From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms

First-order rewriting systems

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Abstract

We define the set of paths associated with a rewriting system and equip it with a structure of partial algebra, a structure of category, and a structure of Artinian ordered set. Next, we consider an extension of the signature associated with the rewriting system and we associate each path with a term in the extended signature. This constitutes a Curry-Howard type mapping. After that we prove that the quotient of the set of paths by the kernel of the Curry-Howard mapping is equipped with a structure of partial algebra, a structure of category, and a structure of Artinian ordered set. Following this we identify a subquotient of the free term algebra in the extended signature that is isomorphic to the algebraic, categorical, and ordered structures on the quotient of paths. This constitutes a Curry-Howard type isomorphism. Additionally, we prove that these two structures are isomorphic to the free partial algebra on paths in a variety of partial algebras for the extended signature.

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1 Introduction

The theory of single-sorted term rewriting systems (which for words has its origins in the works of Thue [23], Dehn [12] and Post [19, 20]), changing one term to another according to certain rewrite rules or productions, is a fundamental field within computer science. Briefly stated, it could be said that rewriting is the root of all computational processes.

On the other hand, the classical Curry-Howard correspondence explains the direct relationship between computer programs and mathematical proofs. More precisely, Curry, in [10], was the first to acknowledge the formal analogy between his combinatory logic and the axioms of a Hilbert–type deduction system for the positive implicational propositional logic. Later on, Howard, in 1969, but published in [17], observed the same formal analogy between Church's λ-calculus and the proof rules of a Gentzen's system of natural deduction for the intuitionistic propositional logic. The Curry-Howard correspondence assigns to each proof in the intuitionistic logic a term in Curry's combinatory logic or in Church's λ-calculus. In other words, the Curry-Howard correspondence consists of the observation that two seemingly unrelated families of formalisms—namely, systems of formal deduction, on the one hand, and models of computation, on the other—are, essentially, the same kind of mathematical object.

What we present here is the first part of the ongoing project presented in [9]. This

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work is the preliminary development of a theory aimed at defining the notions of higher-order many-sorted rewriting systems and higher-order many-sorted categorial algebras and investigating the relationship between them through higher-order many-sorted Curry-Howard isomorphisms. Our frequent use of the qualifier Curry-Howard in this paper is due to the fact that we have been able to represent paths, that is, syntactic derivations between terms for a rewriting system (which have a proof-theoretical flavor) as terms of an algebra relative to a signature associated with the rewriting system. The interested reader can consult all the proofs appearing in this article in [9]. Next to each result, the reader will find the corresponding reference. Recall that our work is framed in the study of syntactic derivation systems in the context of many-sorted algebras. Nevertheless, to facilitate comprehension, in this paper we have opted to present the single-sorted version of our findings. The only prerequisites for reading this work are familiarity with category theory [16, 18], universal algebra [1, 4, 5, 6, 7, 14, 15, 15, 21, 22, 24], the theory of ordered sets [2, 11] and set theory [3, 13]. Nevertheless, regarding set theory, we have adopted the following conventions. An ordinal α is a transitive set that is well-ordered by $\in$, thus $\alpha = \{\beta \mid \beta \in \alpha\}$. The first transfinite ordinal ω_0 will be denoted by $\mathbb{N}$, which is the set of all natural numbers, and, from what we have just said about the ordinals, for every $n \in \mathbb{N}$, $n = \{0, \ldots, n-1\}$.

2 Preliminaries

In this paper, we will use a slight generalization of the notion of algebra.

▶ **Definition 1.** *For $n \in \mathbb{N}$, the category of n-categories will be denoted by* nCat*. Given two n-categories A and B, we will call the morphisms in nCat from A to B* n-functors*. We will denote by $\mathrm{nFunc}(\mathsf{A},\mathsf{B})$ the set of all n-functors from A to B. The set of the finitary operations on an n-category A is $(\mathrm{nFunc}(\mathsf{A}^k,\mathsf{A}))_{k\in\mathbb{N}}$, where, for every $k \in \mathbb{N}$, $\mathsf{A}^k = \prod_{j\in k}\mathsf{A}$ (if $k = 0$, then A^0 is a final n-category).*

