a coalgebraic approach to posets. First of all, we must recall some definitions concerning poset theory.

Definition 2.8. **Poset**
A *partially ordered set* (or *poset*) is a pair $\mathbb{P} = (P, \leq)$ where P is a set and $\leq$ is an order over P.
To each $p \in P$ we can associate two subsets of P:

$$\begin{array}{rl} \text{Its } \textit{downset} & \downarrow p = \{q \in P : \ q \leq p\} \\ \text{Its } \textit{upset} & \uparrow p = \{q \in P : \ p \leq q\} \end{array}$$

Definition 2.9. **Order morphisms**
Given two posets $\mathbb{P}_1 = (P_1, \leq_1)$, $\mathbb{P}_2 = (P_2, \leq_2)$ and $f : P_1 \to P_2$ a map.
We say that f is:

$$\begin{array}{rcl} \textit{Monotone} & \Leftrightarrow & \forall p, q \in P_1 \ (p \leq_1 q \ \Rightarrow \ f(p) \leq_2 f(q)) \\ \textit{Order-reflecting} & \Leftrightarrow & \forall p, q \in P_1 \ (f(p) \leq_2 f(q) \ \Rightarrow \ p \leq_1 q) \\ \textit{Order embedding} & \Leftrightarrow & f \text{ is monotone and order-reflecting.} \\ \textit{Order isomorphism} & \Leftrightarrow & f \text{ is a surjective order embedding.} \end{array}$$

Remark 2.10. **Viewing Posets as $\mathcal{P}$-coalgebras**
Given any poset $\mathbb{P} = (P, \leq)$ we can associate with it, in a very natural way, a $\mathcal{P}$-coalgebra by taking the structure map η that maps each $p \in P$ to its downset (or to its upset).

$$\begin{array}{rccc} \eta : & P & \longrightarrow & \mathcal{P}(P) \\ & p & \longmapsto & \downarrow p \end{array}$$

Remark 2.11. Using Remark 2.1 we can associate to η a relation R. This relation states that for each $p, q \in P$ hold:

$$pRq \ \Leftrightarrow \ q \in \eta(p) \ \Leftrightarrow \ q \in \downarrow p \ \Leftrightarrow \ q \leq p$$

Proposition 2.12. *Let $\mathbb{P}_1 = (P_1, \leq_1)$, $\mathbb{P}_2 = (P_2, \leq_2)$ be two posets with associated $\mathcal{P}$-coalgebras (P_1, η) and (P_2, η) respectively. Let $f : P_1 \to P_2$ be any mapping between the base sets. It holds that f is a $\mathcal{P}$-coalgebra homomorphism if and only if for each $p, q \in P_1$ and each $r \in P_2$*

P1) $p \leq_1 q \implies f(p) \leq_2 f(q)$
P2) $r \leq_2 f(p) \implies \exists q \in P_1 \ (q \leq_1 p$ *and* $f(q) = r)$

Proof. Using Proposition 2.2 and remark 2.11. □

Corollary 2.13. *Let $\mathbb{P}_1 = (P_1, \leq_1)$, $\mathbb{P}_2 = (P_2, \leq_2)$ be two posets with associated $\mathcal{P}$-coalgebras (P_1, η) and (P_2, η) respectively. Let $f : P_1 \to P_2$ be any mapping between the base sets. It holds that f is a $\mathcal{P}$-coalgebra homomorphism if and only if f commutes with downsets, i.e.:*

$$\forall p \in P_1 \quad f[\downarrow p] = \downarrow f(p)$$

Theorem 2.14. *Let $\mathbb{P}_1 = (P_1, \leq_1)$, $\mathbb{P}_2 = (P_2, \leq_2)$ be two posets with associated $\mathcal{P}$-coalgebras (P_1, η) and (P_2, η) respectively. Then the following statements are equivalent:*

a) *f is a $\mathcal{P}$-coalgebra isomorphism.*
b) *f is an order isomorphism.*

PROOF. $\gg$ By assumption f is a bijection. By Prop. 2.12 P1) f is monotone. In order to see that f is order-reflecting, consider any $p, q \in P_1$ such that $f(p) \leq_2 f(q)$, again by Prop. 2.12 P2) there exists some $r \in P_1$ with $r \leq_1 q$ and $f(r) = f(p)$. Since f is injective follows that $r = p$ and hence, $p \leq_1 q$. Finally, f is an order isomorphism.

$\ll$ By assumption f is a monotone bijection. Consider any $p \in P_1$ and $r \in P_2$ such that $r \leq_2 f(p)$, since f is surjective, there exists some $q \in P_1$ such that $f(q) = r$ and using that f is order reflecting, we conclude that $q \leq_1 p$. Therefore applying Prop. 2.12, we conclude that f is a $\mathcal{P}$-coalgebra isomorphism.

□

4. $\mathcal{P}(A \times -) \amalg B$-coalgebras

We want now to introduce more important examples. In this section we will talk about $\mathcal{P}(A \times -) \amalg B$-coalgebras. Let us present this kind of coalgebras and let us characterise its morphisms.

Consider $A, B \in$**Set**. Let us define the endofunctor $F = \mathcal{P}(A \times -) \amalg B$ as:

$$\begin{array}{rccc} \mathcal{P}(A \times -) \amalg B: & \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ & X & \longmapsto & \mathcal{P}(A \times X) \amalg B \end{array}$$

For each $f : X \to Y$ in **Set**, $\mathcal{P}(id_A \times f) \amalg id_B : \mathcal{P}(A \times X) \amalg B \to \mathcal{P}(A \times Y) \amalg B$ is a function that acts on each element of $\mathcal{P}(A \times X) \amalg B$ just by applying f on the elements of X and leaving the rest unchanged.

Now it will important to describe what properties must a function $f : X \to Y$ have in order to be a $\mathcal{P}(A \times -) \amalg B$-coalgebra homomorphism. The next Proposition establishes which are those properties, but first of all we need to fix notation.

REMARK 2.15. Given a $\mathcal{P}(A \times -) \amalg B$-coalgebra (X, α), recall that we can write it as $(X, (R_a)_{a \in A}, \alpha_B)$ with $R_a \subset X \times X$ for each $a \in A$ and $x_1 R_a x_2 \Leftrightarrow \langle a, x_2 \rangle \in \alpha(x_1)$. And $\alpha_B : X \to B$ is a function that maps each $x \in X$ to the element of B according to α. Notice that $\coprod_{a \in A} R_a$ can also play the role of α by setting

$$\coprod_{a \in A} R_a(x_1) = \coprod_{a \in A} (\{a\} \times \{x_2 \in X : \; x_1 R_a x_2\}) = \alpha(x_1)$$

This means that there is a bijection between $\mathcal{P}(A \times X)$ and $\coprod_{a \in A} (\{a\} \times \mathcal{P}(X))$.

PROPOSITION 2.16. *Let $(X, (R_a)_{a \in A}, \alpha_B)$ and $(X', (R'_a)_{a \in A}, \alpha'_B)$ be two $\mathcal{P}(A \times -) \amalg B$-coalgebras. A function $f : X \to X'$ is a $\mathcal{P}(A \times -) \amalg B$-coalgebra homomorphism if and only if:*

P1) $\alpha_B = \alpha'_B f$
P2) $x_1 R_a x_2 \implies f(x_1) R'_a f(x_2)$
P3) $f(x_1) R'_a y \implies \exists x_2 \in X \; (x_1 R_a x_2 \text{ and } f(x_2) = y)$

PROOF. For a $f : X \to X'$ being $\mathcal{P}(A \times -) \amalg B$-coalgebra homomorphism we need that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \mathcal{P}(A \times X) \amalg B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(id_A \times f) \amalg id_B} \\ X' & \xrightarrow[\alpha']{} & \mathcal{P}(A \times X') \amalg B \end{array}$$

That is equivalent to say that the following two diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\alpha_B} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_B} \\ X' & \xrightarrow[\alpha'_B]{} & B \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\alpha} & \mathcal{P}(A \times X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(id_A \times f)} \\ X' & \xrightarrow[\alpha']{} & \mathcal{P}(A \times X') \end{array}$$

The first diagram commutes if and only if *P1)*.
For the second one, recall that there exists a bijection between $\mathcal{P}(A \times X)$ and $\coprod_{a \in A}(\{a\} \times \mathcal{P}(X))$.
Therefore, the second diagram being commutative is equivalent to say that, for each $a \in A$, the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{R_a} & \{a\} \times \mathcal{P}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_{\{a\}} \times \mathcal{P}(f)} \\ X' & \xrightarrow[R'_a]{} & \{a\} \times \mathcal{P}(X') \end{array}$$

Proposition 2.2 says that all the preceding diagrams commute if and only if *P2)* and *P3)* □

COROLLARY 2.17. *For the case $A = 1$ and $B = \emptyset$, Proposition 2.16 reduces to Proposition 2.2.*

5. Multimodal Logic

We present in this section some previous definitions concerning Multimodal Logic, Semantics and Kripke Structures.

DEFINITION 2.18. **Multimodal Language**
Given a set of atomic propositions Prop and an arbitrary set A, the set of all *multimodal formulas* $\mathcal{ML}$ is defined inductively by:

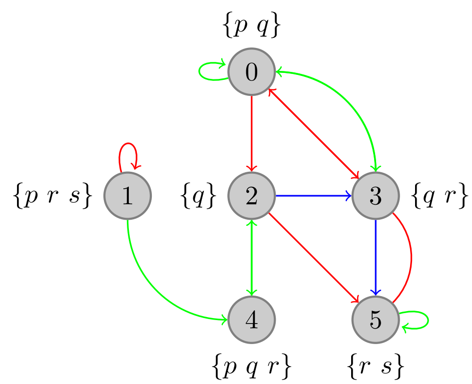
$$\begin{array}{rcl} p \in \mathsf{Prop} & \Rightarrow & p \in \mathcal{ML} \\ \bot \in \mathcal{ML} & & \\ \varphi, \psi \in \mathcal{ML} & \Rightarrow & \varphi \to \psi \in \mathcal{ML} \\ \varphi \in \mathcal{ML},\, a \in A & \Rightarrow & \Box_a \varphi \in \mathcal{ML} \end{array}$$

As usual, $\top$, $\neg$, $\wedge$, $\vee$, can be defined from $\bot$, $\to$. The modal operator $\Diamond_a$ for each $a \in A$ is defined as $\neg\Box_a\neg$.

DEFINITION 2.19. **Kripke Model**
A *Kripke Model* is a triple $(X, (R_a)_{a \in A}, V)$ consisting on a set X, a relation $R_a \subset X \times X$ for each $a \in A$ and a valuation $V : X \to \mathcal{P}(\mathsf{Prop})$.

Elements of X are called *states*. R_a is called the *accesibility relation* according to a. As usual we think of V as a mapping assigning to each possible state the set of atomics propositions holding in x.

Example 2.20. Any Kripke Model $(X, (R_a)_{a\in A}, V)$ can be seen as a Graph with Vertices the states of X and colored edges according to R_a. Next to each state x we write $V(x)$. As an example take:



For $A = 1$ we reduce that construction for the case of the usual modal logic.

We think of A as a set of agents and of $\Box_a\varphi$ as 'agent a knows φ'. Atomic propositions describe the facts agents can know. A Kripke Model $(X, (R_a)_{a\in A}, V)$ can be understood as follows: X is a set of possible worlds and V describes the facts holding in each world; xR_ay means that agent a considers y as an alternative world for x; $x \models \Box_a\varphi$ means that φ holds in all worlds which are considered as alternative worlds by agent a, i.e. a knows φ.

Example 2.21. **Hennessy-Milner Logic**

Consider a multimodal logic without atomic propositions and with modalities $\Box_a$, $a \in A$, where we think of A as a set of actions and $\Box_a\varphi$ as 'φ holds after a'. A Kripke Model is then a transition system $(X, (R_a)_{a\in A})$ (remember that there are no atomic propositions and hence no valuation).

Definition 2.22. **Semantics of Modal Logic**

Given a Kripke Model $(X, (R_a)_{a\in A}, V)$ and $x \in X$ we define:

$$\begin{array}{lcccl}
(X,(R_a)_{a\in A},V,x) & \models & p & \Leftrightarrow & p \in V(x) \\
(X,(R_a)_{a\in A},V,x) & \not\models & \bot & & \\
(X,(R_a)_{a\in A},V,x) & \models & \varphi \to \psi & \Leftrightarrow & \text{if } (X,(R_a)_{a\in A},V,x) \models \varphi \text{ then } (X,(R_a)_{a\in A},V,x) \models \psi \\
(X,(R_a)_{a\in A},V,x) & \models & \Box_a\varphi & \Leftrightarrow & \forall y \in X \text{ such that } xR_ay \text{ then } (X,(R_a)_{a\in A},V,y) \models \varphi
\end{array}$$

When the model $(X, (R_a)_{a\in A}, V)$ is clear from the context, we will write $x \models \varphi$ instead of $(X, (R_a)_{a\in A}, V, x) \models \varphi$. We say that φ *holds* in a model $(X, (R_a)_{a\in A}, V)$, written $(X, (R_a)_{a\in A}, V) \models \varphi$ if and only if $\forall x \in X$ $x \models \varphi$. Finally, φ is *valid*, written $\models \varphi$ if and only if φ holds in all models.

Definition 2.23. **Kripke Frames**

A *Kripke Frame* $(X, (R_a)_{a\in A})$ consists of a set X and a relation $R_a \subset X \times X$ for each $a \in A$. Kripke Models $(X, (R_a)_{a\in A}, V)$, for any $V : X \to \mathcal{P}(X)$, are said to be *based* on $(X, (R_a)_{a\in A})$ and $(X, (R_a)_{a\in A})$ is called the *frame* of the model.

A frame $(X,(R_a)_{a\in A})$ *satisfies* a formula $\varphi \in \mathcal{ML}$ if and only if all models based on the frame $(X,(R_a)_{a\in A})$ satisfy φ:

$$(X,(R_a)_{a\in A}) \models \varphi \;\Leftrightarrow\; \text{for all } V : X \to \mathcal{P}(\mathsf{Prop}) \text{ holds that } (X,(R_a)_{a\in A}, V) \models \varphi$$

6. Bisimulation

Once we have presented all the previous definitions and concepts, it is natural to ask what would be an appropiate notion of morphism for the Kripke structures.

Definition 2.24. **Bisimilar**

Given two Kripke Models $(X,(R_a)_{a\in A}, V)$, $(X',(R'_a)_{a\in A}, V')$ and $x \in X$ and $x' \in X'$. We say that x and x' are *bisimilar* if and only if:

$$\begin{array}{lll} & V(x) = V'(x') & \\ \forall a \in A \quad xR_a y & \Rightarrow & \exists y' \in X' \; (x'R_a y' \text{ and } y, y' \text{ are bisimilar}) \\ \forall a \in A \quad x'R'_a y' & \Rightarrow & \exists y \in X \; (xR_a y \text{ and } y, y' \text{ are bisimilar}) \end{array}$$

The set $B = \{\langle x, x'\rangle : \; x \text{ and } x' \text{ are bisimilar } \} \subset X \times X'$ is called a *bisimulation*. Bisimulations for frames can be obtained as a special case by ignoring the clause concerning the valuations V and V'.

Theorem 2.25. *Given two Kripke Models* $(X,(R_a)_{a\in A}, V)$, $(X',(R'_a)_{a\in A}, V')$ *and* $x \in X$ *and* $x' \in X'$.

$$x,\; x' \textit{ are bisimilar} \Rightarrow \textit{for all } \varphi \in \mathcal{ML}\; (x \models \varphi \Leftrightarrow x' \models \varphi)$$

Proof. Assume x and x' are bisimilar. We will do the proof by induction on the structures of formulas.

$$p \in \mathsf{Prop}) \quad x \models p \quad \Leftrightarrow \quad p \in V(x)$$

$$\begin{array}{lll} \text{By bisimilarity, } V(x) = V'(x') & & \\ & \Leftrightarrow & p \in V'(x') \\ & \Leftrightarrow & x' \models p \end{array}$$

$$\begin{array}{llll} \bot) & x \not\models \bot \quad \text{and} \quad x' \not\models \bot & & \text{therefore} \\ & x \not\models \bot \quad \Leftrightarrow \quad x' \not\models \bot & & \end{array}$$

Assume the statement holds for $\varphi,\; \psi \in \mathcal{MP}$ **(IH)**.

$$\begin{array}{llll} \rightarrow) & x \models \varphi \rightarrow \psi & \Leftrightarrow & \text{if } x \models \varphi \text{ then } x \models \psi \\ & & \Leftrightarrow_{\mathbf{IH}} & \text{if } x' \models \varphi \text{ then } x' \models \psi \\ & & \Leftrightarrow & x' \models \varphi \rightarrow \psi \end{array}$$

$$\Box_a) \quad x \models \Box_a\varphi \quad \Rightarrow \quad \forall y \in X \text{ such that } xR_ay \text{ then } y \models \varphi$$

Consider any $y' \in X'$ such that $x'R_ay'$
By bisimilarity, $\exists y \in X$ (xR_ay and y, y' are bisimilar)
Hence $y \models \varphi$ and by **(IH)** $y' \models \varphi$

$$\begin{array}{ll} \Rightarrow & \forall y' \in X' \text{ such that } x'R_ay' \text{ then } y' \models \varphi \\ \Rightarrow & x' \models \Box_a\varphi \end{array}$$

The other implication is analogous:

$$x' \models \Box_a\varphi \quad \Rightarrow \quad x \models \Box_a\varphi$$

Last case hold for each $a \in A$.

Thus, we conclude that for each $\varphi \in \mathcal{ML}$:

$$x \models \varphi \;\Leftrightarrow\; x' \models \varphi$$

□

Definition 2.26. **Kripke Structure Morphisms**
Given two Kripke Models $(X, (R_a)_{a\in A}, V)$, $(X', (R'_a)_{a\in A}, V')$, a *Kripke model morphism* $f : (X, (R_a)_{a\in A}, V) \rightarrow (X', (R')_{a\in A}, V')$ is a function $f : X \rightarrow X'$ such that its graph $\{(x, f(x)) :\ x \in X\}$ is a bisimulation.
We use the same definition for Kripke Frames.

7. Kripke Structures as $\mathcal{P}(A\times -)\amalg B$-coalgebras

Remark 2.27. **Viewing Kripke Structures as $\mathcal{P}(A\times -)\amalg B$-coalgebras**
Once we have presented $\mathcal{P}(A\times -)\amalg B$-coalgebras and characterised the $\mathcal{P}(A\times -)\amalg B$-coalgebra homomorphisms, let us see how can we use those notions to present Kripke Structures.
Given any Kripke Model $(X, (R_a)_{a\in A}, V)$ we associate them a $\mathcal{P}(A\times -)\amalg B$-coalgebra structure by setting $B = \mathcal{P}(\mathbf{Prop})$ and taking the structure map μ:

$$\begin{array}{rccl} \mu : & X & \longrightarrow & \mathcal{P}(A\times X)\amalg \mathcal{P}(\mathbf{Prop}) \\ & x & \longmapsto & \{\langle a, y\rangle :\ xR_ay\}\amalg V(x) \end{array}$$

And analogously, to each Kripke Frame $(X, (R_a)_{a\in A})$ we can associate in a very natural way a $\mathcal{P}(A\times -)\amalg B$-coalgebra structure by setting $B = \emptyset$ and taking the structure map ρ:

$$\begin{array}{rccl} \rho : & X & \longrightarrow & \mathcal{P}(A\times X) \\ & x & \longmapsto & \{\langle a, y\rangle :\ xR_ay\} \end{array}$$

Remark 2.28. Using Remark 2.15 the relation R_a for each $a \in A$ stated for μ and ρ precisely correspond to the relations in the Kripke Model and in the Kripke Frame, respectively. Furthermore for the Kripke Model case also holds that

$$\mu_B = \mu_{\mathcal{P}(\mathbf{Prop})} = V$$

Theorem 2.29. *Given two Kripke Models* $(X, (R_a)_{a\in A}, V)$ *and* $(X', (R'_a)_{a\in A}, V')$ *together with the structure map* μ *and* $f : X \to X'$*. Then the following are equivalent:*

a) f *is a* $\mathcal{P}(A \times -) \amalg \mathcal{P}(\mathbf{Prop})$*-coalgebra morphism.*
b) f *is a Kripke Model morphism.*

Proof. The proof is done by noticing that the characterisation of $\mathcal{P}(A\times-)\amalg B$-coalgebra morphism done in Proposition 2.16 coincide with the definition of beign a bisimulation stated in 2.24.

□

Corollary 2.30. *Given two Kripke Frames* $(X, (R_a)_{a\in A})$ *and* $(X', (R'_a)_{a\in A})$ *together with the structure map* ρ *and* $f : X \to X'$*. Then the following statements are equivalent:*

a) f *is a* $\mathcal{P}(A \times -)$*-coalgebra morphism.*
b) f *is a Kripke Frame morphism.*

Proof. Any Kripke Frame can be seen as a Kripke Model together with the empty valuation. □

8. Nondeterministic Finite Automaton

Another interesting example of applications of the $\mathcal{P}(A \times -) \amalg B$-coalgebras is in the field of nondeterministic finite automatas.