Let Σ be a signature. A structure of n-categorial Σ-algebra on *an n-category A is a family $F = (F_k)_{k\in\mathbb{N}}$, where, for $k \in \mathbb{N}$, F_k is a mapping from Σ_k to $\mathrm{nFunc}(\mathsf{A}^k,\mathsf{A})$ (if $k = 0$ and $\sigma \in \Sigma_0$, then $F_0(\sigma)$, picks out an object of A and its identity morphism). An* n-categorial Σ-algebra *is a pair (A, F), abbreviated to $\mathbf{A}$, where A is an n-category and F a structure of n-categorial Σ-algebra on A. For a pair $k \in \mathbb{N}$ and a formal operation $\sigma \in \Sigma_k$, in order to simplify the notation, the n-functor $F_n(\sigma)$ from A^n to A will be written simply as $\sigma^{\mathbf{A}}$.*

An n-categorial Σ-homomorphism *from $\mathbf{A}$ to $\mathbf{B}$, where $\mathbf{B} = (\mathsf{B}, G)$, is a triple $(\mathbf{A}, F, \mathbf{B})$, abbreviated to $F\colon \mathbf{A} \longrightarrow \mathbf{B}$, where F is an n-functor from A to B such that, for every $k \in \mathbb{N}$, every $\sigma \in \Sigma_k$ and every family $(\mathsf{a}_j)_{j\in k} \in \mathsf{A}^k$, we have that $F(\sigma^{\mathbf{A}}((\mathsf{a}_j)_{j\in k})) = \sigma^{\mathbf{B}}((F(\mathsf{a}_j))_{j\in k})$. We will denote the category of n-categorial Σ-algebras and n-categorial Σ-homomorphisms by* $\mathsf{nCatAlg}(\Sigma)$.

2.1 Translations

We next introduce, for a Σ-algebra, the concepts of elementary translation and of translation with respect to it.

▶ **Definition 2.** *Let $\mathbf{A}$ be a Σ-algebra. We denote by $\mathrm{Etl}(\mathbf{A})$ the subset of $\mathrm{Hom}(A, A)$ defined as follows: for every mapping $T \in \mathrm{Hom}(A, A)$, $T \in \mathrm{Etl}(\mathbf{A})$ if and only if there is a natural number $n \in \mathbb{N} - 1$, an index $k \in n$, an n-ary operation symbol $\sigma \in \Sigma_n$, a family $(a_j)_{j\in k} \in A^k$, and a family $(a_l)_{l\in n-(k+1)} \in A^{n-(k+1)}$ (recall that $k + 1 = \{0, 1, \ldots, k\}$ and that $n - (k + 1) = \{k + 1, \ldots, n - 1\}$) such that, for every $x \in A$,*

$$T(x) = \sigma^{\mathbf{A}}(a_0, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_{n-1}).$$

We will sometimes use the following presentation of the elementary translations, which consists in adding an underlined space to denote where the variable will be

$$T = \sigma^{\mathbf{A}}(a_0, \dots, a_{k-1}, \underline{\quad}, a_{k+1}, \dots, a_{n-1}).$$

In this case we will say that T is an elementary translation of type σ. *We will call the elements of* $\mathrm{Etl}(\mathbf{A})$ *the* elementary translations for $\mathbf{A}$.

We will denote by $\mathrm{Tl}(\mathbf{A})$ *the subset of* $\mathrm{Hom}(A, A)$ *defined as follows: For every mapping $T \in \mathrm{Hom}(A, A)$, $T \in \mathrm{Tl}(\mathbf{A})$ if and only if there is an $m \in \mathbb{N} - 1$ and a family $(T_j)_{j \in m}$ of elementary translations in $\mathrm{Etl}(\mathbf{A})^m$ for which $T = T_{m-1} \circ \cdots \circ T_0$. For translations, as for words on an alphabet, we have the notion of subtranslation of a translation, which is the counterpart of that of subword of a word. In particular, for a translation as above we will let T' stand for the composition $T_{m-2} \circ \cdots \circ T_0$ and we will call it the* maximal prefix of T, *and we will represent T as $T_{m-1} \circ T'$ or under the form:*

$$T = \sigma^{\mathbf{A}}\left(a_0, \dots, a_{k-1}, T', a_{k+1}, \dots, a_{n-1}\right),$$

where $T_{m-1} = \sigma^{\mathbf{A}}(a_0, \dots, a_{k-1}, \underline{\quad}, a_{k+1}, \dots, a_{n-1})$. The underlined space notation can be extended to translations as well. We will say that T is a translation of type σ *if the elementary translation T_{m-1} is of type σ. We will call m the* height *of T and we will denote this fact by $|T| = m$. In this regard, elementary translations have height 1 and, if T is a translation of height m, i.e., $|T| = m$, then its maximal prefix has height $m - 1$, i.e., $|T'| = m - 1$. We will call the elements of* $\mathrm{Tl}(\mathbf{A})$ *the* translations for $\mathbf{A}$. *Besides the identity mapping id_A will be viewed as an element of $\mathrm{Tl}(\mathbf{A})$. The identity translation has no associated type and we will consider that it has height 0, i.e., $|\mathrm{id}_A| = 0$. Moreover, since the identity mapping id_A is a translation, we agree that an elementary translation T has the identity as maximal prefix.*

The following are characterizations of the congruences on a Σ-algebra.

▶ **Proposition 3** (Prop. 2.7.4). *Let $\mathbf{A}$ be a Σ-algebra and Φ an equivalence relation on A. Then the following conditions are equivalent:*

1. *Φ is a congruence on $\mathbf{A}$.*
2. *Φ is closed under the elementary translations on $\mathbf{A}$, i.e., for every every x, $y \in A$, and every $T \in \mathrm{Etl}(\mathbf{A})$, if $(x, y) \in \Phi$, then $(T(x), T(y)) \in \Phi$.*
3. *Φ is closed under the translations on $\mathbf{A}$, i.e., for every every x, $y \in A$, and every $T \in \mathrm{Tl}(\mathbf{A})$, if $(x, y) \in \Phi$, then $(T(x), T(y)) \in \Phi$.*

3 Paths on terms

In this section we begin by defining the notion of rewriting system.

▶ **Definition 4.** *A rewriting system is an ordered triple $(\Sigma, X, \mathcal{A})$, often abbreviated to $\boldsymbol{\mathcal{A}}$, where Σ is a signature, X a set and $\mathcal{A}$ a subset of $\mathrm{Rwr}(\Sigma, X) = \mathrm{T}_\Sigma(X)^2$, the set of the* rewrite rules with variables in X, *where $\mathrm{T}_\Sigma(X)$ is the underlying set of $\mathbf{T}_\Sigma(X)$, the free Σ-algebra on X. We will call the elements of $\mathrm{Rwr}(\Sigma, X)$* rewrite rules *and we will denote them with lowercase Euler fraktur letters, with or without subscripts, e.g., $\mathfrak{p}$, $\mathfrak{p}_i$, $\mathfrak{q}$, $\mathfrak{q}_i$, etc.*

We next define the notion of path in $\boldsymbol{\mathcal{A}}$ from a term to another.

▶ **Definition 5.** *Let P, Q be terms in $\mathrm{T}_\Sigma(X)$ and $m \in \mathbb{N}$. Then a m-*path in $\boldsymbol{\mathcal{A}}$ from P to Q *is an ordered triple $\mathfrak{P} = ((P_i)_{i \in m+1}, (\mathfrak{p}_i)_{i \in m}, (T_i)_{i \in m})$ in $\mathrm{T}_\Sigma(X)^{m+1} \times \mathcal{A}^m \times \mathrm{Tl}(\mathbf{T}_\Sigma(X))^m$, such that*

1. *$P_0 = P$, $P_m = Q$, and,*
2. *for every $i \in m$, if $\mathfrak{p}_i = (M_i, N_i)$, then $T_i(M_i) = P_i$ and $T_i(N_i) = P_{i+1}$.*

That is, at each step $i \in m$, we consider a rewrite rule $\mathfrak{p}_i$, and a translation for $\mathbf{T}_\Sigma(X)$, T_i, and we require that the translation by T_i of M_i is P_i, whilst the translation by T_i of N_i is P_{i+1}. This statement can also be understood through the use of substitutions. We will be say that P_i contains M_i as a subterm and that P_{i+1} results from substituting one of its subterms M_i for N_i in P_i. This justifies the name rewriting rules for the elements in the family $(\mathfrak{p}_i)_{i\in m}$. On the other hand, we could think of the translations in the family $(T_i)_{i\in m}$ as the contexts in which the rewriting rules are applied.