Definition 2.31. **Nondeterministic Finite Automaton**

A *nondeterministic finite automaton* or *NFA* is a quintuple $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$, where:

- Q is a finite set of *states*.
- Σ is a finite set of symbols, known as *alphabet*. The elements of Σ are called letters.
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is a partial function named *transition function*.
- $q_0 \in Q$ is the *initial* state.
- $F \subseteq Q$ is the set of *final* states.

When the transition function is total we will say that the automaton is completely specified.

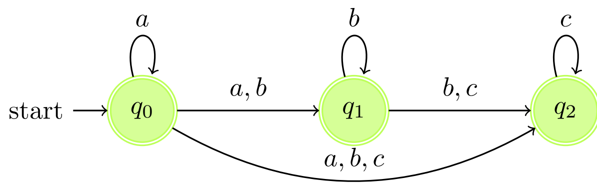
Remark 2.32. One can represent the transition function of a given NFA as a *transition diagram*, i.e., a directed graph in which:

1. The number of nodes is $|Q|$. Each node correspond to one state of Q.
2. For each $q_i, q_j \in Q$ and for each $a_k \in \Sigma$. If $q_j \in \delta(q_i, a_k)$, then the graph has one edge labelled with a_k from the node q_i to the node q_j.
3. The initial state is depicted together with a short entering arrow on it.
4. The final nodes are depicted as two concentric circles.

Example 2.33. Let $\mathbb{M} = (\{q_0, q_1, q_2\}, \{a, b, c\}, \delta, q_0, \{q_0, q_1, q_2\})$ be a NFA with the following definition of δ:

$\delta(q_0, a) = \{q_0, q_1, q_2\}$	$\delta(q_1, a) = \emptyset$	$\delta(q_2, a) = \emptyset$
$\delta(q_0, b) = \{q_1, q_2\}$	$\delta(q_1, b) = \{q_1, q_2\}$	$\delta(q_2, b) = \emptyset$
$\delta(q_0, c) = \{q_2\}$	$\delta(q_1, c) = \{q_2\}$	$\delta(q_2, c) = \{q_2\}$

The corresponding transition diagram is given by:



Definition 2.34. **Extended transition function**

In order to define the behaviour of a NFA on a string it is necessary to extend the transition function to a function acting on states and strings. Therefore, we define the *extended transition function* $\hat{\delta} : Q \times \Sigma^\star \to \mathcal{P}(Q)$ in the following way:

$\forall q \in Q,\ x \in \Sigma^\star,\ a \in \Sigma$:
1. $\hat{\delta}(q, \lambda) = \{q\}$
2. $\hat{\delta}(q, xa) = \bigcup_{p \in \hat{\delta}(q,x)} \delta(p, a)$

Item 1. means that a NFA can not change its state until it gets a symbol; Item 2. states the recursive definition of $\hat{\delta}$ on non-empty strings.

Remark 2.35. Notice that the behaviour of the extended transition function behaves as the initial one when we restrict it to the domain $Q \times \Sigma$. Therefore we will write δ instead of $\hat{\delta}$. Moreover we introduce some useful conventions for the study of NFA:

$p, q, \cdots$ are reserved for states. The initial state will be q_0.
$a, b, \cdots$ are reserved for letters of the alphabet.
$w, x, y, z, \cdots$ are reserved for strings.

We will write $p \xrightarrow{a} q$ for $q \in \delta(p, a)$. This notation can also be generalized for strings, $p \xrightarrow{x} q$.

Definition 2.36. **Accepted Language of a NFA**

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ be a NFA, and let $x \in \Sigma^\star$ be a string. We say that x is *accepted* by $\mathbb{M}$ whenever $\delta(q_0, x) \cap F \neq \emptyset$ holds. We define the *accepted language* of the NFA $\mathbb{M}$ as:

$$L(\mathbb{M}) = \{x \in \Sigma^\star :\ \delta(q_0, x) \cap F \neq \emptyset\}$$

9. Bisimilarity on NFA

Definition 2.37. **Nature of a state**

Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ be a NFA. The mapping $N : Q \to \mathcal{P}(2)$ is defined for each $p \in Q$ as:

$0 \in N(p)$ if and only if $p = q_0$.
$1 \in N(p)$ if and only if $p \in F$.
$\emptyset$ otherwise.

We say that $N(p)$ is the *nature of the state p*.

Definition 2.38. **Bisimilar states**

Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two NFA over the same alphabet Σ. We say that the states $p_1 \in Q_1$ and $p_2 \in Q_2$ are *bisimilar* if and only if:

$$N(p_1) = N(p_2)$$

$$\forall a \in \Sigma \qquad p_1 \xrightarrow{a} q_1 \ \Rightarrow\ \exists q_2 \in Q_2\ (p_2 \xrightarrow{a} q_2 \text{ and } q_1, q_2 \text{ are bisimilar})$$

$$\forall a \in \Sigma \qquad p_2 \xrightarrow{a} q_2 \;\Rightarrow\; \exists q_1 \in Q_1 \ (p_1 \xrightarrow{a} q_1 \text{ and } q_1, q_2 \text{ are bisimilar})$$

The set $B = \{\langle p_1, p_2\rangle :\ p_1, p_2 \ \textit{are bisimilars}\}$ is called a *bisimulation* between $\mathbb{M}_1$ and $\mathbb{M}_2$.

It is straightforward to see that the definition can be rewritten changing the letter $a \in \Sigma$ for any string $x \in \Sigma^\star$. This is due to the recursive definition of δ on strings.

THEOREM 2.39. *Let $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ and $\mathbb{M}' = (Q', \Sigma, \delta', q_0', F')$ be two NFA. If q_0 and q_0' are bisimilar, then $L(\mathbb{M}) = L(\mathbb{M}')$.*

PROOF. Let us check that $L(\mathbb{M}) \subseteq L(\mathbb{M}')$. Consider any $x \in L(\mathbb{M})$. We distinguish two cases:

$|x| = 0$ Therefore $x = \lambda$. Hence $\delta(q_0, \lambda) = \{q_0\} \subseteq F$ which means that q_0 is also a final state. Therefore $1 \in N(q_0)$. Since q_0 and q_0' are bisimilars, we get that $1 \in N(q_0')$, i.e., q_0' is also a final state and also λ is accepted by $\mathbb{M}'$.

$|x| \geq 0$ Therefore it holds that $\delta(q_0, x) \cap F \neq \emptyset$. That is to say that there exists some $p \in Q$ such that $q_0 \xrightarrow{x} p$ and $p \in F$. Since p is final it holds that $1 \in N(p)$. Since q_0 and q_0' are bisimilar, applying Definition 2.38 one can find some $p' \in Q'$ such that $q_0' \xrightarrow{x} p'$ with p and p' bisimilar, i.e., $1 \in N(p) = N(p')$, thus $p' \in F'$. Finally, $p' \in \delta'(q_0', x) \cap F'$ which means that $x \in L(\mathbb{M}')$.

The inclusion $L(\mathbb{M}') \subseteq L(\mathbb{M})$ has a similar proof. □

10. NFA as $\mathcal{P}(A \times -) \amalg B$-coalgebras

REMARK 2.40. **Viewing NFA as $\mathcal{P}(A \times -) \amalg B$-coalgebras**

At this point, one can easily picture which will be the associated coalgebra structure associated to a given NFA, $\mathbb{M}$. Given any NFA $\mathbb{M} = (Q, \Sigma, \delta, q_0, F)$ we associate to it a $\mathcal{P}(A \times -) \amalg B$-coalgebra structure by setting $A = \Sigma$ and $B = \mathcal{P}(2)$ and taking the structure map τ:

$$\begin{array}{rccl} \tau: & Q & \longrightarrow & \mathcal{P}(\Sigma \times Q) \amalg \mathcal{P}(2) \\ & p & \longmapsto & \{\langle a, q\rangle :\ p \xrightarrow{a} q\} \amalg N(p) \end{array}$$

11. $(- \times A)^B$-coalgebras

We introduce in this section the last example of coalgebras: the $(- \times A)^B$-coalgebras. Let us present this kind of coalgebras and let us characterise its morphisms. But first of all, we need to fix notation.

REMARK 2.41. Consider $X, Y \in \mathbf{Set}$. We denote by X^Y the set of functions from Y to X, i.e.:

$$X^Y = \{g :\ g : Y \to X\}$$

Thus, we are able to define a new kind of endofunctor. Consider $A, B \in \mathbf{Set}$. Let us define the endofunctor $F = (- \times A)^B$ as:

$$\begin{array}{rccc} (- \times A)^B: & \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ & X & \longmapsto & (X \times A)^B \end{array}$$

The endofunctor $(- \times A)^B$ maps each $f : X \to Y$ to $(f \times id_A)^B$, where the last mapping acts as:

$$\begin{array}{rccc} (f \times id_A)^B: & (A \times X)^B & \longrightarrow & (A \times Y)^B \\ & g & \longmapsto & (f \times id_A) \circ g \end{array}$$

As in every example shown before, it is useful to characterise how are the $(- \times A)^B$-coalgebra homomorphisms. But before we do that, let us fix again some notation.

REMARK 2.42. Let (X, α) be a $(- \times A)^B$-coalgebra. Since α maps the elements of X to mappings from B to $A \times X$, let us use, for each $x \in X$, α_x instead of $\alpha(x)$ in order to avoid extra brackets.

Notice that α_x is a mapping from a set to a product of sets.

Let us denote b α_x^i to the respective compositions with each projection, i.e.:

$$\alpha_x^i = \pi_i \ \alpha_x \qquad i \in \{1, 2\}$$

PROPOSITION 2.43. *Let (X, α), (Y, β) be two $(- \times A)^B$-coalgebras. Let $f : X \to Y$ be any mapping between them. It holds that f is a $(- \times A)^B$-coalgebra homomorphism if and only if for each $x \in X$ holds:*

P1) $f\alpha_x^1 = \beta_{f(x)}^1$

P2) $\alpha_x^2 = \beta_{f(x)}^2$

PROOF. Assume f is a $(- \times A)^B$-coalgebra homomorphism, then f makes the following diagram commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \alpha \big\downarrow & & \big\downarrow \beta \\ (X \times A)^B & \xrightarrow[(f \times ct_A)^B]{} & (Y \times A)^B \end{array}$$

Thus, for each $x \in X$ holds that $(f \times ct_A)^B \ \alpha_x = \beta_{f(x)}$. The equality also holds for its projections, therefore we conclude that P1) and P2) must hold. Conversely, if we assume that P1) and P2) hold we get again the equality written before and consequently the diagram commutes. □

12. Turing Machines as $(- \times A)^B$-coalgebras

We introduce here some previous definitions.

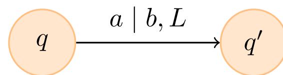
DEFINITION 2.44. **Turing Machine**
A *Turing Machine* is a 6-tuple $\mathbb{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_+)$ where:

- Q is a finite set of states.
- Σ is a finite alphabet, called the *Input Alphabet*.
- Γ is a finite alphabet that contains Σ and the blank symbol, $\square \in \Gamma$. Γ is called the *Tape Alphabet*.
- $q_0 \in Q$. The *Initial State*.
- $q_+ \in Q$. The *Accepting State*.
- $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, N, R\}$ is a partial function (possibly undefined for some elements) called the *Transition Function*.

One can imagine a Turing Machine as a device that can read an input. Then, it processes that input according to δ and when it reaches the accepting state, it writes a final result on an output tape.

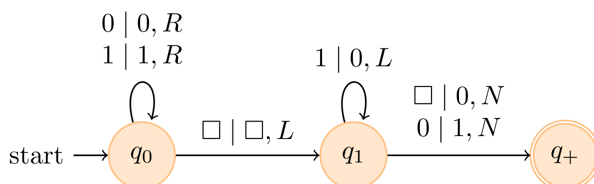
REMARK 2.45. Let $\mathbb{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_+)$ be any Turing Machine. In order to represent a Turing Machine, we will depict a set of vertices according to the states in Q and arrows between them according to δ

For example, if δ is such that $\delta(\langle q, a\rangle) = \langle q', b, L\rangle$, we will depict:



The special case of the states q_0 and q_+ will be respectively depict as:

EXAMPLE 2.46. We will depict the Turing Machine that adds one to any input given in binary. For this example: $Q = \{q_0, q_1, q_+\}$ $\Sigma = \{0, 1\}$ $\Gamma = \Sigma \cup \{\square\}$



REMARK 2.47. **Viewing Turing Machines as $(- \times A)^B$-coalgebras**

Let $\mathbb{M} = (Q, \Sigma, \Gamma, \delta, q_0, q_+)$ be a Turing Machine. We associate to $\mathbb{M}$ a $(-\times A)^B$-coalgebra structure by setting $A = \Gamma \times \{L, N, R\}$ and $B = \Gamma$ and taking the structure map:

$$\begin{array}{rccl} \delta: & Q & \longrightarrow & (Q \times \Gamma \times \{L, N, R\})^\Gamma \\ & q & \longmapsto & \delta_q \end{array}$$

Where δ_q is defined as usual:

$$\begin{array}{rccl} \delta_q: & \Gamma & \longrightarrow & (Q \times \Gamma \times \{L, N, R\}) \\ & a & \longmapsto & \delta(\langle q, a\rangle) \end{array}$$

CHAPTER 3

Bisimulations

1. Generalisation of Bisimulation

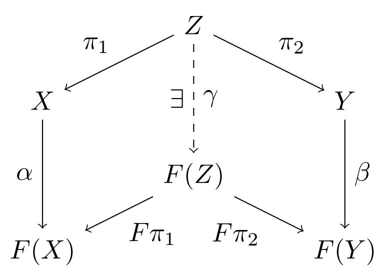
We introduce in this section a more general definition of bisimulation on the category $\mathcal{X} = \mathbf{Set}$[1]

DEFINITION 3.1. **Bisimulation**

Let F be any endofunctor over **Set**. Let (X, α), (Y, β) be F-coalgebras.

A subset $Z \subset X \times Y$ of the cartesian product of X and Y is called a F-*bisimulation* if there exists a structure map $\gamma : Z \to F(Z)$ such that the projections from Z to X and Y are F-coalgebra homomorphisms.

That means that (Z, γ) makes the following diagram commute:



We shall also say, making explicit reference to the structure map, that (Z, γ) is a bisimulation between (X, α) and (Y, β). We will denote by $B(X, Y)$ the set of all bisimulations between X and Y.

If $(X, \alpha) = (Y, \beta)$, then (Z, γ) is called a bisimulation on (X, α). We will write $B(X)$ instead of $B(X, X)$. A *bisimulation equivalence* is a bisimulation that is also an equivalence relation.

Two states $x \in X$, $y \in Y$ are called *bisimilar* if there exists a bisimulation Z with $\langle x, y \rangle \in Z$.

It will be useful to characterise the bisimulations between $\mathcal{P}$-coalgebras. Remember that given any $\mathcal{P}$-coalgebra, (X, α), using Remark 2.1 one can write (X, R_α) instead of (X, α)

PROPOSITION 3.2. *Let (X, R_α) and (Y, R_β) be two $\mathcal{P}$-coalgebras. Consider any $Z \subseteq X \times Y$. It holds that Z is a bisimulation between X and Y if and only if for each $\langle x_0, y_0 \rangle \in Z$ the following two properties hold:*

a) $\forall x \in X \ x_0 R_\alpha x \implies \exists y_x \in Y \ \ (\langle x, y_x \rangle \in Z$ *and* $y_0 R_\beta y_x)$

b) $\forall y \in Y \ y_0 R_\beta y \implies \exists x_y \in X \ \ (\langle x_y, y \rangle \in Z$ *and* $x_0 R_\alpha x_y)$

Notice that this Proposition requires the Axiom of Choice.

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[1] See [**Hug01**] for a definition on arbitrary categories.

PROOF. $\gg$ Assume towards a contradiction that (Z,γ) is a bisimulation but one of these properties does not hold. Without loss of generality, assume that property a) does not hold, that is there exists some $\langle x_0, y_0\rangle \in Z$ for which,

$$\exists x \in X \ x_0 R_\alpha x \text{ but } \forall y \in Y \quad (\langle x, y\rangle \notin Z \text{ or } \neg(y_0 R_\beta y))$$

Since $x_0 R_\alpha x$ holds, it follows that there must be at least some $y_x \in Y$ such that $\langle x, y_x\rangle \in \gamma(\langle x_0, y_0\rangle)$ (otherwise the left hand side of the diagram in Definition 3.1 does not commute). Notice that $y_x \in Y$, therefore applying the previous statement, it could happen that

- $\langle x, y_x\rangle \notin Z$; Which contradicts that $\langle x, y_x\rangle \in \gamma(\langle x, y_x\rangle) \subseteq Z$
- $\neg(y_0 R_\beta y_x)$); Which contradicts that (Z,γ) is a bisimulation since the right hand side of the diagram in Definition 3.1 does not commute.

$\ll$ By hypothesis for each $\langle x_0, y_0\rangle \in Z$, using the Axiom of Choice, for each $x \in X$ such that $x_0 R_\alpha x$ we can take some $y_x \in Y$ with $\langle x, y_x\rangle \in Z$ and $y_0 R_\beta y_x$. Using again the Axiom of Choice, for each $y \in Y$ such that $y_0 R_\beta y$ we can take some $x_y \in X$ with $\langle x_y, y\rangle \in Z$ and $x_0 R_\alpha x_y$.

Just take the structure map for Z:

$$\begin{array}{rccl} \gamma: & Z & \longrightarrow & \mathcal{P}(Z) \\ & \langle x_0, y_0\rangle & \longmapsto & \left(\bigcup_{x_0 R_\alpha x} \{\langle x, y_x\rangle\}\right) \cup \left(\bigcup_{y_0 R_\beta y} \{\langle x_y, y\rangle\}\right) \end{array}$$

In order to see that Z is a bisimulation between X and Y, we must see that the diagram of Definition 3.1 commutes. We will do the proof just for the left hand side of the diagram, the other is quite analogous. Take any $\langle x_0, y_0\rangle \in Z$. Notice that $\pi_1\alpha(\langle x_0, y_0\rangle) = \alpha(x_0) = R_\alpha(x_0)$. On the other side, $F\pi_1\gamma(\langle x_0, y_0\rangle) = (\bigcap_{x_0 R_\alpha x}\{x\}) \cap (\bigcap_{y_0 R_\beta y}\{x_y\}) = R_\alpha(x_0) \cap (\bigcap_{y_0 R_\beta y}\{x_y\})$. Notice that by the choice of each x_y, it holds that $x_0 R_\alpha x_y$, therefore the sets $F\pi_1\gamma(\langle x_0, y_0\rangle)$ and $\pi_1\alpha(\langle x_0, y_0\rangle)$ are equal.

□

EXAMPLE 3.3. Let $(X, (R_a)_{a\in A}, V)$ and $(X', (R'_a)_{a\in A}, V')$ be two Kripke Models considered as coalgebras with structure map μ, as seen on Remark 2.27. Let $f : X \to X'$ be a $\mathcal{P}(A \times -) \amalg \mathcal{P}(\mathbf{Prop})$-coalgebra morphism. Then there is a bisimulation for X and X'.

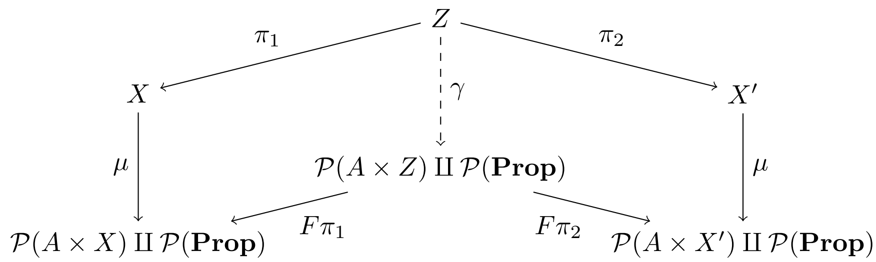
PROOF. Consider the set

$$Z = \{\langle x, f(x)\rangle : \ x \in X\} \subset X \times X'$$

together with the structure map:

$$\begin{array}{rccl} \gamma: & Z & \longrightarrow & \mathcal{P}(A \times Z) \amalg \mathcal{P}(\mathbf{Prop}) \\ & \langle x, f(x)\rangle & \longmapsto & \{\langle a, \langle y, f(y)\rangle\rangle : \ x R_a y\} \amalg V(x) \end{array}$$

Let us see that (Z,γ) makes the following diagram commute:



Let's see it works for the right side of the diagram. Conside any $\langle x, f(x)\rangle \in Z$, then holds:

$$\begin{array}{rcl} \mu\pi_2(\langle x, f(x)\rangle) & = & \{\langle a, y'\rangle : \ f(x)R'_a y'\} \amalg V'(f(x)) \\ F\pi_2\gamma(\langle x, f(x)\rangle) & = & \{\langle a, f(y)\rangle : \ xR_a y\} \amalg V(x). \end{array}$$

By Theorem 2.29 and Definition 2.24 we know that $V(x) = V'(f(x))$ and by Proposition 2.16 we obtain the different inclusions on the other sets. Therefore $\mu\pi_2 = F\pi_2\gamma$

The left side of the diagram is even easier. □

We finish this section by showing another important example.

Example 3.4. Let F be an endofunctor on **Set**. Let (X, α), (Y, β) be two F-coalgebras. It holds that the empty set, $\emptyset \subseteq X \times Y$, is a bisimulation between X and Y.

Proof. Just take γ as the empty mapping. The diagram of 3.1 trivially commutes. □

2. Basic Results

The Example 3.3 (and also the Theorem 2.29) could be seen as an immediate consequence of the following Theorem.

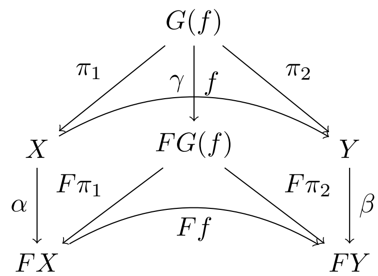
Theorem 3.5. *Let F be an arbitrary endofunctor over* **Set**. *Let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be an arbitrary map. It holds that f is a F-coalgebra homomorphism if and only if $G(f)$ is a bisimulation between (X, α) and (Y, β), where $G(f)$ represents the graph of the function f:*

$$G(f) = \{\langle x, f(x)\rangle : x \in X\} \subset X \times Y.$$

Proof. $\gg$ By assumption f is a F-coalgebra homomorphism therefore holds that $\beta f = F f \alpha$

Notice that π_1 is a bijection and so is $F\pi_1$, so we can take $(F\pi_1)^{-1}$. Notice that $(F\pi_1)^{-1} = F(\pi_1^{-1})$. Let γ be the structure map for $G(f)$ defined as $\gamma = (F\pi_1)^{-1}\alpha\pi_1$.

By composition holds that γ is a structure map. Let us check that γ makes the following diagram commute:



Obviously γ makes the left side of the diagram commutative.
For the right side, consider any $\langle x, f(x)\rangle \in G(f)$.

$$\begin{array}{rcl} \beta\pi_2(\langle x, f(x)\rangle) & = & \beta f(x) \\ \\ F\pi_2\gamma(\langle x, f(x)\rangle) & = & F\pi_2(F\pi_1^{-1})\alpha\pi_1(\langle x, f(x)\rangle) \\ & = & F\pi_2 F(\pi_1^{-1})\alpha(x) \\ & = & F(\pi_2\pi_1^{-1})\alpha(x) \\ & = & Ff\alpha(x) \\ & = & \beta f(x) \end{array}$$

Hence $(G(f), \gamma)$ is a bisimulation between (X, α) and (Y, β).

$\ll$ On the converse, assume $G(f)$ is a bisimulation between (X, α) and (Y, β). Therefore one can find an structure map $(G(f), \gamma)$ that makes the preceding diagram commute. Since π_1 is bijective it has an inverse π_1^{-1} which is a F-coalgebra homomorphism by Property 1.7. Since $f = \pi_2\pi_1^{-1}$, follows that f is a composition of F-coalgebra homomorphisms and hence so is f. □

The previous theorem would give us some basic results.

Corollary 3.6. *Let F be any endofunctor over* **Set**. *Let (X, α) be any F-coalgebra. Then the diagonal Δ_X is a bisimulation on (X, α).*

Proof. Notice that $id_X : X \to X$ is a F-coalgebra homomorphism. Therefore by previous Theorem (3.5) its Graph is a Bisimulation on (X, α). Notice that $G(id_X) = \Delta_X$. □

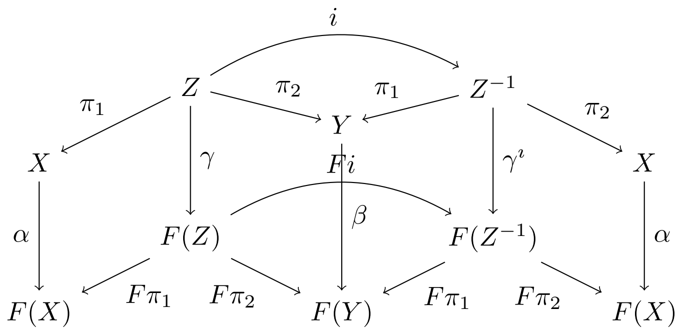
Theorem 3.7. *Let F be any endofunctor over* **Set**. *Let (X, α) and (Y, β) be any two F-coalgebras and let (Z, γ) be a bisimulation between (X, α) and (Y, β). Then the inverse of Z, Z^{-1}, is a bisimulation between (Y, β) and (X, α).*

Proof. Let i be the natural bijection between Z and Z^{-1}:

$$\begin{array}{rlcl} i: & Z & \longrightarrow & Z^{-1} \\ & \langle x, y\rangle & \longmapsto & \langle y, x\rangle \end{array}$$

Consider the structure map $\gamma^\imath = Fi\;\gamma\; i^{-1}$. $(Z^{-1}, \gamma^\imath)$ turns a F-coalgebra.

We must check that $(Z^{-1}, \gamma^\imath)$ is a bisimulation between (Y, β) and (X, α), therefore it must make the following diagram commute:



Consider any $\langle y, x\rangle \in Z^{-1}$:

Left hand side	Right hand side
$\beta\pi_1(\langle y, x\rangle) = \beta(y)$	$\alpha\pi_2(\langle y, x\rangle) = \alpha(x)$
$F\pi_1\gamma^\imath(\langle y, x\rangle) = F\pi_1 Fi\ \gamma\ i^{-1}(\langle y, x\rangle) = F(\pi_1\ i)\gamma(\langle x, y\rangle) = F\pi_2\gamma(\langle x, y\rangle) = \beta\pi_2(\langle x, y\rangle) = \beta(y)$	$F\pi_2\gamma^\imath(\langle y, x\rangle) = F\pi_2 Fi\ \gamma\ i^{-1}(\langle y, x\rangle) = F(\pi_2\ i)\gamma(\langle x, y\rangle) = F\pi_1\gamma(\langle x, y\rangle) = \alpha\pi_1(\langle x, y\rangle) = \alpha(x)$

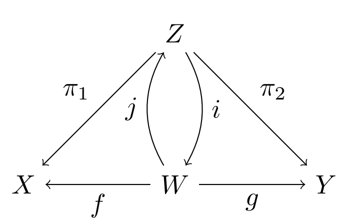
Therefore, Z^{-1} is a bisimulation between Y and X. □

THEOREM 3.8. *Let F be an endofunctor over* **Set**. *Let (X, α), (Y, β) and (W, δ) be three F-coalgebras. Consider $f : W \to X$ and $g : W \to Y$ be two F-coalgebra homomorphisms. Then the image $\langle f, g\rangle(W) = \{\langle f(w), g(w)\rangle : w \in W\}$ is a bisimulation between X and Y.*

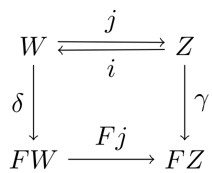
PROOF. Let us set $Z = \langle f, g\rangle(W)$.
Consider the mapping

$$j : \begin{array}{ccc} W & \longrightarrow & Z \\ w & \longmapsto & \langle f(w), g(w)\rangle \end{array}$$

Clearly j is a surjective mapping, so one can find using the Axiom of Choice any right inverse for j, namely i. That means that $j\ i = id_Z$. And thus, the following diagram commutes:



Define the structure map $\gamma = Fj\ \delta\ i$, then (Z, γ) turns a F-coalgebra.



Let us prove that (Z, γ) is a bisimulation for X and Y by checking that it makes commute the diagram from the definition 3.1 of bisimulation. Take any $w \in W$:

Left hand side

$$\begin{aligned} \alpha\pi_1(\langle f(w), g(w)\rangle) &= \alpha\ f(w) \\ &= Ff\ \delta(w) \end{aligned}$$

$$\begin{aligned} F\pi_1\gamma(\langle f(w), g(w)\rangle) &= F\pi_1\ Fj\ \delta\ i(\langle f(w), g(w)\rangle) \\ &= F(\pi_1\ j)\delta\ (i\ j)(w) \\ &= Ff\ \delta(w) \end{aligned}$$

The other side is analogous. □

Theorem 3.9. *Let F be an endofunctor over* **Set**.

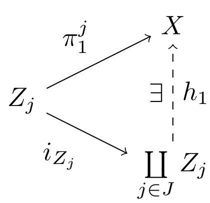
Let (X, α) and (Y, β) be two F-coalgebras and let $\{Z_j \subset X \times Y\}_{j\in J}$ be a family of bisimulations between X and Y. Then the union of the family is also a bisimulation between X and Y.

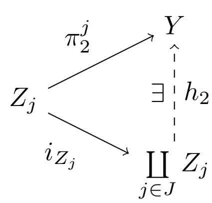
Proof. Consider the coproduct of the whole family $\coprod_{j\in J} Z_j$

It is a F-coalgebra by remark 1.15.

Notice that for each $j \in J$, Z_j is a bisimulation between X and Y, therefore each projection $\pi_1^j : Z_j \to X$ and $\pi_2^j : Z_j \to Y$ is a F-coalgebra homomorphism.

By the universal property of the coproduct 1.14, there exists h_1 and h_2 making the following diagrams commute:





Notice that h_1 and h_2 are precisely the componentwise projections.

Applying Theorem 3.8 holds that $\langle h_1, h_2\rangle(\coprod_{j\in J} Z_j)$ is a bisimulation between X and Y.

Let us prove that $\langle h_1, h_2\rangle(\coprod_{j\in J} Z_j) = \bigcup_{j\in J} Z_j$.

- $\subseteq$ Any element of $\langle h_1, h_2\rangle(\coprod_{j\in J} Z_j)$ has the form $\langle h_1(z), h_2(z)\rangle$ for some $z \in \coprod_{j\in J} Z_j$. Hence, there exists $j_0 \in J$ for which $z \in Z_{j_0} \subset X \times Y$ holds. Notice that $z = \langle z_1, z_2\rangle$ with $z_1 \in X$ and $z_2 \in Y$, therefore:

$$\langle h_1(z), h_2(z)\rangle = \langle h_1(\langle z_1, z_2\rangle), h_2(\langle z_1, z_2\rangle)\rangle = \langle z_1, z_2\rangle = z \in Z_{j_0} \subseteq \bigcup_{j\in J} Z_j$$

- $\supseteq$ Analogous.

□

Corollary 3.10. *Let F be an endofunctor over* **Set**. *Let (X, α) and (Y, β) be two F-coalgebras. Then it holds that $B(X, Y)$ is a complete lattice for the inclusion order, with least upper bound and greatest lower bound given by:*

$$\bigvee_{j \in J} Z_j = \bigcup_{j \in J} Z_j$$

$$\bigwedge_{j \in J} Z_j = \bigcup \{Z : \ Z \in B(X, Y) \text{ and } Z \subseteq \bigcap_{j \in J} Z_j\}$$

We will refer to $\mathbb{B}(X, Y)$ when we explicitly want to remark the lattice structure of $B(X, Y)$. This corollary also proves, as a particular case, the existence of the greatest bisimulation between X and Y, denoted by $X \between Y$. It is the union of all the bisimulations:

$$X \between Y = \bigcup B(X, Y)$$

We end this section by showing a useful result concerning quotients.

Proposition 3.11. *Let F be an endofunctor over* **Set**. *Let (X, α) be a F-coalgebra. Let Z be a bisimulation equivalence on X. Let $\pi_Z : X \to X/Z$ be the quotient mapping. Then there exists a unique map structure $\gamma_Z : X/Z \to F(X/Z)$ that turns π_Z into a F-coalgebra homomorphism.*

Proof. We will prove that $(X/Z, \pi_Z)$ is a coequalizer of $\pi_1 : Z \to X$ and $\pi_2 : Z \to X$, so we must show that the properties of Definition 1.16 hold

1. $\pi_Z \pi_1 = \pi_Z \pi_2$
   It is equivalent to say that for each $\langle x_1, x_2 \rangle \in Z$, $[x_1]_Z = [x_2]_Z$ which trivially holds in X/Z
2. Analogous to proof done in Proposition 1.17

Therefore, $(X/Z, \pi_Z)$ is the coequalizer of the projections mappings. Moreover, since Z is a bisimulation it holds that the projections are also F-coalgebra homomorphisms. Using Proposition 1.18 we conclude that there exists a unique map structure $\gamma_Z : X/Z \to F(X/Z)$ that turns π_Z into a F-coalgebra homomorphism. □

3. Pullbacks and Bisimulations

In the following section we will introduce some useful results concerning endofunctors that preserve weak pullbacks.

Theorem 3.12. *Let F :***Set***→***Set*** *be an endofunctor that preserves weak pullbacks. Let (X, α), (Y, β) and (Z, γ) be three F-coalgebras and let $f : X \to Z$, $g : Y \to Z$ be two F-coalgebra homomorphisms. Then the weak pullback (P, π_1, π_2) of (f, g) from Proposition 1.20 is a bisimulation between X and Y.*

Proof. Since F preserves weak pullbacks and (P, π_1, π_2) is a pullback for (f, g), then $(FP, F\pi_1, F\pi_2)$ is a pullback for (Ff, Fg).

Consider the triple $(P, \alpha\pi_1, \beta\pi_2)$. Let us see that $Ff\ \alpha\pi_1 = Fg\ \beta\pi_2$

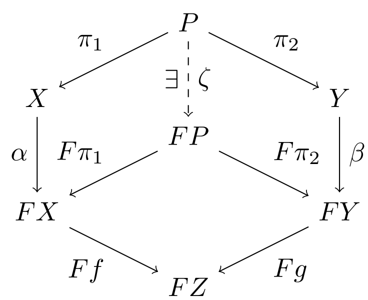
Consider any $\langle x, y \rangle \in P$

$$\begin{aligned} Ff\ \alpha\pi_1(\langle x, y\rangle) &= Ff\alpha(x) \\ &= \gamma f(x) \end{aligned}$$

$$\begin{aligned} Fg\ \beta\pi_2(\langle x, y\rangle) &= Fg\beta(y) \\ &= \gamma g(y) \end{aligned}$$

Last term comes from the fact that f and g are F-coalgebra homomorphims. Notice finally that by definition of P $f(x) = g(y)$, so we conclude that $Ff\ \alpha\pi_1 = Fg\ \beta\pi_2$.

Now by the 2nd property of $(FP, F\pi_1, F\pi_2)$ being a pullback for (Ff, Fg) one can find a mediating mapping $\zeta : P \to FP$ such that the following diagram commutes:

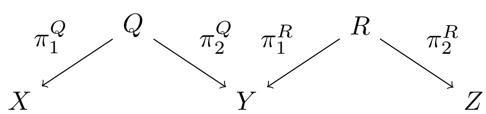


Therefore, P is a bisimulation between X and Y. □

Theorem 3.13. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks. Let (X, α), (Y, β), (Z, γ) be three F-coalgebras. Let Q be a bisimulation between X and Y and let R be a bisimulation between Y and Z.*

Then the composition $Q \circ R$ is a bisimulation between X and Z.

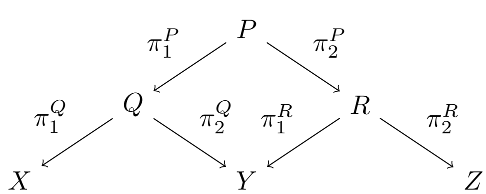
Proof. Since Q and R are bisimulations, the following diagram holds and all the mappings are F-coalgebra homomorphisms.



Let us focus on Q, R and the mappings π_2^Q and π_1^R.

By Theorem 3.12, the pullback (P, π_1^P, π_2^P) of (π_2^Q, π_1^R) is a bisimulation between Q and R.

Hence, the following diagram commutes and all the mappings are F-coalgebras



Notice that by construction

$$\begin{aligned} P &= \{\langle\langle x, y_1\rangle, \langle y_2, z\rangle\rangle \in Q \times R : \ \pi_2^Q(\langle x, y_1\rangle) = \pi_1^R(\langle y_2, z\rangle)\} \\ &= \{\langle\langle x, y_1\rangle, \langle y_2, z\rangle\rangle \in Q \times R : \ y_1 = y_2\} \end{aligned}$$

Now consider $f = \pi_1^Q \pi_1^P$ and $g = \pi_2^R \pi_2^P$.
They are F-coalgebra homomorphisms by composition.
By Theorem 3.8 $\langle f, g\rangle(P)$ is a bisimulation between X and Z.
Let us prove that $\langle f, g\rangle(P) = Q \circ R$

- $\subseteq$ Take any $\langle\langle x, y_1\rangle, \langle y_2, z\rangle\rangle \in P$. It holds that $f(\langle\langle x, y_1\rangle, \langle y_2, z\rangle\rangle) = x$ and $g(\langle\langle x, y_1\rangle, \langle y_2, z\rangle\rangle) = z$. The elements of $\langle f, g\rangle(P)$ are of the form $\langle x, z\rangle$ for which there is some $y \in Y$ ($y = y_1 = y_2$) such that $\langle x, y\rangle \in Q$ and $\langle y, z\rangle \in R$. Therefore $\langle x, z\rangle \in Q \circ R$
- $\supseteq$ Analogous

$\square$

Corollary 3.14. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra. Then $X \between X$ is a bisimulation equivalence.*

Proof. As seen on Corollary 3.10, $X \between X$ is a bisimulation. Also holds:

- $\Delta_X \subseteq X \between X$
  Since Δ_X is a bisimulation by Corollary 3.6 and using the fact that $X \between X$ is the greatest bisimulation on X.
- $(X \between X)^{-1} \subseteq X \between X$
  Since $(X \between X)^{-1}$ is a bisimulation by Theorem 3.7 and using the fact that $X \between X$ is the greatest bisimulation on X.
- $(X \between X) \circ (X \between X) \subseteq X \between X$
  Since $(X \between X) \circ (X \between X)$ is a bisimulation by Theorem 3.13 and using the fact that $X \between X$ is the greatest bisimulation on X.

$\square$

Remark 3.15. Let F be an endofunctor that preserves weak pullbacks. Let (X, α) be a F-coalgebra. It holds:

$$X \between X = \{\langle x_1, x_2\rangle \in X \times X : \ x_1 \text{ and } x_2 \text{ are bisimilar } \}$$

Corollary 3.16. *Let F be an endofunctor that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras. Let $f : X \to Y$ be any F-coalgebra homomorphism. It holds that the Kernel of f, denoted by $Kerf$, is a bisimulation equivalence. Where:*

$$Kerf = \{\langle x_1, x_2\rangle \in X \times X : \ f(x_1) = f(x_2)\}$$

Proof. Since f is a F-coalgebra homomorphism, we can use Theorem 3.5 to conclude that $G(f)$ is a bisimulation between X and Y. Moreover, using Theorem 3.7 we get that $G(f)^{-1}$ is a bisimulation between Y and X. Using now Theorem 3.13 we conclude that $G(f) \circ G(f)^{-1}$ is a bisimulation on X.

Notice that

$$\begin{array}{rcl} G(f) & = & \{\langle x, f(x)\rangle : \ x \in X\} \\ G(f) \circ G(f)^{-1} & = & \{\langle x_1, x_2\rangle : \ \exists y \in Y \ (\langle x_1, y\rangle \in G(f) \ \textit{and} \ \langle y, x_2\rangle \in G(f)^{-1})\} \end{array}$$

It holds that $(\langle x_1, y\rangle \in G(f)$ *and* $\langle y, x_2\rangle \in G(f)^{-1})$ if and only if $y = f(x_1) = f(x_2)$. Thus, $G(f) \circ G(f)^{-1} = Kerf$ and we conclude that $Kerf$ a bisimulation on X. By definition of $Kerf$ one can easily check that it is also an equivalence relation on X. $\square$

Proposition 3.17. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be a F-coalgebra homomorphism between them. It holds that:*

1. *If Z is a bisimulation on X, then $f(Z)$ is a bisimulation on Y. Where:*
$$f(Z) = \{\langle f(x_1), f(x_2)\rangle : \langle x_1, x_2\rangle \in Z\}$$
2. *If Z is a bisimulation on Y, then $f^{-1}(Z)$ is a bisimulation on X. Where:*
$$f^{-1}(Z) = \{\langle x_1, x_2\rangle : \langle f(x_1), f(x_2)\rangle \in Z\}$$

Proof. Proof for 1.

Since f is a F-coalgebra homomorphism, applying Theorem 3.5, $G(f)$ is a bisimulation between X and Y. Also by Theorem 3.7, $G(f)^{-1}$ is a bisimulation between Y and X. Using Theorem 3.13, we conclude that the composition $G(f)^{-1} \circ Z \circ G(f)$ is a bisimulation on Y.

Let us see, step by step, what is this composition:

$$
\begin{aligned}
G(f) &= \{\langle x, f(x)\rangle : x \in X\} \\
G(f)^{-1} &= \{\langle f(x), x\rangle : x \in X\} \\
Z \circ G(f) &= \{\langle t_1, t_2\rangle : \exists x \in X(\langle t_1, x\rangle \in Z \textit{ and } \langle x, t_2\rangle \in G(f))\} \\
&= \{\langle t_1, f(x_2)\rangle : \langle t_1, x_2\rangle \in Z\} \\
G(f)^{-1} \circ (Z \circ G(f)) &= \{\langle t_1, t_2\rangle : \exists x \in X(\langle t_1, x\rangle \in G(f)^{-1} \textit{ and } \langle x, t_2\rangle \in Z \circ G(f))\} \\
&= \{\langle f(x_1), f(x_2)\rangle : \langle x_1, x_2\rangle \in Z\} \\
&= f(Z)
\end{aligned}
$$

Thus, $f(Z)$ is a bisimulation on Y.

The proof for 2. is analogous, just notice that $f^{-1}(Z) = G(f) \circ Z \circ G(f)^{-1}$ □

Remark 3.18. We can prove a stronger version of the first point of the preceding Theorem by removing the assumption of F being an endofunctor that preserves weak pullbacks. This due to the fact that $f(Z) = \langle f\pi_1, f\pi_2\rangle(Z)$ and we can apply Theorem 3.8.

We will end this section by checking how the preceding construction interacts with some special kinds of bisimulations.

Proposition 3.19. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras. Let $f : X \to Y$ be a F-coalgebra homomorphism. It holds:*

1. $f(\Delta_X) = \Delta_{f(X)}$
2. *For each Z bisimulation on X holds that $f(Z^{-1}) = f(Z)^{-1}$*
3. *If f is a F-coalgebra embedding, for each Z_1, Z_2 bisimulations on X holds that $f(Z_1 \circ Z_2) = f(Z_1) \circ f(Z_2)$*

Proof. This proof reduces to show equality between sets. Let us write in each case what this sets consist of:

For 1.

$$
\begin{aligned}
f(\Delta_X) &= \{\langle f(x_1), f(x_2)\rangle : \langle x_1, x_2\rangle \in \Delta_X\} \\
&= \{\langle f(x), f(x)\rangle : x \in X\} = \Delta_{f(X)}
\end{aligned}
$$

For 2.

$$\begin{aligned} f(Z^{-1}) &= \{\langle f(x_1), f(x_2)\rangle : \langle x_1, x_2\rangle \in Z^{-1}\} \\ &= \{\langle f(x_2), f(x_1)\rangle : \langle x_1, x_2\rangle \in Z\} \quad = \quad f(Z)^{-1} \end{aligned}$$

For 3.

$$\begin{aligned} f(Z_1 \circ Z_2) &= \{\langle f(x_1), f(x_3)\rangle : \exists x_2 \in X(\langle x_1, x_2\rangle \in Z_1 \textit{ and } \langle x_2, x_3\rangle \in Z_2)\} \\ f(Z_1) \circ f(Z_2) &= \{\langle f(x_1), f(x_3)\rangle : \exists y \in Y(\langle f(x_1), y\rangle \in f(Z_1) \textit{ and } \langle y, f(x_3)\rangle \in f(Z_2))\} \end{aligned}$$

We want to see that

$$f(Z_1 \circ Z_2) = f(Z_1) \circ f(Z_2)$$

- $\subseteq$ This always hold. Just take $y = f(x_2)$
- $\supseteq$ In that inclusion we need f to be injective. Notice that $\langle f(x_1), y\rangle \in f(Z_1)$, so y must be of the form $f(x_2)$ for some $x_2 \in X$ such that $\langle x_1, x_2\rangle \in Z_1$. On the other side, $\langle y, f(x_3)\rangle \in f(Z_2)$, so y must be of the form $f(x_2')$ for some $x_2' \in X$ such that $\langle x_2', x_3\rangle \in Z_2$. Since f is injective we conclude that $x_2 = x_2'$, therefore, $\langle x_1, x_3\rangle \in Z_1 \circ Z_2$

□

CHAPTER 4

Associated Dioid

1. Previous Definitions on Semiring Theory

Definition 4.1. **Semigroup**
A *semigroup* is a 2-tuple, $\mathbb{M} = (M, +)$ where:

- M is a nonempty set.
- $+ : M \times M \to M$ is an *associative operation*.
  That is, for each $m_1, m_2, m_3 \in M$ it holds:

$$m_1 + (m_2 + m_3) = (m_1 + m_2) + m_3$$

An element $m \in M$ is said to be *idempotent* if $m + m = m$.

Definition 4.2. **Monoid**
A *monoid* is a 3-tuple, $\mathbb{M} = (M, +, 0)$ where:

- $(M, +)$ is a semigroup.
- $0 \in M$ is an *identity* element. That is, for each $m \in M$ it holds:

$$0 + m = m = m + 0$$

We say that a monoid is *commutative* if for each $m_1, m_2 \in M$ holds that:

$$m_1 + m_2 = m_2 + m_1$$

A monoid $\mathbb{M} = (M, +, 0)$ is *partially-ordered* iff there exists a partial order relation $\leq$ defined on M satisfying that for each $m_1, m_2, m_3 \in M$, it holds:

$$m_1 \leq m_2 \Rightarrow m1 + m_3 \leq m_2 + m_3$$

$$m_1 \leq m_2 \Rightarrow m_3 + m_1 \leq m_3 + m_2$$

Remark 4.3. Any idempotent commutative monoid $(M, +, 0)$ can be equipped with the *natural order relation*. For all $m_1, m_2 \in M$, we define:

$$m_1 \leq m_2 \Leftrightarrow m_1 + m_2 = m_2$$

Definition 4.4. **Hemiring**
An *hemiring* is a 4-tuple, $\mathbb{S} = (S, +, \cdot, 0,)$ where:

- $(S, +, 0)$ is a commutative monoid.
- $(S, \cdot)$ is a semigroup.
- distributes over $+$, i.e., $\forall s, t, u \in S$

$$s(t + u) = (st) + (su)$$

$$(t + u)s = (ts) + (us)$$

- For all $s \in S$, $s0 = 0 = 0s$.

Definition 4.5. **Semiring** A *semiring* is a 5-tuple, $\mathbb{S} = (S, +, \cdot, 0, 1)$ where:

- $(S, +, \cdot, 0)$ is a hemiring.

- $(S, \cdot, 1)$ is a monoid.

We say that $\mathbb{S}$ is a *commutative semiring* when $(S, \cdot, 1)$ is a commutative monoid. We say that $\mathbb{S}$ is an *idempotent semiring*, or *dioid*, if $+$ is idempotent. In that case, $\mathbb{S}$ can be equipped with the natural order relation. We say that it is *complete* if it is a complete lattice with the natural order relation.

We say that $\mathbb{S}$ is a *multiplicatively idempotent*, if $\cdot$ is idempotent.

EXAMPLE 4.6. **Boolean Matrices**
Consider $M_n(\mathbb{B}_1)$ the set of all square boolean matrices of size $n \in \mathbb{N}$. $M_n(\mathbb{B}_1)$ together with the usual sum and product of matrices forms a complete idempotent semiring.

PROPOSITION 4.7. *Let $\mathbb{S}$ be an idempotent semiring. The operations $+$ and are compatible with $\leq$ in the sense:*
For each $s, t, u, v \in S$,

$$s \leq t,\ u \leq v \ \Rightarrow\ s + u \leq t + v$$

$$s \leq t \ \Rightarrow\ su \leq tu,\ us \leq ut$$

PROOF. Assume that $s \leq t$ and $u \leq v$. It holds:

$$(s + u) + (t + v) = (s + t) + (u + v) = t + v$$

Last statement is equivalent to say that $s + u \leq t + v$.

For the multiplication,

$$(su) + (tu) = (s + t)u = tu$$

Last statement is equivalent to say that $su \leq tu$.

□

2. Structure of $B(X)$

In the following let F be an endofunctor over **Set**. Given a F-coalgebra (X, α) we denote by $B(X)$, as before, the set of all bisimulations on (X, α)

PROPOSITION 4.8. *The set $B(X)$ together with the union of sets forms a commutative idempotent monoid with identity element $\emptyset$.*

PROOF. Theorem 3.9 states that the union of bisimulations is a bisimulation. In example 3.4 we have seen that $\emptyset$ is a bisimulation. As we know, the union of sets is commutative, associative, idempotent and has $\emptyset$ as the identity element.

Thus, $(B(X), \cup, \emptyset)$ forms a commutative idempotent monoid with identity element. □

REMARK 4.9. Every commutative idempotent monoid with identity element is a join-semilattice with zero and viceversa.

Assume from now that F in this chapter preserves weak pullbacks. We will see that we will get a richer structure on $B(X)$.

PROPOSITION 4.10. *The set $B(X)$ together with the composition of relations forms a monoid.*

Proof. The composition of relations is always associative.

Theorem 3.13 states that the composition of two bisimulations is a bisimulation.

Corollary 3.6 states that the diagonal relation is a bisimulation. Notice that Δ_X is the neutral element for the composition of relations. Thus, $(B(X), \circ, \Delta_X)$ forms a monoid. □

Proposition 4.11. *The composition of relations is compatible with the lattice structure on $\mathcal{P}(X \times X)$. Also for each Z_1, Z_2, $Z_3 \in \mathcal{P}(X \times X)$ holds:*

$$Z_1 \subseteq Z_2 \;\Rightarrow\; Z_3 \circ Z_1 \subseteq Z_3 \circ Z_2$$
$$Z_1 \subseteq Z_2 \;\Rightarrow\; Z_1 \circ Z_3 \subseteq Z_2 \circ Z_3$$

Proof. We will prove only the first statement, the other one is quite similar.

Notice that:

$$Z_3 \circ Z_1 = \{\langle x_3, x_1\rangle : \exists x_0 \in X(\langle x_3, x_0\rangle \in Z_3 \text{ and } \langle x_0, x_1\rangle \in Z_1)\} \subseteq Z_3 \circ Z_2$$

Last equality holds because $\langle x_0, x_1\rangle \in Z_1 \subseteq Z_2$ □

Proposition 4.12. *The composition of relations distributes over the union. That is to say that for each Z_1, Z_2, $Z_3 \in \mathcal{P}(X \times X)$ it holds that:*

$$Z_1 \circ (Z_2 \cup Z_3) = (Z_1 \circ Z_2) \cup (Z_1 \circ Z_3)$$

Proof. We must check the two inclusions.

- $\subseteq$ Let $\langle x_1, y\rangle \in Z_1 \circ (Z_2 \cup Z_3)$. We know that there exists some $x' \in (Z_2 \cup Z_3)$ for which it holds that $\langle x_1, x'\rangle \in Z_1$ and $\langle x', y\rangle \in (Z_2 \cup Z_3)$. If $\langle x', y\rangle \in Z_2$ holds, then $\langle x_1, y\rangle \in Z_1 \circ Z_2 \subseteq (Z_1 \circ Z_2) \cup (Z_1 \circ Z_3)$. If $\langle x', y\rangle \in Z_3$ holds, then $\langle x_1, y\rangle \in Z_1 \circ Z_3 \subseteq (Z_1 \circ Z_2) \cup (Z_1 \circ Z_3)$. And we get one inclusion.
- $\supseteq$ Using Proposition 4.11

$$Z_2 \subseteq Z_2 \cup Z_3 \;\Rightarrow\; Z_1 \circ Z_2 \subseteq Z_1 \circ (Z_2 \cup Z_3)$$
$$Z_3 \subseteq Z_2 \cup Z_3 \;\Rightarrow\; Z_1 \circ Z_3 \subseteq Z_1 \circ (Z_2 \cup Z_3)$$

And finally,

$$(Z_1 \circ Z_2) \cup (Z_1 \circ Z_3) \subseteq Z_1 \circ (Z_2 \cup Z_3)$$

□

Theorem 4.13. *The set $B(X)$ togheter with the union of bisimulations and the composition of bisimulations forms a semiring.*

Proof. By Proposition 4.8, $(B(X), \cup, \emptyset)$ is a commutative monoid with identity element $\emptyset$.

By Proposition 4.10, $(B(X), \circ, \Delta_X)$ is a monoid with identity element Δ_X.

By Proposition 4.12, the composition of relations distributes over the union of sets.

Finally, $\emptyset$ annihilates $B(X)$ with respect to composition of relations. That is to say that for each $Z \in B(X)$ holds:

$$Z \circ \emptyset = \emptyset \circ Z = \emptyset$$

Thus, $(B(X), \cup, \circ, \emptyset, \Delta_X)$ forms a semiring. □

Remark 4.14. Semirings in which the correspondent addition operation is idempotent are called idempotent semirings or dioids. In our case, $B(X)$ is a dioid since the union of sets is idempotent.

Definition 4.15. **Associated Dioid**
Let (X, α) be a F-coalgebra.
We define its *associated diod*, denoted by $\pi(X, \alpha)$, to the dioid $(B(X), \cup, \circ, \emptyset, \Delta_X)$

Theorem 4.16. *Let (X, α) be a F-coalgebra. Then holds:*

$$|\pi(X, \alpha)| \leq 2^{|X|^2}$$

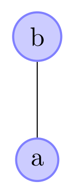
Proof. Notice that each $Z \in \pi(X, \alpha)$ is a bisimulation on X, so $Z \subseteq X \times X$ and therefore holds that $B(X) \subseteq \mathcal{P}(X \times X)$. Hence

$$|\pi(X, \alpha)| = |B(X)| \leq |\mathcal{P}(X \times X)| = 2^{|X \times X|} = 2^{|X|^2}$$

□

Corollary 4.17. *Let (X, α) be a F-coalgebra. If X is finite, so is $\pi(X, \alpha)$*

Example 4.18. Let us calculate the associated dioid for the following Poset $(P, \leq)$

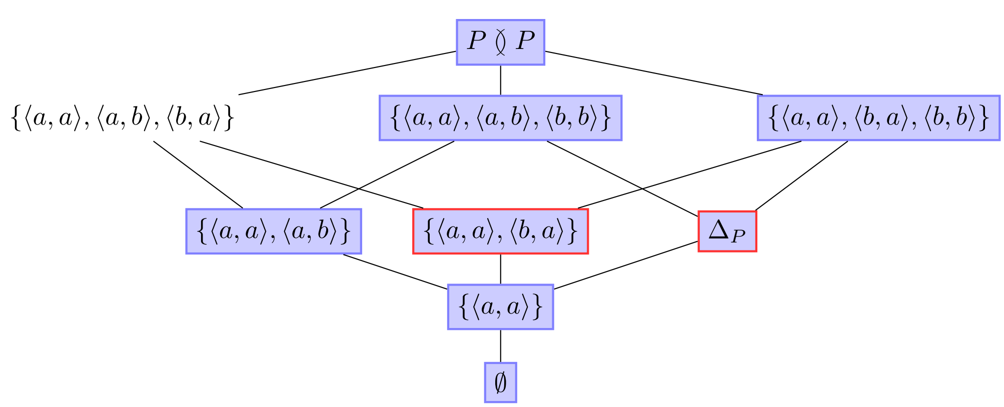


as $\mathcal{P}$-coalgebra, in the sense of the construction shown in 2.10.
Notice first that by the preceding Corollary $|\pi(P, \eta)| \leq 2^{|X|^2} = 16$
Let us prove a previous lemma, which tells us which are all the possible bisimulations on P.

Lemma 4.19. *Let $(P, \leq)$ be any Poset. Let Z be any subset of $P \times P$. It holds that $Z \in B(P)$ if and only if for each $\langle p, q \rangle \in Z$ the following two properties hold:*
a) $\forall p \in P \; p \leq p_0 \implies \exists q_p \in P \quad (\langle p, q_p \rangle \in Z$ *and* $q_p \leq q_0)$
b) $\forall q \in P \; q \leq q_0 \implies \exists p_q \in P \quad (\langle p_q, q \rangle \in Z$ *and* $p_q \leq p_0)$

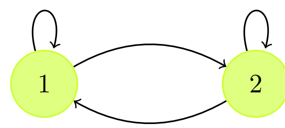
Proof. It follows from Proposition 3.2 and Remark 2.10. □

Consequently, we can conclude that $|\pi(P, \eta)| = 9$
We present the Hasse Diagram of all possible bisimulations on P:



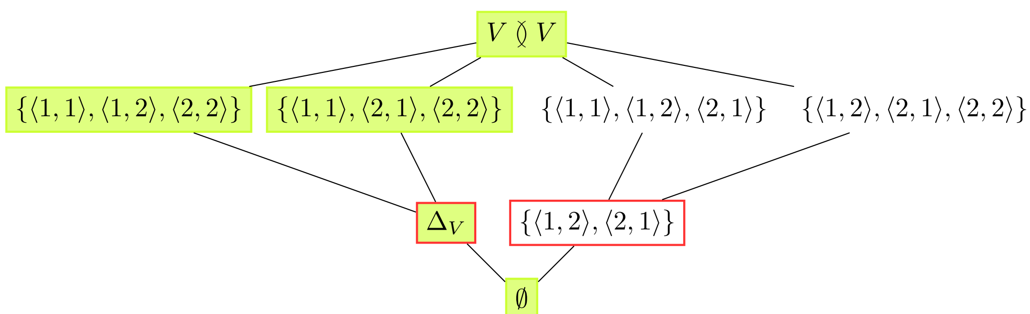
We have coloured in blue all the idempotent elements of the multiplicative monoid and bordered in red all the bisimulations of the form graph of some F-coalgebra homomorphism.

Example 4.20. Let us calculate the associated dioid for the following Graph (V, ξ) as a $\mathcal{P}$-coalgebra in the sense of 2.6:



It also holds $|\pi(V, \xi)| \leq 2^{|V|^2} = 16$

The dioid is given by:



We have coloured in green all the idempotent elements of the multiplicative monoid and bordered in red all the bisimulations of the form graph of some F-coalgebra homomorphism.

3. Usefulness

The associated dioid is a useful tool to characterize F-coalgebras. The following proposition tell us how can we use that dioid.

Proposition 4.21. *Let (X, α), (Y, β) be two F-coalgebras. If $f : X \to Y$ is a F-coalgebra isomorphism between X and Y, then*

$$\pi(X, \alpha) \cong \pi(Y, \beta)$$

Proof. Just take as dioid isomorphism the mapping f

$$\begin{array}{rccc} f: & \pi(X, \alpha) & \longrightarrow & \pi(Y, \beta) \\ & Z & \longmapsto & f(Z) \end{array}$$

Notice that every mapping respects the union of sets, thus $f(Z_1 \cup Z_2) = f(Z_1) \cup f(Z_2)$. Since f is injective, we can use the 3rd point of Proposition 3.19 to conclude that f respects the composition. Moreover, since f is surjective we can apply the 1st point of the same Proposition to conclude that

$$f(\Delta_X) = \Delta_{f(X)} = \Delta_Y$$

Therefore it maps the unit of $\pi(X, \alpha)$ to the unit of $\pi(Y, \beta)$ and it also holds that $f(\emptyset) = \emptyset$. Thus, we conclude that $\pi(X, \alpha) \cong \pi(Y, \beta)$

□

CHAPTER 5

SubCoalgebras

1. Basic Facts

DEFINITION 5.1. **Subcoalgebra**

Let F be an arbitrary functor over **Set**. Let (X, α) be a F-coalgebra. Let $W \subseteq X$ be any subset of X.

We say that W is a *subcoalgebra* of X, written $W \leq X$, if there exists an structure map α_W on W such that turns the inclusion mapping $i : W \to X$ into a F-coalgebra homomorphism. That is to say that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\ i\ } & X \\ {\scriptstyle \alpha_W}\downarrow & & \downarrow{\scriptstyle \alpha} \\ FW & \xrightarrow[\ Fi\ ]{} & FX \end{array}$$

PROPOSITION 5.2. *Let F be an arbitrary functor over* **Set**.

Let (X, α) be a F-coalgebra. Let $W \subseteq X$ be any subset of X.

If there exists a structure map on W, namely α_W, that turns the inclusion mapping into a F-coalgebra homomorphism, then α_W is uniquely determined.

PROOF. For the case $W = \emptyset$ it is straightforward to see that α_W is the empty mapping and we can conclude that it is uniquely determined.

Now assume that $W \neq \emptyset$. Let α'_W be another structure map on W that makes the diagram of 5.1 commute. Hence:

$$Fi\alpha_W = \alpha i = Fi\alpha'_W$$

Since i is mono, by Proposition 1.4 so is Fi, therefore $\alpha_W = \alpha'_W$ □

So far, one can easily see that the empty set, $\emptyset$, and X are always subcoalgebras of (X, α).

DEFINITION 5.3. **Minimal Coalgebras**

Let F be any endofunctor over **Set**. A F-coalgebra (X, α) is called *minimal* if it does not have any proper subcoalgebra (i.e., different from $\emptyset$ and X).

Subcoalgebras can be characterized in terms of bisimulations as follows:

PROPOSITION 5.4. *Let F be an arbitrary functor over* **Set**.

Let (X, α) be a F-coalgebra. Let $W \subseteq X$ be any subset of X.

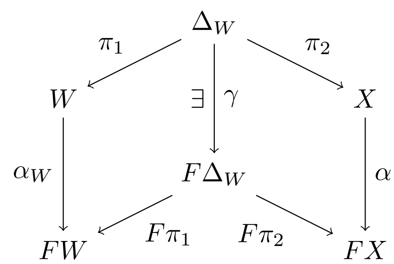
$W \leq X$ if and only if the diagonal of W, Δ_W, is a bisimulation on X.

PROOF. $\gg$ Assume that $W \leq X$, therefore there exists an structure map α_W that turns the inclusion mapping into a F-coalgebra homomorphism. Therefore, by

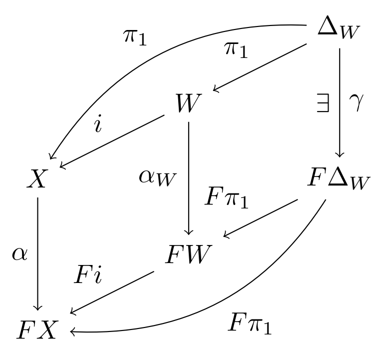
Theorem 3.5 the graph of i is a bisimulation between (W, α_W) and (X, α). Notice that

$$G(i) = \{\langle w, i(w)\rangle \ : \ w \in W\} = \Delta_W$$

So there exists an structure map $\gamma : \Delta_W \to F\Delta_W$ such that the following diagram commutes:



If Δ_W must be a bisimulation on X, we must be able to replace (W, α_W) by (X, α) in the left hand side of the preceding diagram. Notice also that α_W turns the inclusion mapping into a F-coalgebra homomorphism, so we obtain the following commutative diagram:



Notice that $\pi_1 = \pi_1 i$ and therefore, $F\pi_1 Fi = F(\pi_1 i) = F\pi_1$. We can get analogous results for the other projection.

So finally, Δ_W is a bisimulation on X.

$\ll$ For the converse. Assume that Δ_W is a bisimulation on X. So there exists a structure map γ and that makes the diagram of Definition 3.1 commute. So we get the equality $F\pi_1\gamma = \alpha\pi_1$

So if we take

$$\begin{array}{rccl} \alpha_W : & W & \longrightarrow & FW \\ & w & \longmapsto & F\pi_1\gamma(\langle w, w\rangle) \end{array}$$

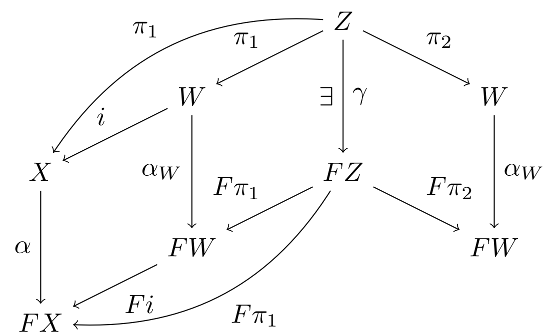
We can conclude that $\Delta_W = G(i)$ is a bisimulation between W and X and therefore using Theorem 3.5 we see that α_W turns the including map i into a F-coalgebra homomorphism. □

The kind of proof done in the first part of the previous statement can be easily extended to each bisimulation on W. Thus, we get the following Proposition:

Proposition 5.5. *Let F be an arbitrary endofunctor on* **Set**. *Let (X, α) be a F-coalgebra and let (W, α_W) be any subcoalgebra of X. It holds that every bisimulation on W is also a bisimulation on X. That is to say that:*

$$B(W) \subseteq B(X)$$

Proof. Let Z be any bisimulation on W, we must check that $Z \in B(X)$. As before, since Z is a bisimulation on W, it makes the following diagram commute:



Notice that α_W turns the inclusion mapping into a F-coalgebra homomorphism. We have depicted that fact in the preceding diagram. As in the previous Proposition, $\pi_1 = \pi_1 i$ and therefore, $F\pi_1 Fi = F(\pi_1 i) = \pi_1$ and also for the other projection. Thus, Z is a bisimulation on X.

□

2. More on Semiring Theory

With the last proposition we can determine the relation between the associated dioid of W and the associated dioid of X.

Definition 5.6. **Subhemiring, Subsemiring**

Let $\mathbb{S} = (S, +, \cdot, 0)$ be a hemiring. Let H be a subset of S. We say that H is a *subhemiring* of $\mathbb{S}$ if $0 \in H$ and it is closed under $+$ and $\cdot$. We will denote this fact by $H \preceq S$.

Moreover, if S is a semiring and $1 \in H$ we say that H is a *subsemiring*. We will denote this fact by $H \leq S$.

Remark 5.7. Let F be an arbitrary endofunctor on **Set**. Let (X, α) be a F-coalgebra and let $W \leq X$ be any subcoalgebra. It holds that $\pi(W, \alpha_W) \preceq \pi(X, \alpha)$.

3. Pullbacks and Subcoalgebras

In this section we will work with endofunctors that preserve weak pullbacks. This extra assumption turns into extra nice properties on the subcoalgebras of a given coalgebra.

Proposition 5.8. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra and let (W, α_W) be any subcoalgebra of X. It holds that:*

$$\Delta_W \circ B(X) \circ \Delta_W = B(W)$$

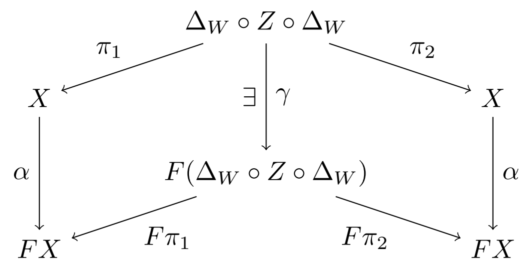
Where $\Delta_W \circ B(X) \circ \Delta_W = \{\Delta_W \circ Z \circ \Delta_W : \ Z \in B(X)\}$

Proof. We must check the two inclusions.

$\subseteq$ Let Z be any bisimulation on X.

Since W is a subcoalgebra of X, we know by Proposition 5.4 that its diagonal is a bisimulation on X. Moreover, since F preserves weak pullbacks, we know that the composition of bisimulations is a bisimulation by Theorem 3.13. Hence, $\Delta_W \circ Z \circ \Delta_W$ is a bisimulation on X.

Therefore, the following diagram commutes:

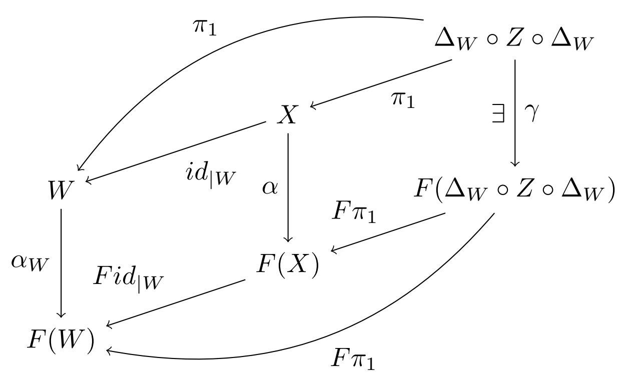


We must notice two important facts:

i. $\Delta_W \circ Z \circ \Delta_W \subseteq W \times W$

ii. Since W is a subcoalgebra of X, we get that the inclusion mapping i is a F-coalgebra homomorphism. Moreover, $id_{|W} : X \to W$ is also a F-coalgebra homomorphism by Proposition 1.7, since it is an inverse mapping of i.

Thus, we get the following commutative diagram:



Notice that $\pi_1 = \pi_1 id_{|W}$ and therefore, $F\pi_1 Fi = F(\pi_1 id_{|W}) = F\pi_1$. We can get analogous results for the other projection.

So finally, $\Delta_W \circ Z \circ \Delta_W$ is a bisimulation on W.

$\supseteq$ Let Z_W be any bisimulation on W. By Proposition 5.5 it holds that Z_W is also a bisimulation on X. One can easily check that $Z_W = \Delta_W \circ Z_W \circ \Delta_W$

Thus, $B(W) \subseteq \Delta_W \circ B(X) \circ \Delta_W$

□

It also holds that when the considered endofunctor preserves weak pullbacks, we get that the image and the inverse image of subcoalgebras are still subcoalgebras.

Proposition 5.9. *Let F be an endofunctor over* **Sets** *that preserves weak pullbacks. Let (X, α), (Y, β) be two F-coalgebras and let $f : X \to Y$ be any F-coalgebra homomorphism between them. It holds:*

1. *If $W \subseteq X$ is a subcoalgebra of X, then $f(W)$ is a subcoalgebra of Y.*
2. *If $W \subseteq Y$ is a subcoalgebra of Y, then $f^{-1}(W)$ is a subcoalgebra of X.*

PROOF. We will do the proof for the first statement. The proof for the other statement is quite similar.

Assume $W \subseteq X$ is a subcoalgebra of X, then by Proposition 5.4 it holds that Δ_W is a bisimulation on X. Using Proposition 3.17 we get that $f(\Delta_W)$ is a bisimulation on Y. Moreover, using Proposition 3.19 it holds that $f(\Delta_W) = \Delta_{f(W)}$, therefore $\Delta_{f(W)}$ is a bisimulation on Y. Finally, using again Proposition 5.4 we conclude that $f(W)$ is a subcoalgebra of Y. □

REMARK 5.10. Using Remark 3.18, we can get a stronger version of the first statement of the preceding theorem by not requiring F to preserve weak pullbacks.

We finish this chapter by stating also an important Theorem about the structure of all the subcoalgebras.

THEOREM 5.11. *Let F be an endofunctor over* **Set** *that preserves weak pullbacks.*

Let (X, α) be a F-coalgebra. The collection of all subcoalgebras of X is a complete lattice in which least upper bounds and greatest lower bounds are given by union and intersection.

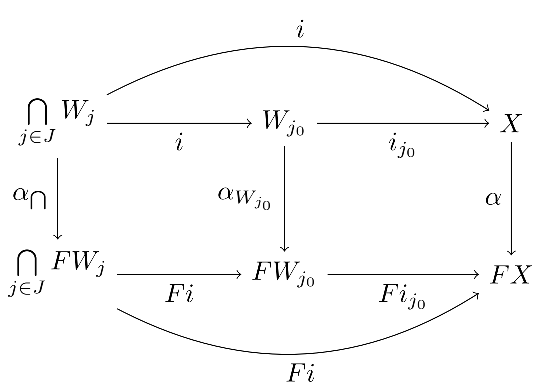
PROOF. Let $\{W_j : j \in J\}$ be a family of subcoalgebras of X. We want to see that the union and the intersection of the family are again subcoalgebras of X.

$\bigcup$: Notice that for each $j \in J$, W_j is a subcoalgebra of X. Therefore, applying Proposition 5.4, it holds that Δ_{W_j} is a bisimulation on X. Thus, $\{\Delta_{W_j} : j \in J\}$ is a set of bisimulations on X. Applying Theorem 3.9 it holds that $\bigcup_{j \in J} \Delta_{W_j}$ is also a bisimulation on X. Notice that

$$\Delta_{\bigcup\limits_{j \in J} W_j} = \bigcup_{j \in J} \Delta_{W_j}$$

therefore, applying again Proposition 5.4, we get that $\bigcup_{j \in J} W_j$ is a subcoalgebra of X.

$\bigcap$: By Proposition 1.23, since F preserves weak pullbacks, it also preserves intersections. More specifically, F transforms the (generalized) pullback diagram of the intersection of the subsets $\{W_j : j \in J\}$ into a pullback diagram of the sets FW_j for each $j \in J$. In particular, for a fixed $j_0 \in J$ it holds:



Which means that the inclusion mapping $i : \bigcap_{j\in J} W_j \to X$ is a F-coalgebra homomorphism, thus $\bigcap_{j\in J} W_j$ is a subcoalgebra of X.

□

The preceding Theorem allow us to give the following definitions of the smallest subcoalgebra generated by a subset and the greatest subcoalgebra that contains a subset.

DEFINITION 5.12. **Smallest Subcoalgebra containing Y**
Let F be an endofunctor that preserves weak pullbacks. Let (X, α) be a F-coalgebra and let $Y \subseteq X$ be any subset of X. The *subcoalgebra generated by* Y, denoted by $\langle Y\rangle$, is defined as

$$\langle Y\rangle = \bigcap\{W : W \text{ is a subcoalgebra of } X \text{ and } Y \subseteq W\}$$

$\langle Y\rangle$ is the smallest subcoalgebra of X containing Y.
If $X = \langle Y\rangle$ for some subset Y of X, then X is said to be generated by Y
The subcoalgebra generated by a singleton set $\{x\}$ is denoted by $\langle x\rangle$

We also get its dual notion:

DEFINITION 5.13. **Greatest Subcoalgebra contained in Y**
Let F be an endofunctor that preserves weak pullbacks. Let (X, α) be a F-coalgebra and let $Y \subseteq X$ be any subset of X. The *greatest subcoalgebra contained in* Y, denoted by $[Y]$, is defined as

$$[Y] = \bigcup\{W : W \text{ is a subcoalgebra of } X \text{ and } W \subseteq Y\}$$

We finish this chapter with two Properties stating the characterisations of a system that can be reduced to a single state - coalgebra.

PROPOSITION 5.14. *Let F be an endofunctor that preserves weak pullbacks. Let (X, α) be a F-coalgebra. Let $x_0 \in X$ be an arbitrary element on X. If $ct_{x_0} : X \to X$ is a F-coalgebra homomorphism, where:*

$$\begin{array}{rccc} ct_{x_0}: & X & \longrightarrow & X \\ & X & \longmapsto & x_0 \end{array}$$

then $\{x_0\}$ is a subcoalgebra of X.

PROOF. Since ct_{x_0} is a F-coalgebra homomorphism, we get by Theorem 3.8 that $\langle ct_{x_0}, ct_{x_0}\rangle(X)$ is a bisimulation on X. Notice that:

$$\langle ct_{x_0}, ct_{x_0}\rangle(X) = \{\langle x_0, x_0\rangle\} = \Delta_{\{x_0\}}$$

Applying Proposition 5.4 we get that $\{x_0\}$ is a subcoalgebra of X. □

PROPOSITION 5.15. *Let F be an endofunctor that preserves weak pullbacks. Let (X, α) be a F-coalgebra. Let $x_0 \in X$ be an arbitrary element on X. If $ct_{x_0} : X \to X$ is a F-coalgebra homomorphism, then $X \between X = X^2$.*

PROOF. We must check the two inclusions.

$\subseteq$ It always hold.

$\supseteq$ Let $x, y \in X$. We must check that $\langle x, y\rangle \in X \between X$. Notice that $G(ct_{x_0})$, the graphic of ct_{x_0}, is a bisimulation on X applying Theorem 3.5, thus we get that $\langle x, x_0\rangle, \langle y, x_0\rangle \in G(ct_{x_0})$, moreover it holds that $G(ct_{x_0})^{-1}$ is again a bisimulation applying Theorem 3.7 and $\langle x_0, y\rangle \in G(ct_{x_0})^{-1}$. Since

F preserves weak pullbacks, we get that the composition of bisimulations is again a bisimulation and since $X \between X$ is the greatest bisimulation on X we get that:

$$\langle x, y \rangle \in G(ct_{x_0}) \circ G(ct_{x_0})^{-1} \subseteq X \between X$$

$\square$

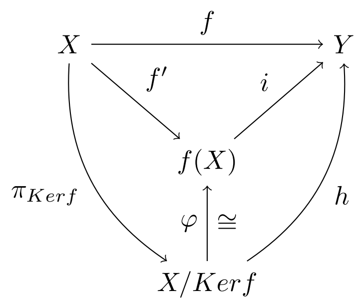
CHAPTER 6

Isomorphism Theorems

1. 1st Isomorphism Theorem

THEOREM 6.1. **1st Isomorphism Theorem**
Let F be an endofunctor over **Set** *that preserves weak pullbacks.*

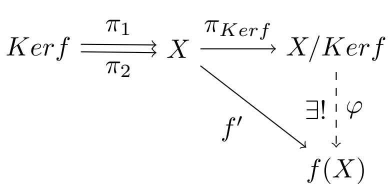
Let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be a F-coalgebra homomorphism. Then there is the following factorization of f:

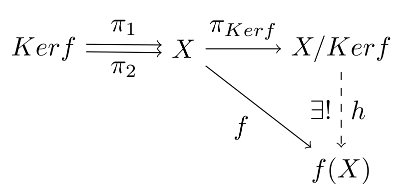


where i is the inclusion homomorphism, h is a F-coalgebra monomorphism, f' is a F-coalgebra epimorphism with $f(x) = f'(x)$ for each $x \in X$ and π_{Kerf} is the quotient homomorphism.

PROOF. X is subcoalgebra of itself, so using Theorem 5.9, $f(X)$ is a subcoalgebra of Y and hence the inclusion mapping i turns into a F-coalgebra monomorphism. Notice that $f = if'$, so using Proposition 1.8, f' is a surjective F-coalgebra homomorphism, i.e., an epimorphism.

On the other hand, Corollary 3.16 states that $Kerf$ is a bisimulation equivalence on X, and using Proposition 3.11 π_{Kerf} turns into a F-coalgebra epimorphism. Moreover in the proof of that theorem we prove that $(X/Kerf, \pi_{Kerf})$ is a coequalizer of the F-coalgebra homomorphisms $\pi_1 : Kerf \to X$ and $\pi_2 : Kerf \to X$, so by the universal property we obtain the mappings:





Again by Theorem 5.9, h and φ are F-coalgebra homomorphisms.

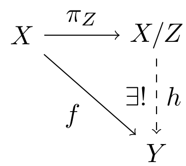
It is important to remark how φ acts on the elements of $X/Kerf$. It follows from the construction of the coequalizers done in the Proposition 1.17 that φ is defined as:

$$\begin{array}{rccc} \varphi: & X/Kerf & \longrightarrow & f(X) \\ & [x]_{Kerf} & \longmapsto & f(x) \end{array}$$

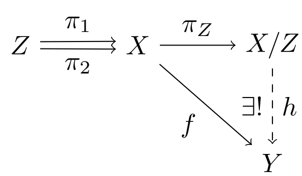
Clearly φ is surjective and injective, and hence φ becomes a F-coalgebra isomorphism. Finally $h = i\varphi$ where φ is a bijection and i is an injective mapping, therefore we conclude that h is an injective mapping, and thus it is a F-coalgebra monomorphism. □

Notice that the argument used before to obtain the F-coalgebra homomorphisms h and φ can be generalized in the following way:

THEOREM 6.2. *Let F be an endofunctor that preserves weak pullbacks.*
Let (X, α) and (Y, β) be two F-coalgebras and let $f : X \to Y$ be a F-coalgebra homomorphism. Let Z be a bisimulation equivalence on X contained in $Kerf$, that is $Z \subseteq Kerf$. Then there is a unique homomorphism $h : X/Z \to Y$ such that $f = h\pi_Z$. That is to say that h makes the following diagram commute:



PROOF. Using Proposition 3.11 π_Z, turns into a F-coalgebra epimorphism. Moreover in the proof of that theorem we prove that $(X/Z, \pi_Z)$ is a coequalizer of the F-coalgebra homomorphisms $\pi_1 : Z \to X$ and $\pi_2 : Z \to X$, so by the universal property we obtain the unique mapping:



Notice that $f = h\pi_Z$ and by Theorem 5.9, h is a F-coalgebra homomorphism. □

2. 2nd Isomorphism Theorem

The 2nd Isomorphism Theorem states that there is a one-to-one correspondence between subcoalgebras of a quotient of a given coalgebra and quotients of subcoalgebras of the coalgebra considered.

THEOREM 6.3. **2nd Isomorphism Theorem**
Let F be an endofunctor over **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra. Let $W \subseteq X$ be a subcoalgebra of X and let Z be a bisimulation equivalence on X. Let W^Z be defined as*

$$W^Z = \{x \in X : \exists w \in W(\langle x, w\rangle \in Z)\}$$

The following facts hold:

1. *W^Z is a subcoalgebra of X*
2. *$Z \cap (W \times W)$ is a bisimulation equivalence on W*
3. *$W/(Z \cap (W \times W)) \cong W^Z/Z$*

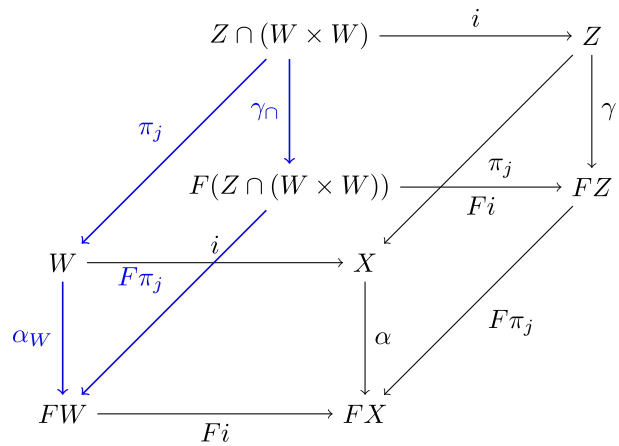
Proof. We will do a proof for each item:

1. Let us see that $W^Z = \pi_1\pi_2^{-1}(W)$

$$\begin{array}{rcl} \pi_2^{-1}(W) & = & \{\langle x_1, x_2\rangle \in Z : \ x_2 \in W\} \\ \pi_1\pi_2^{-1}(W) & = & \{x_1 \in X : \ \exists x_2 \in W(\langle x_1, x_2\rangle \in Z)\} \quad = \quad W^Z \end{array}$$

   Therefore, using Proposition 5.9 we conclude that W^Z is a subcoalgebra of X.

2. Notice that $Z\cap(W\times W) = \pi_1^{-1}(W)\cap\pi_2^{-1}(W)$, so using again Proposition 5.9 and Theorem 5.11, we conclude that $Z \cap (W \times W)$ is a subcoalgebra of Z. And we get the following commutative diagram for each projection mapping π_j, for $j \in \{1, 2\}$:



   Notice that the part of the diagram coloured in blue correspond to the Definition of $Z \cap (W \times W)$ being a bisimulation on W. Since Z is an equivalence relation on X and $W \subseteq X$ it is straightforward to see that $Z \cap (W \times W)$ is also an equivalence relation on W. Finally, $Z \cap (W \times W)$ is a bisimulation equivalence on W.

3. Consider the quotient homomorphism, $\pi_Z : X \to X/Z$ and let $\pi_{Z|W} : W \to X/Z$ be its restriction to W. Notice that $\pi_{Z|W} = \pi_Z i$ is a F-coalgebra homomorphism since it is the composition of two F-coalgebra homomorphism. Using 1st Isomorphism Theorem 6.1 we get that

$$W/Ker\pi_{Z|W} \cong \pi_{Z|W}(W)$$

   Notice that:

$$\begin{array}{rcl} Ker\pi_{Z|W} & = & \{\langle x_1, x_2\rangle \in W \times W : \ \pi_{Z|W}(x_1) = \pi_{Z|W}(x_2)\} \\ & = & \{\langle x_1, x_2\rangle \in W \times W : \ [x_1]_Z = [x_2]_Z\} \\ & = & \{\langle x_1, x_2\rangle \in W \times W : \ \langle x_1, x_2\rangle \in Z\} \\ & = & Z \cap (W \times W) \end{array}$$

$$\begin{array}{rcl} \pi_{Z|W}(W) & = & \{\pi_{Z|W}(x) : \ x \in W\} \\ & = & \{[x]_Z : \ x \in W\} \\ & = & \pi_Z(W^Z) \\ & = & W^Z/Z \end{array}$$

   And finally, we get that:

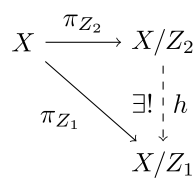
$$W/Z \cap (W \times W) \cong W^Z/Z$$

□

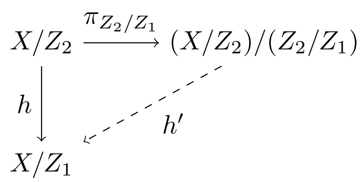
3. 3rd Isomorphism Theorem

THEOREM 6.4. **3rd Isomorphism Theorem**
Let F be an endofunctor over **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra and let Z_1 and Z_2 be two bisimulation equivalences on X such that $Z_2 \subseteq Z_1$. It holds:*

1. *There is a unique F-coalgebra homomorphism $h : X/Z_2 \to Z_1$ such that $h\pi_{Z_2} = \pi_{Z_1}$. That is to say that the following diagram commutes:*



2. *Let Z_2/Z_1 denote $Kerh$. It holds that Z_2/Z_1 is a bisimulation equivalence on X/Z_2 and induces a F-coalgebra isomorphism $h' : (X/Z_2)/(Z_2/Z_1) \to X/Z_1$ such that $h = h'\pi_{Z_2/Z_1}$. That is to say that the following diagram commutes:*



PROOF. We will do a proof for each item:

1. It is a consequence of Theorem 6.2. Take $Y = X/Z_1$, $f = \pi_{Z_1}$ and $Z = Z_2$. Notice that $Ker\pi_{Z_1} = Z_1$ and the assumption $Z_2 \subseteq Z_1$ holds.
2. First of all, notice that from previous item, we have that Using the 1st Isomorphism Theorem 6.1, it holds that $h\pi_{Z_2} = \pi_{Z_1}$, since all the projections are surjective, so is h. Using the 1st Isomorphism Theorem 6.1, it holds that $(X/Z_2)/(Z_2/Z_1) = (X/Z_2)/Kerh \cong h(X/Z_2) = X/Z_1$. Also from the same Theorem we get that $h = h'\pi_{Z_2/Z_1}$.

□

CHAPTER 7

Simple Coalgebras

1. Simple Coalgebras

DEFINITION 7.1. **Simple Coalgebras**
Let F be an endofunctor on **Set** that preserves weak pullbacks. We say that a F-coalgebra, (X, α), is *simple* if it has no proper quotients. That is to say, if Z is a bisimulation equivalence on X, then $X/Z \cong X$.

Next theorem give us equivalent characterisations:

THEOREM 7.2. *Let F be an endofunctor on* **Set**. *Let (X, α), (Y, β) be two F-coalgebras. The following statements are equivalent:*

1. *(X, α) is a simple F-coalgebra.*
2. *Every F-coalgebra epimorphism $f : X \to Y$ is a F-coalgebra isomorphism.*
3. *Let Z be a bisimulation on X, then $Z \subseteq \Delta_X$. This characterisation is known as the coinduction proof principle.*
4. *Δ_X is the only bisimulation equivalence on X.*
5. *Let $f : Y \to X$ and $g : Y \to X$ be two F-coalgebra homomorphisms, then $f = g$.*
6. *The quotient homomorphism $\pi_{\between} : X \to X/(X \between X)$ is a F-coalgebra isomorphism.*
7. *Any F-coalgebra homomorphism, $f : X \to Y$, is injective.*

PROOF. We will prove the following equivalences:

1.≫2. Let $f : X \to Y$ be any F-coalgebra epimorphism. By the 1st Isomorphism Theorem 6.1 it holds that

$$X/Kerf \cong f(X)$$

As seen on Corollary 3.16, $Kerf$ is a bisimulation equivalence on X. Since X is simple, it holds that $X/Kerf \cong X$.

Since f is surjective, it holds that $f(X) = Y$. Thus, we get that $X \cong Y$ by f, then f is a F-coalgebra isomorphism.

2.≫1. Let Z be any bisimulation equivalence on X. By Proposition 3.11, the quotient mapping $\pi_Z : X \to X/Z$ is a F-coalgebra epimorphism. By assumption, π_Z is an F-coalgebra isomorphim, thus we get that $X/Z \cong X$, i.e., (X, α) is simple.

1.≫4. Let Z be any bisimulation equivalence on X. By assumption $X/Z \cong X$, then $Z = \Delta_X$. Therefore, Δ_X is the only bisimulation equivalence on X.

4.≫1. Let Z be any bisimulation equivalence on X. By assumption, $Z = \Delta_X$, thus we get that $X/\Delta_X \cong X$, i.e., (X, α) is simple.

3.$\gg$4. Let Z be any bisimulation equivalence on X. Since Z is reflexive, it holds that $\Delta_X \subseteq Z$. Moreover, by assumption, we get that $Z \subseteq \Delta_X$. Thus, $Z = \Delta_X$, i.e., Δ_X is the only bisimulation equivalence on X.

4.$\gg$3. Let Z be any bisimulation on X. It holds that $Z \subseteq X \between X$, since $X \between X$ is the greatest bisimulation on X. Moreover, by Corollary 3.14, it holds that $X \between X$ is a bisimulation equivalence on X. By assumption, $X \between X = \Delta_X$. Therefore, $Z \subseteq \Delta_X$.

3.$\gg$5. Let $f : Y \to X$ and $g : Y \to X$ be two F-coalgebra homomorphisms. By Theorem 3.8, $\langle f, g\rangle(Y) = \{\langle f(y), g(y)\rangle : \ y \in Y\}$ is a bisimulation on X. By assumption, $Z \subseteq \Delta_X$, thus $f = g$.

5.$\gg$3. Let Z be a bisimulation on X. By definition of bisimulation, the projection mappings $\pi_1 : Z \to X$ and $\pi_2 : Z \to X$ are F-coalgebra homomorphisms. By assumption, $\pi_1 = \pi_2$, thus we get that $Z \subseteq \Delta_X$.

2.$\gg$6. As a particular case.

6.$\gg$4. Let Z be a bisimulation equivalence on X. By assumption, $\pi_{\between} : X \to X/(X \between X)$ is a F-coalgebra isomorphism. Notice that $Z \subseteq X \between X$ since $X \between X$ is the greatest bisimulation on X. Moreover, $X \between X = Ker\pi_{\between}$. Thus, applying Theorem 6.2 there exists a unique F-coalgebra homomorphism h, $h : X/Z \to X/(X \between X)$ such that $h\pi_Z = \pi_{\between}$. Since $\pi_{\between}$ is an isomorphism, we get that π_Z is injective, then $Z \subseteq \Delta_X$.

7.$\gg$1. Let Z be a bisimulation equivalence on X. By Proposition 3.11, the quotient mapping $\pi_Z : X \to X/Z$ is a F-coalgebra epimorphism. By assumption, π_Z is also injective, therefore π_Z is a F-coalgebra isomorphism. That is to say that $X \cong X/Z$.

4$\gg$7. By Corollary 3.16, $Kerf$ is a bisimulation equivalence on X. By assumption $\Delta_X = Kerf$, i.e., f is injective.

$\square$

Every coalgebra can be made simple by taking the quotient with respect to its greatest bisimulation. This is a consequence of the following Proposition:

Proposition 7.3. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra and let Z be a bisimulation equivalence on X. It holds:*

$$X/Z \text{ is simple if and only if } Z = X \between X$$

Proof. First of all notice that by Proposition 3.11, $\pi_Z : X \to X/Z$ is a F-coalgebra homomorphism. We will prove the two implications:

$\gg$ Let Z' be any bisimulation on X, we will prove that $Z' \subseteq Z$. By definition of bisimulation, the projection mappings $\pi_1 : Z' \to X$ and $\pi_2 : Z' \to X$ are F-coalgebra homomorphisms. Moreover, the compositions $\pi_Z\pi_1$ and $\pi_Z\pi_2$ are F-coalgebras homomorphisms from Z' to X/Z. Since X/Z is simple, by the 5th characterisation of Theorem 7.2, we get that $\pi_Z\pi_1 = \pi_Z\pi_2$, i.e., $Z' \subseteq Z$. In particular, it holds that $X \between X \subseteq Z \subseteq X \between X$. Thus, we conclude that $Z = X \between X$.

$\ll$ Let Z' be any bisimulation on X/Z, we will prove that $Z' \subseteq \Delta_{X/Z}$. By Proposition 3.17 it holds that $\pi_Z^{-1}(Z')$ is a bisimulation on X. Since $X \between X$ is the greatest bisimulation on X, it holds that $\pi_Z^{-1}(Z') \subseteq Z$. Thus, we get that $Z' \subseteq \pi_Z(Z) = \Delta_{X/Z}$. Finally, by the 3rd characterisation of Theorem 7.2 it holds that X/Z is simple.

□

2. Subcoalgebras and Simple Coalgebras

PROPOSITION 7.4. *Every subcoalgebra of a simple coalgebra is simple. Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a simple F-coalgebra. Let W be a subcoalgebra of X, then W is simple.*

PROOF. Let $Z \in B(W)$ be any bisimulation on W. By Proposition 5.5 it holds that $Z \in B(X)$. Since X is simple by 3rd charaterization of Theorem 7.2 it holds that $Z \subseteq \Delta_X$. Notice however that $Z \subseteq W \times W$, thus $Z \subseteq \Delta_W$ and applying again Theorem 7.2 it holds that W is simple. □

PROPOSITION 7.5. *There exists a bijection between the set of bisimulations of a simple coalgebra and the set of its subcoalgebras. Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a simple F-coalgebra, then there exists a bijection between $Sub(X) := \{W \subseteq X : W \text{ is a subcoalgebra of } X\}$ and $B(X)$.*

PROOF. Define the bijection as:

$$\begin{array}{rccc} \varphi : & Sub(X) & \longrightarrow & B(X) \\ & W & \longmapsto & \Delta_W \end{array}$$

It is well-defined applying Proposition 5.4. The diagonal of a subcoalgebra is a bisimulation on the upper set. We must check it is a bijection.

- Injec. Let W_1 and W_2 be two subcoalgebras of X such that $\varphi(W_1) = \varphi(W_2)$. Then it holds that $\Delta_{W_1} = \Delta_{W_2}$, thus we get that necessarily it holds that $W_1 = W_2$.
- Surj. Let Z be any bisimulation on $B(X)$. Since X is simple it holds by Theorem 7.2 that $Z \subseteq \Delta_X$. Then there exists some $W \subseteq X$ for which $Z = \Delta_W$. Notice that W is a subcoalgebra of X since its diagonal is a subcoalgebra of X (Proposition 5.4).

□

3. More on Semiring Theory

DEFINITION 7.6. **Diagonal elements**

Let $\mathbb{S}$ be an idempotent semiring. We say that $s \in S$ is a *diagonal element* if it holds that $s \leq 1$. We denote by Δ_S the set of all diagonal elements.

PROPOSITION 7.7. *Let $\mathbb{S}$ be an idempotent semiring. Let $s, t \in \Delta_S$. Then, $s + t \in \Delta_S$ and $st \in \Delta_S$.*

PROOF. Take any $s, t \in \Delta_S$. It holds that $s, t \leq 1$.
For $+$, applying Proposition 4.7:

$$s \leq 1,\ t \leq 1 \ \Rightarrow\ s + t \leq 1 + 1 = 1$$

For $\cdot$, applying Proposition 4.7:

$$s \leq 1 \ \Rightarrow\ st \leq 1t = t \leq 1$$

□

COROLLARY 7.8. *Let $\mathbb{S}$ be an idempotent semiring, then $\Delta_S \leq S$.*

PROPOSITION 7.9. *Let $\mathbb{S}$ be an idempotent semiring.*

$$\nabla_S = \{0\} \cup \{s \in S : \ s \geq 1\} \leq S$$

DEFINITION 7.10. **Infinite Element**
Let $\mathbb{S} = (S, +, \cdot, 0)$ be an hemiring. We say that $s \in S$ is *infinite* if for each $t \in S$ holds:

$$t + s = s$$

If such element exists, it is unique.

DEFINITION 7.11. **Simple**
Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be a semiring. We say that $\mathbb{S}$ is *simple* if 1 is infinite.

COROLLARY 7.12. *Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be an idempotent semiring. It holds:*

$$\mathbb{S} \textit{ is simple } \Leftrightarrow S = \Delta_S$$

PROPOSITION 7.13. *Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be a semiring. The following statements are equivalent:*

1) *$\mathbb{S}$ is simple.*
2) *For all $s, t \in S$, $s = st + s$*
3) *For all $s, t \in S$, $s = ts + s$*
4) *For all $s, t, u \in S$, $st = st + s + u + t$.*

PROOF. We will only check the equivalency between 1) and 2).

1) $\gg$ 2) Assuming that $\mathbb{S}$ is simple, we get that $1 + t = 1$. Therefore:

$$s = s1 = s(1 + t) = s + st$$

2) $\gg$ 1) Let $t \in S$. By assumption, $1 = 1 + 1t = 1 + t$ therefore 1 is infinite.

□

REMARK 7.14. Identities 2) and 3) of Proposition 7.13 are noncommutative versions of *"absorption laws"* familiar from the axiomatic algebraic definitions of lattices. Because of them, simple semirings are sometimes refered to as *distributive pseudolattices*.

COROLLARY 7.15. *Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be a semiring. The following statements are equivalent:*

1) *$\mathbb{S}$ is simple and multiplicatively idempotent.*
2) *For all $s, t, u \in S$, $(s + t)(s + u) = s + tu$*
3) *For all $s, t \in S$, $s + t = s \ \Leftrightarrow \ st = t = ts$*

PROOF. We will only check the equivalency between 1) and 2).

1) $\gg$ 2) Assume 1). By Proposition 7.13 we get:

$$(s + t)(s + u) = s^2 + su + ts + tu = (s + su) + ts + tu = (s + ts) + tu = s + tu$$

2) $\gg$ 1) Assume 2). Let $s \in S$. It holds:

$$s^2 = (s + 0)(s + 0) = s + 0 = s$$

Now let $s, t \in S$. It holds:

$$s + st = s^2 + st = (s + 0)(s + t) = s + 0t = s$$

Therefore, applying Proposition 7.13, $\mathbb{S}$ is simple.

□

COROLLARY 7.16. *A commutative semiring is a bounded distributive lattice iff it is a simple multiplicatively idempotent semiring.*

PROPOSITION 7.17. *Let $\mathbb{S} = (S, +, \cdot, 0, 1)$ be a simple semiring. Consider an element $s \in S$ and define:*

$$S(s) = \{0\} \cup \{t \in S : \ t + s = 1\}$$

For each $s, t \in S$ holds:

1) $S(s) \leq S$
2) $S(s) \cap S(t) = S(st)$.

PROOF. We will prove the 2 statements.

1) $0 \in S(s)$ and since S is simple, we get that $1 + s = 1$, therefore $1 \in S(s)$. $S(s)$ is additively closed. Let $t, u \in S(s)$ (we can assume they are different from zero) it holds:

$$(u + t) + s = u + (t + s) = u + 1 = 1$$

In order to check that it is multiplicatively closed, we will use Prop. 7.13:

$$\begin{aligned} 1 &= 1 + s \\ &= (t + s)(u + s) + s \\ &= tu + ts + su + s^2 + s \\ &= tu + ts + su + (s^2 + s) \\ &= tu + ts + (su + s) \\ &= tu + (ts + s) \\ &= tu + s \end{aligned}$$

2) We will check the two inclusions:

$\subseteq$ Take any $u \in S(s) \cap S(t)$ (assume $u \neq 0$). As in 1):

$$1 = 1 + u = (s + u)(t + u) + u = st + u$$

$\supseteq$ Take any $u \in S(st)$ (assume $u \neq 0$). It holds:

$$1 = 1 + s = u + st + s = u + (st + s) = u + s$$

We have used Prop. 7.13. Analogous for t.

□

REMARK 7.18. Notice that $S(0) = \{0, 1\}$ and $S(1) = S$.

Once we have presented the previous definitions on semiring theory we apply those concepts to simple coalgebras:

PROPOSITION 7.19. *Let F be an endofunctor on* **Set**. *Let (X, α) be an F-coalgebra. It holds:*

$$(X, \alpha) \textit{ is simple } \Leftrightarrow \pi(X, \alpha) \textit{ is simple}$$

PROOF. We just need to apply Theorem 7.2. (X, α) is simple if and only if for each $Z \in B(X)$ it holds that $Z \subseteq \Delta_X$ if and only if $B(X) = \Delta_{B(X)}$ if and only if $\pi(X, \alpha)$ is simple. □

REMARK 7.20. Let F be an endofunctor on **Set**. Let (X, α) be a simple F-coalgebra, notice that $\pi(X, \alpha)$ is simple and multiplicatively idempotent, therefore on $\pi(X, \alpha)$ we can apply the statements of Corollary 7.15. Moreover, applying Corollary 7.16 we get that $\pi(X, \alpha)$ is a complete bounded distributive lattice.

4. Bisimulation Permutability

We introduce here an important notion that applies also to simple coalgebras: the permutability of bisimulations. We begin this section by showing the following property on diagonal bisimulations.

PROPOSITION 7.21. *Let F be an endofunctor on* **Set**. *Let (X, α) be an F-coalgebra. For each $Z_1, Z_2 \in \Delta_{B(X)}$ it holds that:*

$$Z_1 \circ Z_2 = Z_1 \cap Z_2$$

PROOF. We must check the two inclusions:

- $\subseteq$ Let $Z_1, Z_2 \in \Delta_{B(X)}$. Take any $\langle x_1, x_3 \rangle \in Z_1 \circ Z_2$, therefore, there exists an element $x_2 \in X$ such that $\langle x_1, x_2 \rangle \in Z_1$ and $\langle x_2, x_3 \rangle \in Z_2$. Since $Z_1, Z_2 \subseteq \Delta_X$, we get that $x_1 = x_2 = x_3$. Therefore $\langle x_1, x_3 \rangle \in Z_1$ and $\langle x_1, x_3 \rangle \in Z_2$ so we can conclude that $\langle x_1, x_3 \rangle \in Z_1 \cap Z_2$.
- $\supseteq$ Let $\langle x, y \rangle \in Z_1 \cap Z_2$. It holds that $\langle x, y \rangle \in Z_1$ and $\langle x, y \rangle \in Z_2$. Since $Z_1, Z_2 \subseteq \Delta_X$, we get that $x = y$. Therefore, $\langle x, y \rangle \in Z_1 \circ Z_2$.

□

COROLLARY 7.22. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a simple F-coalgebra. Then the finite intersection of bisimulations on X is again a bisimulation on X.*

COROLLARY 7.23. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a simple F-coalgebra. Then on $\pi(X, \alpha)$ we get that $\circ$ is commutative.*

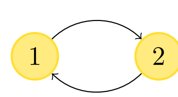
COROLLARY 7.24. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a simple F-coalgebra. It holds:*

$$\pi(X, \alpha) \cong (B(X), \cup, \cap, \emptyset, \Delta_X)$$

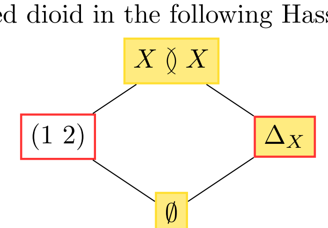
DEFINITION 7.25. **Bisimulation Permutability**
Let F be an endofunctor on **Set** that preserves weak pullbacks and let (X, α) be a F-coalgebra. We say that (X, α) has *bisimulation permutability* if for each $Z_1, Z_2 \in B(X)$ it holds:

$$Z_1 \circ Z_2 = Z_2 \circ Z_1$$

REMARK 7.26. Each simple F-coalgebra has bisimulation permutability by Proposition 7.21, but the converse does not hold. Take as counterexample the following Directed Graph (X, α):



We depict its associated dioid in the following Hasse Diagram:



As always, we present the idempotent bisimulations in yellow and we border in red the bisimulations of the form $G(f)$ for some endomorphism $f : X \to X$. The notation (1 2) is standard for presenting permutations. It is straightforward to see that each pair of bisimulations permute and (X, α) is not simple since 1 is bismilar to 2.

We introduce here the main results on bisimulation permutability:

PROPOSITION 7.27. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α) be a F-coalgebra with bisimulation permutability. Let $f, g : X \to X$ be F-coalgebra endomorphisms then $fg = gf$.*

PROOF. By Theorem 3.5 we get that the graphs of f and g are bisimulations on X. Applying bisimulation permutability, we get that:

$$G(fg) = G(g) \circ G(f) = G(f) \circ G(g) = G(gf)$$

Since the graph of fg equals the graph of gf we obtain the desired result. □

PROPOSITION 7.28. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α) be a F-coalgebra with bisimulation permutability. Then it holds that each endomorphism on X is an automorphism.*

PROOF. Let $f : X \to X$ be an arbitrary F-coalgebra endomorphism. By Theorem 3.5 we get that $G(f)$ is a bisimulation on X and by Theorem 3.7 we also get that $G(f)^{-1}$ is a bisimulation on X. Applying bisimulation permutability, we get that:

$$Kerf = G(f) \circ G(f)^{-1} = G(f)^{-1} \circ G(f) = \Delta_{f(X)}$$

Moreover we get that

$$\Delta_X \subseteq Kerf = \Delta_{f(X)} \subseteq \Delta_X$$

Therefore:

1. $Kerf = \Delta_X$, thus f is injective.
2. $\Delta_{f(X)} = \Delta_X$, thus $f(X) = X$ and f is surjective.

□

The preceding result can also be derived by the following results:

PROPOSITION 7.29. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and let $f, g : Y \to X$ be two F-coalgebra homomorphisms, then:*

$$f(Kerg) = g(Kerf)$$

PROOF. The following sets are bisimulations on X:

$$G(f)^{-1} \circ G(g) \in B(X) \quad ---- \quad G(g)^{-1} \circ G(f) \in B(X)$$

Applying bisimulation permutability we get that:

$$\begin{aligned}
[G(f)^{-1} \circ G(g)] \circ [G(g)^{-1} \circ G(f)] &= [G(g)^{-1} \circ G(f)] \circ [G(f)^{-1} \circ G(g)] \\
G(f)^{-1} \circ [G(g) \circ G(g)^{-1}] \circ G(f) &= G(g)^{-1} \circ [G(f) \circ G(f)^{-1}] \circ G(g) \\
G(f)^{-1} \circ [Kerg] \circ G(f) &= G(g)^{-1} \circ [Kerf] \circ G(g) \\
f(Kerg) &= g(Kerf)
\end{aligned}$$

□

COROLLARY 7.30. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X,α), (Y,β) be two F-coalgebras. Assume that (X,α) has bisimulation permutability and let $f,g : Y \to X$ be two F-coalgebra homomorphisms, then:*

$$f \text{ is injective } \Leftrightarrow \quad g \text{ is injective}$$

PROOF. By previous Proposition, $f(Kerg) = g(kerf)$. Assume that f is injective let us check that so is g. Let $y_1, y_2 \in Y$ such that $g(y_1) = g(y_2)$, then $\langle y_1, y_2\rangle \in Kerg$. Therefore, $\langle f(y_1), f(y_2)\rangle \in f(Kerg) = g(Kerf) = g(\Delta_Y) = \Delta_{g(Y)}$. Hence $f(y_1) = f(y_2)$ because the pair belong to a diagonal. Since f is injective we get that $y_1 = y_2$. Analogous for the other implication. □

PROPOSITION 7.31. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X,α), (Y,β) be two F-coalgebras. Assume that (X,α) has bisimulation permutability and let $f,g : X \to Y$ be two F-coalgebra homomorphisms, then:*

$$f^{-1}(\Delta_{g(X)}) = g^{-1}(\Delta_{f(X)})$$

REMARK 7.32. In order to obtain a better understanding on the preceding sets, let $\langle x_1, x_2\rangle \in f^{-1}(\Delta_{g(X)})$, then $\langle f(x_1), f(x_2)\rangle \in \Delta_{g(X)}$. Notice that $\Delta_{g(X)} \subseteq \Delta_Y$, thus $f(x_1) = f(x_2)$, therefore

$$f^{-1}(\Delta_{g(X)}) \subseteq Kerf$$

In the conditions of the preceding proposition it holds:

$$f^{-1}(\Delta_{g(X)}) = g^{-1}(\Delta_{f(X)}) \subseteq Kerf \cap Kerg$$

COROLLARY 7.33. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X,α), (Y,β) be two F-coalgebras. Assume that (X,α) has bisimulation permutability and let $f,g : X \to Y$ be two F-coalgebra homomorphisms, then:*

$$f \text{ is surjective } \Leftrightarrow \quad g \text{ is surjective}$$

PROOF. By previous Proposition, $f^{-1}(\Delta_{g(X)}) = g^{-1}(\Delta_{fX})$. Assume that f is surjective, then we get that $\Delta_{f(X)} = \Delta_Y$ and thus we obtain:

$$f^{-1}(\Delta_{g(X)}) = g^{-1}(\Delta_{f(X)}) = g^{-1}(\Delta_Y) = Kerg$$

Let $y \in Y$ be an arbitary element of Y. Since f is surjective, there exists some $x \in X$ such that $f(x) = y$. Notice that $\langle x, x\rangle \in \Delta_X \subseteq Kerg$. Therefore $\langle x, x\rangle \in f^{-1}(\Delta_{g(X)})$. This means that $\langle f(x), f(x)\rangle = \langle y, y\rangle \in \Delta_{g(X)}$. Since y was arbitary, we get that $\Delta_Y \subseteq \Delta_{g(X)} \subseteq \Delta_Y$. Then $g(X) = Y$, which means that g is surjective. Analogous for the other implication. □

PROPOSITION 7.34. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X,α) be a F-coalgebra with bisimulation permutability. Let Z_1 and Z_2 be two bisimulation equivalences on X, then $Z_1 \circ Z_2$ is also a bisimulation equivalence.*

PROOF. Since F preserves weak pullbacks, we know by Theorem 3.13 that $Z_1 \circ Z_2$ is a bisimulation. We need to show that is also an equivalence relation.

- $\Delta_X \subseteq Z_1 \circ Z_2$
  We will use Proposition 4.11. Notice that $\Delta_X \subseteq Z_2$, then $Z_1 \circ \Delta_X \subseteq Z_1 \circ Z_2$. Notice that $\Delta_X \subseteq Z_1 = Z_1 \circ \Delta_X \subseteq Z_1 \circ Z_2$.

- $(Z_1 \circ Z_2)^{-1} \subseteq Z_1 \circ Z_2$

  We develop the first term and we apply that Z_1 and Z_2 are equivalence relation and also that X has bisimulation permutability:

$$(Z_1 \circ Z_2)^{-1} = Z_2^{-1} \circ Z_1^{-1} \subseteq Z_2 \circ Z_1 = Z_1 \circ Z_2$$

- $(Z_1 \circ Z_2)^2 \subseteq Z_1 \circ Z_2$

  As before:

$$\begin{aligned}(Z_1 \circ Z_2) \circ (Z_1 \circ Z_2) &= (Z_1 \circ Z_2) \circ (Z_2 \circ Z_1) \\ &= Z_1 \circ (Z_2 \circ Z_2) \circ Z_1 \\ &= Z_1 \circ (Z_2 \circ Z_1) \\ &= Z_1 \circ (Z_1 \circ Z_2) \\ &= (Z_1 \circ Z_1) \circ Z_2 \\ &= Z_1 \circ Z_2\end{aligned}$$

Then $Z_1 \circ Z_2$ is a bisimulation equivalence.

□

PROPOSITION 7.35. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α) be a F-coalgebra with bisimulation permutability. Let $x, y \in X$ be two bisimilar states, then $\langle x \rangle = \langle y \rangle$.*

PROOF. Assume that x and y are bisimilar, hence we can find some $Z \in B(X)$ such that $\langle x, y \rangle \in Z$. Notice that for each $W \leq X$ subcoalgebra of X it holds:

$$\Delta_W \circ Z = \{\langle x, y \rangle \in Z : \; x \in W\}$$

$$Z \circ \Delta_W = \{\langle x, y \rangle \in Z : \; y \in W\}$$

In particular $\langle x \rangle$ and $\langle y \rangle$ are subcoalgebras of X and we have that $\langle x, y \rangle \in \Delta_{\langle x \rangle} \circ Z$ and $\langle x, y \rangle \in Z \circ \Delta_{\langle y \rangle}$. Applying bisimulation permutability on X we have:

$$\langle x, y \rangle \in \Delta_{\langle x \rangle} \circ Z = Z \circ \Delta_{\langle x \rangle} \;\Rightarrow\; y \in \langle x \rangle$$

$$\langle x, y \rangle \in Z \circ \Delta_{\langle y \rangle} = \Delta_{\langle y \rangle} \circ Z \;\Rightarrow\; x \in \langle y \rangle$$

Thus, $\langle x \rangle \subseteq \langle y \rangle$ and $\langle y \rangle \subseteq \langle x \rangle$. □

COROLLARY 7.36. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α) be a F-coalgebra with bisimulation permutability. Let $x, y \in X$ be two bisimilar states and $W \leq X$ be any subcoalgebra of X. It holds:*

$$x \in W \quad \Leftrightarrow \quad y \in W$$

PROOF. We will just check one implication, the other is analogous.

≫ Assume $x \in W$, then $\langle x \rangle \subseteq W$. Since x and y are bisimilar and applying previous Proposition we get $y \in \langle y \rangle = \langle x \rangle \subseteq W$.

□

THEOREM 7.37. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and let $f, g : Y \to X$ be two F-coalgebra homomorphisms. Let $W \subseteq Y$ be any subcoalgebra of Y, then it holds $f(W) = g(W)$.*

Proof. Since W is a subcoalgebra of Y, the inclusion $i : W \to Y$ is an F-coalgebra homomorphism. Moreover the compositions $fi, gi : W \to X$ are also F-coalgebra homomorphisms. Applying Theorem 3.8 $\langle fi, gi\rangle(W)$ is a bisimulation on X, therefore for any $w \in W$, $fi(w) = f(w)$ is bisimilar to $gi(w) = g(w)$. Moreover $f(W)$ and $g(W)$ are subcoalgebras of X by Proposition 5.9. Then by the preceding Corollary 7.36 we get that:

$$f(w) \in f(W) \quad \Leftrightarrow \quad g(w) \in f(W)$$
$$f(w) \in g(W) \quad \Leftrightarrow \quad g(w) \in g(W)$$

Therefore, $f(W) = g(W)$. □

Corollary 7.38. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and let $f, g : Y \to X$ be two F-coalgebra homomorphisms. It holds*

$$f(Y) = g(Y).$$

Corollary 7.39. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and let $f, g : Y \to X$ be two F-coalgebra homomorphisms. It holds*

$$Y/Kerf \cong Y/Kerg.$$

Corollary 7.40. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α) be a F-coalgebra with bisimulation permutability. Let $Z \in B(X)$ be any bisimulation on X. It holds*

$$\pi_1(Z) = \pi_2(Z).$$

Theorem 7.41. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and (Y, β) is simple. Let $f : Y \to X$ be a F-coalgebra homomorphism. Let $y \in Y$ and $x \in X$, it holds:*

$$f(y) \textit{ is bisimilar to } x \quad \Leftrightarrow \quad x = f(y)$$

Proof. We will check the two implications:

- $\ll$ Trivial.
- $\gg$ Since $f(y)$ is bisimilar to x and $f(Y)$ is a subcoalgebra of X, we get by Corollary 7.36 that $x \in f(Y)$. Therefore there must exist some $y' \in Y$ such that $f(y') = x$. But then y' is bisimilar to x and since being bisimilar is transitive, we get that y' is bisimilar to y. Since (Y, β) is simple we get that $y' = y$, thus $x = f(y)$.

□

Corollary 7.42. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks and let (X, α), (Y, β) be two F-coalgebras. Assume that (X, α) has bisimulation permutability and (Y, β) is simple. Let $f, g : Y \to X$ be two F-coalgebra homomorphisms. Then $f = g$.*

Proof. For each $y \in Y$, it holds that $f(y)$ and $g(y)$ are bisimilar, thus $f(y) = g(y)$ by the preceding Theorem. □

CHAPTER 8

Final Coalgebras

1. Final Coalgebras

DEFINITION 8.1. **Final Coalgebras**

Let F be an endofunctor on **Set**. We say that a F-coalgebra, (X, α), is *final* if for any other F-coalgebra (Y, β) there exists a unique F-coalgebra homomorphism $f_Y : Y \to X$.

THEOREM 8.2. *Let F be an endofunctor on* **Set**. *Let (X, α) be a final F-coalgebra, then α is a F-coalgebra isomorphism.*

PROOF. This result arises from the fact that $(FX, F\alpha)$ is also a F-coalgebra and $\alpha : X \to FX$ is a F-coalgebra homomorphism. In fact we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\quad\alpha\quad} & FX & \xrightarrow{\quad f_{FX}\quad} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle\alpha} \\ FX & \xrightarrow[\quad F\alpha\quad]{} & F(FX) & \xrightarrow[\quad Ff_{FX}\quad]{} & FX \end{array}$$

The right hand side of the diagram comes from the fact that (X, α) is final. By composition of F-coalgebra homomorphisms, we get that $f_{FX}\alpha$ is a F-coalgebra homomorphism from X to X. Notice that since (X, α) is final, this composition must be equal to the identity of X, $f_{FX}\alpha = Id_X$. By Proposition 1.7, if we reverse the mappings we obtain again F-coalgebra homomorphisms and we get that $\alpha f_{FX} = Id_{FX}$. Thus, α is a bijection with inverse f_{FX} □

Last theorem tell us that X is a fixed point for the endofunctor F.

THEOREM 8.3. *Final coalgebras, if they exist, are uniquely determined up to isomorphism.*

THEOREM 8.4. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a final F-coalgebra, then (X, α) is simple.*

PROOF. Let Z be a bisimulation on X. By definition, the projections $\pi_1, \pi_2 : Z \to X$ are F-coalgebra homomorphisms. By finality of X, we get that $\pi_1 = \pi_2$. That is to say that $Z \subseteq \Delta_X$ for an arbitrary bisimulation $Z \in B(X)$. By a characterisation on Theorem 7.2 we get that (X, α) is simple. □

A final coalgebra can be considered as a universal domain of canonical representatives for bisimulation equivalence classes in the following way.

PROPOSITION 8.5. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Assume that F has a final coalgebra (X, α). Let (Y, β) be an arbitrary F-coalgebra. For each $y_1, y_2 \in Y$ holds:*

$$\langle y_1, y_2 \rangle \in Y \between Y \Leftrightarrow f_Y(y_1) = f_Y(y_2)$$

PROOF. We must check the two implications:

$\gg$ By assumption, $\langle y_1, y_2 \rangle \in Y \between Y$. Notice that for $i = 1, 2$, the mappings $f_Y \pi_i : Y \between Y \to X$ are F-coalgebra homomorphisms by composition. Therefore, since X is final, $f_Y \pi_1 = f_Y \pi_2$. In particular, $f_Y(y_1) = f_Y(y_2)$.

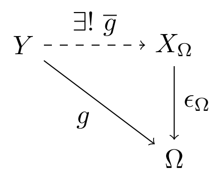
$\ll$ Notice that $f_Y : Y \to X$ is a F-coalgebra homomorphism with $f_Y(y_1) = f_Y(y_2)$. By Corollary 3.16, $Ker f_Y$ is a bisimulation on Y and $\langle y_1, y_2 \rangle \in Ker f_Y$. Since $Y \between Y$ is the greatest bisimulation on Y we get that $\langle y_1, y_2 \rangle \in Y \between Y$.

□

2. Cofree Coalgebras

DEFINITION 8.6. **Cofree Coalgebras**

Let F be an arbitrary endofunctor over **Set**. Let Ω be any set. A F-coalgebra (X_Ω, α) with a map $\epsilon_\Omega : X_\Omega \to \Omega$ is called *cofree over* Ω if for any other F-coalgebra (Y, β) and any map $g : Y \to \Omega$ there exists exactly one F-coalgebra homomorphism $\overline{g} : Y \to X$ with $\epsilon_\Omega\ \overline{g} = g$. That is to say that the following diagram commutes:



The set Ω is often thought of as a set of "colours" and g as a colouring.

REMARK 8.7. For $\Omega = 1$, (X_1, α) is the final F-coalgebra.

LEMMA 8.8. [GS01b]

Let F be any endofunctor over **Set**. *Let Ω be any set. If (X_Ω, α) is a cofree coalgebra over Ω, then for each $\Theta \subseteq \Omega$ there exists a cofree coalgebra (W_Θ, β) over Θ. Moreover, $W_\Theta \leq X_\Omega$.*

COROLLARY 8.9. *If there is no final F-coalgebra, then there can be no cofree coalgebra over Ω, unless $\Omega = \emptyset$.*

PROPOSITION 8.10. *Let F be any endofunctor over* **Set**. *Let $(\Theta_i)_{i \in I}$ be a family of sets, let Ω be a set larger than their cartesian product. If (X_Ω, α) exists, then:*

$$\prod_{i \in I} W_{\Theta_i} \cong W_{\prod_{i \in I} \Theta_i}$$

3. An Application of Coinduction

The existence of a final coalgebra for an specific endofunctor allow us to define concepts and mappings in an analogous way of what we do for algebras. In the following chapter we will see how we can use it for the endofunctor $A \times -$ on the category **Set**.

Let A be an arbitrary set, we define the endofunctor $G = A \times -$ as:

$$\begin{array}{rccl} A\times - : & \mathbf{Set} & \longrightarrow & \mathbf{Set} \\ & X & \longmapsto & A\times X \\ & f & \longmapsto & id_A \times f \end{array}$$

Let (X, α) be an arbitrary G-coalgebra. Therefore X is an arbitrary set and α is its structure map:

$$\begin{array}{rccl} \alpha : & X & \longrightarrow & A\times X \\ & x & \longmapsto & \alpha(x) \end{array}$$

The structure map can be splitted in two functions $X \to A$ and $X \to X$ which we will call $value : X \to A$ and $next : X \to X$. With these operations we can do two things, given an element of $x \in X$:

1) Produce an element in A, namely $value(x)$.
2) Produce a next element in X, namely $next(x)$.

Now we can repeat 1) and 2) and therefore form another element in A, namely $value(next(x))$. By preceding in this way we can get for each element $x \in X$ an infinite sequence $(a_0, a_1, a_2, \cdots) \in A^{\mathbb{N}}$ of elements $a_n = value(next^{(n)}(x)) \in A$, where $next^{(0)}(x)$ denotes x. This sequence of elements that x gives rise to is what we can observe about x.

In order to better understand this coalgebra, let us characterise the G-coalgebra homomorphisms:

Proposition 8.11. *Let (X, α) and (Y, β) be two G-coalgebras. Let $f : X \to Y$ be an arbitrary mapping. It holds that f is a G-coalgebra homomorphism if and only if for each $x \in X$ holds:*

$$value(x) = value(f(x))$$

$$f(next(x)) = next(f(x))$$

It will be important also to characterise the bisimilarity upon states:

Proposition 8.12. *Let (X, α) and (Y, β) be two G-coalgebras. Consider some $x \in X$ and $y \in Y$, it holds that x and y are bisimilar if and only if:*

$$value(x) = value(y)$$

$$next(x) \text{ and } next(y) \text{ are bisimilar}$$

Furthermore, we get an important proposition concerning bisimilar states:

Proposition 8.13. *Let (X, α) and (Y, β) be two G-coalgebras. Let $x \in X$ and $y \in Y$ be two bisimilar states. It holds for each $n \in \mathbb{N}$:*

$$value(next^{(n)}(x)) = value(next^{(n)}(y))$$

The most important property of this coalgebra is the existence of a final coalgebra. Let us check how it works.

Consider as base the set $A^{\mathbb{N}}$, i.e., the set of infinite sequences upon A. Each element of $A^{\mathbb{N}}$ can be written as $(a_i)_{i\in\mathbb{N}}$. We define on this set a coalgebra structure by taking the structure map:

$$\begin{array}{rccl} \alpha : & A^{\mathbb{N}} & \longrightarrow & A\times A^{\mathbb{N}} \\ & (a_i)_{i\in\mathbb{N}} & \longmapsto & \langle a_0, (a_{i+1})_{i\in\mathbb{N}}\rangle \end{array}$$

For this specific coalgebra we will write $\alpha = \langle head, tail\rangle$, instead of the generic notation $\alpha = \langle value, next\rangle$.

Proposition 8.14. *The G-coalgebra $(A^{\mathbb{N}}, \alpha)$ is simple.*

Proof. Let $a = (a_i)_{i\in\mathbb{N}}$, $b = (b_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$ be two states. Assume they are bisimilar. Then by Proposition 8.13 it holds that for each $n \in \mathbb{N}$:

$$value(next^{(n)}(a)) = value(next^{(n)}(b))$$

Notice that $head(tail^{(n)}(a)) = a_n = b_n = head(tail^{(n)}(b))$. Therefore $a = b$.

This means that each bisimulation Z on $A^{\mathbb{N}}$ is contained in the diagonal $\Delta_{A^{\mathbb{N}}}$. Therefore, applying Theorem 7.2 we get that $(A^{\mathbb{N}}, \alpha)$ is a simple F-coalgebra. □

Proposition 8.15. *The G-coalgebra $(A^{\mathbb{N}}, \alpha)$ is final.*

Proof. Take any G-coalgebra, (Y, β). We must check that there exists a unique G-coalgebra homomorphism from Y to $A^{\mathbb{N}}$. Let us define the followin mapping:

$$\begin{array}{rccl} f_Y : & Y & \longrightarrow & A^{\mathbb{N}} \\ & y & \longmapsto & (value(next^{(i)}(y)))_{i\in\mathbb{N}} \end{array}$$

Let us check that it is a G-coalgebra homomorphism. Notice that it holds:

$$head(f_Y(y)) = value(next^{(0)}(y)) = value(y)$$

$$f_Y(next(y)) = (value(next^{(i)}(next(y))))_{i\in\mathbb{N}} =$$
$$= (value(next^{(i+1)}(y)))_{i\in\mathbb{N}} = tail(f_Y(y))$$

Therefore, applying Proposition 8.11 we get that f_Y is a G-coalgebra homomorphism. In order to see that f_Y is unique. Assume there is another G-coalgebra homomorphism $g_Y : Y \to A^{\mathbb{N}}$. We have seen on Proposition 8.14 that $(A^{\mathbb{N}}, \alpha)$ is simple, also by Theorem 7.2 we get that $f_Y = g_Y$.

Thus, $(A^{\mathbb{N}}, \alpha)$ is final. □

Once we have presented the final coalgebra for the endofunctor G we will present some cases where this finality is used.

Example 8.16. **Definitions on $A^{\mathbb{N}}$ via Coinduction**

Let us start with a simple example, which involves defining the constant sequence $const(a) = (a, a, a, \cdots) \in A^{\mathbb{N}}$ by coinduction for some $a \in A$. We shall define this constant as a function $const(a) : 1 \to A^{\mathbb{N}}$, where $1 = \{\star\}$ is a singleton set. We must produce a coalgebra structure $\alpha : 1 \to G(1) = A \times 1$, in such a way that $const(a)$ arises by repetition. In this case the only thing we want to observe is the element $a \in A$ itself, and so we simply define as coalgebra structure:

$$\begin{array}{rccl} \alpha : & 1 & \longrightarrow & A \times 1 \\ & \star & \longmapsto & \langle a, \star \rangle \end{array}$$

Indeed, $const(a)$ arises in the following finality diagram:

$$\begin{array}{ccc} 1 & \xrightarrow{\quad const(a) \quad} & A^{\mathbb{N}} \\ {\scriptstyle \alpha}\downarrow & & \downarrow \cong\ {\scriptstyle \langle head, tail \rangle} \\ A \times 1 & \xrightarrow[\quad id_A \times const(a) \quad]{} & A \times A^{\mathbb{N}} \end{array}$$

It expresses that $head(const(a)) = a$ and $tail(const(a)) = const(a)$.

Next example will give us a mapping $even : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that for each sequence $a = (a_i)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ it returns the even elements. That is to say:

$$even(a) = (a_{2i})_{i \in \mathbb{N}}$$

As before, we must define a coalgebra structure on $A^{\mathbb{N}}$ in such a way that *even* arises by finality. In this case, we must focus our attention only on the even elements of a given sequence and disard the others. Let us consider the following coalgebra structure:

$$\begin{array}{rccl} \beta : & A^{\mathbb{N}} & \longrightarrow & A \times A^{\mathbb{N}} \\ & a & \longmapsto & \langle head(a), tail^{(2)}(a) \rangle \end{array}$$

As before, *even* arises in the following diagram:

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{\quad even \quad} & A^{\mathbb{N}} \\ \beta \downarrow & & \cong \downarrow \langle head, tail \rangle \\ A \times A^{\mathbb{N}} & \xrightarrow[\quad id_A \times even \quad]{} & A \times A^{\mathbb{N}} \end{array}$$

It means that *even* is the unique mapping for which it holds that:

$$head(even(a)) = head(a)$$

$$tail(even(a)) = even(tail^{(2)}(a))$$

For the next example, we want to define a similar mapping, $odd : A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that for each sequence $a = (a_i)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ we get only the odd elements, i.e.:

$$odd(a) = (a_{2i+1})_{i \in \mathbb{N}}$$

For this case we can proceed as before by setting a coalgebra structure on $A^{\mathbb{N}}$ and obtain *odd* by finality. Otherwise one can just define odd as the composition $even \cdot tail$.

Next example will show us how to define a more complex operation on infinite sequences. We will define a function $merge : A^{\mathbb{N}} \times A^{\mathbb{N}} \to A^{\mathbb{N}}$ such that for any two given sequences $a = (a_i)_{i \in \mathbb{N}}$ and $b = (b_i)_{i \in \mathbb{N}}$ of $A^{\mathbb{N}}$, it returns a sequence $c = (c_i)_{i \in \mathbb{N}}$ such that:

$$c_i = a_{i/2} \text{ if } i \in 2\mathbb{N} \text{ and}$$

$$c_i = b_{(i-1)/2} \text{ if } i \in 2\mathbb{N} + 1$$

It means that $merge(a, b)$ is the sequence that alternates a component of a and a component of b.

As before, *merge* will arise by finality of $A^{\mathbb{N}}$. For this case, we must set a coalgebra structure on $A^{\mathbb{N}} \times A^{\mathbb{N}}$ such that it alternates the observation of each sequence. We define:

$$\begin{array}{rccl} \delta : & A^{\mathbb{N}} \times A^{\mathbb{N}} & \longrightarrow & A \times (A^{\mathbb{N}} \times A^{\mathbb{N}}) \\ & \langle a, b \rangle & \longmapsto & \langle head(a), \langle b, tail(a) \rangle \rangle \end{array}$$

Notice that in the definition of δ we switch the order of the elements of the tuple. As we said, *merge* arises in the following diagram:

$$\begin{array}{ccc} A^{\mathbb{N}} \times A^{\mathbb{N}} & \xrightarrow{\quad merge \quad} & A^{\mathbb{N}} \\ \delta \downarrow & & \cong \downarrow \langle head, tail \rangle \\ A \times (A^{\mathbb{N}} \times A^{\mathbb{N}}) & \xrightarrow[id_A \times merge]{} & A \times A^{\mathbb{N}} \end{array}$$

It means that *merge* is the unique mapping for which it holds that:

$$head(merge(a, b)) = head(a)$$

$$tail(merge(a, b)) = merge(b, tail(a))$$

EXAMPLE 8.17. **A Proof on $A^{\mathbb{N}}$ via Coinduction**

Last definition of operations on infinite sequences will allow us to present a proof that uses coinduction. Using the definitions of *merge*, *even* and *odd*, we will prove that for any given sequence $a = (a_i)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ it holds that:

$$merge(even(a), odd(a)) = a$$

We will do this proof in some steps:

1. Consider $A^{\mathbb{N}}$ together with the G-coalgebra structure $\alpha = \langle head, tail \rangle$.
2. $f = merge \cdot \langle even, odd \rangle : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a G-coalgebra homomorphism.

Here we must check the statements of Proposition 8.11. Take any $a = (a_i)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$. It holds:

$$\begin{array}{rcl} head(f(a)) & = & head(merge(even(a)), odd(a))) \\ & = & head(even(a)) \\ & = & head(a) \\ \\ tail(f(a)) & = & tail(merge(even(a)), odd(a))) \\ & = & merge(odd(a), tail(even(a))) \\ & = & merge(even(tail(a)), odd(tail(a))) \\ & = & f(tail(a)) \end{array}$$

Therefore, applying Proposition 8.11, f is a G-coalgebra homomorphism.

3. $(A^{\mathbb{N}}, \alpha)$ is a final G-coalgebra. Since the identity mapping $id_{A^{\mathbb{N}}}$ is also a G-coalgebra homomorphism from $A^{\mathbb{N}}$ to $A^{\mathbb{N}}$ we get that $f = id_{A^{\mathbb{N}}}$.
4. Finally, we get the expected result. For each $a \in A^{\mathbb{N}}$ it holds

$$merge(even(a), odd(a)) = id_{A^{\mathbb{N}}}(a) = a$$

So far we have considered G-coalgebras parametrised by an arbitrary set A. In the following examples we will take special choices of the set A.

EXAMPLE 8.18. **Decimal Representation**

For this example we wil take $A = 10 = \{0, 1, \cdots, 9\}$ and we will consider as base the set $X = [0, 1]$. We define on X a G-coalgebra by taking as structure map:

$$\begin{array}{rccc} \tau : & [0,1] & \longrightarrow & A \times [0,1] \\ & x & \longmapsto & \langle d, 10x - d \rangle \end{array}$$

Where $d \in 10$ is the unique natural number such that $d \leq 10x < d + 1$. Notice that τ extracts from each number the next decimal number of its decimal representation.

For this case, the G-coalgebra $(10^{\mathbb{N}}, \langle head, tail \rangle)$ is final. Therefore, there exists a unique G-coalgebra homomorphism $f : [0, 1] \to 10^{\mathbb{N}}$ that makes the following diagram commute:

$$
\begin{array}{ccc}
[0,1] & \xrightarrow{\quad f \quad} & 10^{\mathbb{N}} \\
\tau \downarrow & & \cong \downarrow \langle head, tail \rangle \\
10 \times [0,1] & \xrightarrow[\quad id_{10} \times f \quad]{} & 10 \times 10^{\mathbb{N}}
\end{array}
$$

As we expected f returns the decimal representation of all the elements of $[0, 1]$. One important result that we obtain via coinduction is the proof of the unicity of the decimal representation. As example of application we get that:

$$f(1/2) = (5, 0, 0, \cdots) \qquad f(1/3) = (3, 3, 3, \cdots) \qquad f(1/\pi) = (3, 1, 8, \cdots)$$

CHAPTER 9

On Bisimilarity

We return at this point to the notion of bisimilarity. We have seen on Chapter 3 that the bisimilarity turns into an equivalence relation on Coalgebras for an endofunctor that preserves weak pullbacks. Our intention in this chapter is to give a further view on this topic.

1. Towards a general Theory

First of all, we introduce a general notion of bisimilarity on coalgebras.

DEFINITION 9.1. **Bisimilarity on Coalgebras**
Let F be an endofunctor on **Set** that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras. We say that X is bisimilar to Y if and only if:

1. $\forall x \in X\ \exists y \in Y$ such that x and y are bisimilar.
2. $\forall y \in Y\ \exists x \in X$ such that x and y are bisimilar.

In that case, we will write $X \sim Y$.

PROPOSITION 9.2. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Then the relation of bisimilarity on Coalgebras is an equivalence relation on* $\mathbf{CoAlg(F)}$.

PROOF. We must check it is an equivalence relation.

Refl. Let (X, α) be any coalgebra in $\mathbf{CoAlg(F)}$. Take any $x \in X$, we know by Corollary 3.6 that the diagonal of X is a bisimulation on X, thus $\langle x, x\rangle \in \Delta_X$ and x is bisimilar to itself.

Simm. Let (X, α) and (Y, β) be two F-coalgebras that are bisimilar. We must chek that $Y \sim X$.

Take any $y \in Y$, since $X \sim Y$ by definition of bisimilarity on coalgebras, there exists some $x \in X$ such that x and y are bisimilar. That means that there exists some bisimulation Z between X and Y such that $\langle x, y\rangle \in Z$. Notice that by Theorem 3.7 it holds that Z^{-1} is a bisimulation between Y and X, moreover, $\langle y, x\rangle \in Z^{-1}$ and we get that y and x are bisimilar.

For $x \in X$ we can get a similar proof.

Trans. Let (X_1, α_1), (X_2, α_2) and (X_3, α_3) be three F-coalgebras. Such that $X_1 \sim X_2$ and $X_2 \sim X_3$. We must check that $X_1 \sim X_3$.

Take any $x_1 \in X_1$, since $X_1 \sim X_2$, we know there exists some $x_2 \in X_2$ such that x_1 and x_2 are bisimilar. That means there exists some Z_1, bisimulation between X_1 and X_2, such that $\langle x_1, x_2\rangle \in Z_1$. We focus our attention now on $x_2 \in X_2$, since since $X_2 \sim X_3$, we know there exists some $x_3 \in X_3$ such that x_2 and x_3 are bisimilar. That means there exists some Z_2, bisimulation between X_2 and X_3, such that $\langle x_2, x_3\rangle \in Z_2$. Since

F preserves weak pullbacks, using theorem 3.13, we know that $Z_1 \circ Z_2$ is a bisimulation between X_1 and X_3. Notice that it holds that $\langle x_1, x_3 \rangle \in Z_1 \circ Z_3$, thus x_1 and x_3 are bisimilar.

For $x_3 \in X_3$ we can get a similar proof.

□

THEOREM 9.3. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras. Let $f : X \to Y$ be a surjective F-coalgebra homomorphism. Then X is bisimilar to Y.*

PROOF. Notice that f is a F-coalgebra homomorphism between X and Y, thus using Theorem 3.5 it holds that its graph $G(f)$ is a bisimulation between X and Y. Let us check that $X \sim Y$.

Take any $x \in X$ notice that $\langle x, f(x) \rangle \in G(f)$, thus x and $f(x)$ are bisimilar.

Take any $y \in Y$, since f is surjective, we know there exists some $x \in X$ such that $f(x) = y$, therefore $\langle x, y \rangle \in G(f)$, thus x and y are bisimilar.

Finally, $X \sim Y$. □

COROLLARY 9.4. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) be a F-coalgebra. It holds:*

$$X \sim X/(X \between X)$$

PROOF. By Proposition 3.11 we know that there exists a unique map structure $\gamma_{\between}$ that turns $(X/X \between X, \gamma_{\between})$ into a F-coalgebra. Moreover it turns the projection mapping $\pi_{\between} : X \to X/(X \between X)$ into a F-coalgebra homomorphism. Notice that $\pi_{\between}$ is surjective. Applying Theorem 9.3 it holds that $X \sim X/(X \between X)$. □

COROLLARY 9.5. *Let F be an endofunctor that preserves weak pullbacks, then any F-coalgebra is bisimilar to a simple F-coalgebra.*

2. Bisimilarity and Simple Coalgebras

LEMMA 9.6. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras, assume (X, α) is simple. Let $x, x' \in X$ and $y \in Y$. It holds:*

$$\left.\begin{array}{r} x \text{ and } y \text{ are bisimilar} \\ x' \text{ and } y \text{ are bisimilar} \end{array}\right| \Rightarrow \quad x = x'$$

PROOF. Since x and y are bisimilar there exists some bisimulation Z between X and Y such that $\langle x, y \rangle \in Z$. Since x' and y are bisimilar, there exists some bisimulation Z' between X and Y such that $\langle x', y \rangle \in Z'$. Notice that $\langle x', x \rangle \in Z' \circ Z^{-1}$. It holds that $Z' \circ Z^{-1}$ is a bisimulation on X, since X is simple, by Theorem 7.2 it holds that $\langle x', x \rangle \in \Delta_X$, therefore $x = x'$. □

THEOREM 9.7. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras, assume (X, α) is simple and $X \sim Y$. Then there exists an F-coalgebra epimorphism $f : Y \to X$.*

PROOF. From the fact that $X \sim Y$, take any $x \in X$, we know there exists some $y_x \in Y$ such that x and y_x are bisimilar. Assume the pair $\langle x, y_x \rangle \in Z_x$. On the other hand, take any $y \in Y$, we know there exists some $x_y \in X$ such that x_y and y are bisimilar. Assume the pair $\langle x_y, y\rangle \in Z_y$. Define

$$Z = \bigcup_{x\in X} Z_x \cup \bigcup_{y\in Y} Z_y$$

It holds that Z is a bisimulation between X and Y since so is each component, Theorem 3.9. Thus, each projection π_1 and π_2 becomes an F-coalgebra homomorphism.

Consider $\pi_2 : Z \to Y$, it holds that π_2 is an F-coalgebra epimorphism. For each $y \in Y$ we know that $\langle x_y, y\rangle \in Z_y \subseteq Z$ and $y = \pi_2(\langle x_y, y\rangle)$. Applying the 1st Isomorphim Theorem 6.1, we get that

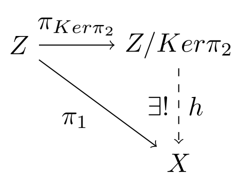
$$Z/Ker\pi_2 \cong \pi_2(Z) = Y$$

Assume that φ give us that F-coalgebra isomorphism.

Let us focus now on $Ker\pi_2$.

$$Ker\pi_2 = \{\langle\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\rangle \in Z^2 :\ \pi_2(\langle x_1, y_1\rangle) = \pi_2(\langle x_2, y_2\rangle)\} =$$
$$= \{\langle\langle x_1, y\rangle, \langle x_2, y\rangle\rangle \in Z^2 :\ x_1, x_2 \in X,\ y \in Y\}$$

Notice that each x_1 and y of that form are bisimilar since their pair belongs to Z which is a bisimulation between X and Y. It occurs the same for X_2 and y. Since X is simple, applying Lemma 9.6 we get that $x_1 = x_2$. That means that $Ker\pi_2 \subseteq Ker\pi_1$. Applying Theorem 6.2, it holds that there exists a unique F-coalgebra homomorphism h such that $\pi_1 = h\pi_{Ker\pi_2}$. That is to say that h makes the following diagram commute:



Notice that π_1 is a surjective mapping for the same reasoning used for π_2. Since π_1 and $\pi_{Ker\pi_2}$ are surjective mappings, so is h. Define $f = h\varphi^{-1}$, it is an F-coalgebra homomorphism from Y to X. Moreover it is surjetive since so is h. □

Corollary 9.8. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras, assume (X, α) is simple. Then the following statements are equivalent:*

- *There exists some F-coalgebra epimorphism, $f : Y \to X$.*
- *$X \sim Y$.*

Corollary 9.9. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two simple F-coalgebras with $X \sim Y$. Then $X \cong Y$.*

Proof. Applying the preceding Theorem, there exists some F-coalgebra epimorphism $f : Y \to X$. Since Y is simple, one characterisation of Theorem 7.2 states that f is a F-coalgebra isomorphism. □

Corollary 9.10. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two simple F-coalgebras. It holds:*

$$X \sim Y \Leftrightarrow X \cong Y$$

Theorem 9.11. *Let F be an endofunctor on* **Set** *that preserves weak pullbacks. Let (X, α) and (Y, β) be two F-coalgebras. It holds:*

$$X \sim Y \Leftrightarrow X/(X \between X) \cong Y/(Y \between Y)$$

Proof. We must check the two implications:

- ≫ Assume $X \sim Y$. We know by Corollary 9.4 we get that $X \sim X/(X \between X)$ and $Y \sim Y/(Y \between Y)$. Notice that $X/(X \between X)$ and $Y/(Y \between Y)$ are simple coalgebras by Proposition 7.3. By transitivity of $\sim$, it holds that $X/(X \between X) \sim Y/(Y \between Y)$. Finally Corollary 9.10 states that $X/(X \between X) \cong Y/(Y \between Y)$.
- ≪ Assume $X/(X \between X) \cong Y/(Y \between Y)$ where $X/(X \between X)$ and $Y/(Y \between Y)$ are simple coalgebras by Proposition 7.3. Applying Corollary 9.10 it holds that $X/(X \between X) \sim Y/(Y \between Y)$. Finally by Corollary 9.4 and transitivity of $\sim$ we get that $X \sim Y$.

□

Bibliography

[Hug01] J. Hughes, *A Study of Categories of Algebras and Coalgebras*, Ph.D. Thesis, Pittsburgh, 2001.

[Kur01] A. Kurz, *Coalgebras and Modal Logic*, Lecture Notes, Amsterdam, 2001.

[BS81] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag Graduate Texts in Mathematics, 1981.

[WR83] D.R. White and K.P. Reitz, *Graphs and Semigroup Homomorphism on Networks of Relations*, Social Networks, Volume 5, Issue 2, 1983.

[Rut00] J.J.M.M. Rutten, *Universal coalgebra: a theory of systems*, Elsevier, Theoretical Computer Science, Amsterdam, 2000.

[Ven05] Y. Venema, *Algebras and Coalgebras*, Institute for Logic, Language and Computation, Univ. of Amsterdam, 2005.

[Mad06] R. Maddux, *Relational Algebras*, Elsevier, Studies in Logic and Foundations of Mathematics, 2006.

[KS04] F. Kamüller, J.W. Sanders, *Idempotent Relation in Isabelle/HOL*, ISSN 1436-9915 Bericht-Nr. 2004-04.

[RS09] J. Rhodes, B. Steinberg, *The q-theory of Finite Semigroups*, Springer, Springer Monographics in Mathematics, 2009.

[CGQ03] G. Cohen, S. Gaubert, J.P. Quadrat, *Duality and separation theorems in idempotent semimodules*, Elsevier, Linear Algebra and its Applications, 2003.

[GPR98] P. García, T. Pérez, J. Ruiz, E. Segarra, J.M. Sempere, M. Vázquez de Parga, *Apunts sobre la teoria d'autómats i llenguatges formals*, Monografies de la Universitat Politècnia de València, 1998.

[Jac05] B. Jacobs, *Introduction to Coalgebra. Towards Mathematics of States and Observations*, Draft Copy, Version 1.00, 22nd April 2005.

[SR11] D. Sangiorgi, J. Rutten, *Advanced Topics in Bisimulation and Coinduction*, Cambridge Tracts in Theoretical Computer Science, 52.

[VOO02] J. van Oosten *Basic Category Theory*, Department of Mathematics, Utrecht University, July 2002.

[Gol99] J.S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, 1999.

[GS01b] H.P. Gumm, T. Schröder, *Product of Coalgebras*, Algebra Universalis, 2001.

[Cli10] J. Climent Vidal, *Teoría de Conjuntos*, Lecture Notes, Universitat de València, 2010.