These paths will be variously depicted as $\mathfrak{P}\colon P \longrightarrow Q$ or

$$\mathfrak{P}\colon P_0 \xrightarrow{(\mathfrak{p}_0,T_0)} P_1 \xrightarrow{(\mathfrak{p}_1,T_1)} \dots P_{m-2} \xrightarrow{(\mathfrak{p}_{m-2},T_{m-2})} P_{m-1} \xrightarrow{(\mathfrak{p}_{m-1},T_{m-1})} P_m$$

*For every $i \in m$, we will say that P_{i+1} is $(\mathfrak{p}_i, T_i)$-*directly derivable *or, when no confusion can arise,* directly derivable *from P_i. For every $i \in m+1$, the term P_i will be called a* 0-constituent *of the m-path $\mathfrak{P}$. The term P_0 will be called the* $(0,1)$-source *of the path $\mathfrak{P}$, the term P_m will be called the* $(0,1)$-target *of the path $\mathfrak{P}$, and we will say that $\mathfrak{P}$ is a* path from P_0 to P_m. *The* length *of a m-path $\mathfrak{P}$ in $\boldsymbol{\mathcal{A}}$, denoted by $|\mathfrak{P}|$, is m and we will say that $\mathfrak{P}$ has m* steps. *If $|\mathfrak{P}| = 0$, then we will say that $\mathfrak{P}$ is a* $(1,0)$-identity path. *This happens if, and only if, there exists a term P in $\mathrm{T}_\Sigma(X)$ such that $\mathfrak{P} = ((P), \lambda, \lambda)$, identified to (P, λ, λ), where, by abuse of notation, we have written (λ, λ) for the unique element of $\mathcal{A}^0 \times \mathrm{Tl}(\mathbf{T}_\Sigma(X))^0$. This path will be called the* $(1,0)$-identity path on P. *If $|\mathfrak{P}| = 1$, then we will say that $\mathfrak{P}$ is a* one-step *path. We will denote by $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ the set of all possible paths in $\boldsymbol{\mathcal{A}}$. We define the mappings*

1. $\mathrm{ip}^{(1,X)}$ *the mapping from X to $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ that sends $x \in X$ to the $(1,0)$-identity path on x; by*
2. $\mathrm{sc}^{(0,1)}$ *the mapping from $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ to $\mathrm{T}_\Sigma(X)$ that sends a path to its $(0,1)$-source; by*
3. $\mathrm{tg}^{(0,1)}$ *the mapping from $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ to $\mathrm{T}_\Sigma(X)$ that sends a path to its $(0,1)$-target; and by*
4. $\mathrm{ip}^{(1,0)\sharp}$ *the mapping that sends a term P to the $(1,0)$-identity path on P.*

These mappings are depicted in the diagram of Figure 3a.

We next define the partial operation of 0-composition of paths.

▶ **Definition 6.** *Let $\mathfrak{P}$, $\mathfrak{Q}$ be paths in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, where, for a unique $m \in \mathbb{N}$, $\mathfrak{P}$ is a path in $\boldsymbol{\mathcal{A}}$ of the form $\mathfrak{P} = ((P_i)_{i\in m+1}, (\mathfrak{p}_i)_{i\in m}, (T_i)_{i\in m})$, and, for a unique $n \in \mathbb{N}$, $\mathfrak{Q}$ is a path in $\boldsymbol{\mathcal{A}}$ of the form $\mathfrak{Q} = ((Q_j)_{j\in n+1}, (\mathfrak{q}_j)_{j\in n}, (U_j)_{j\in n})$, such that $\mathrm{sc}^{(0,1)}(\mathfrak{Q}) = \mathrm{tg}^{(0,1)}(\mathfrak{P})$.*

Then the 0-composite *of $\mathfrak{P}$ and $\mathfrak{Q}$, denoted by $\mathfrak{Q} \circ^0 \mathfrak{P}$, is the ordered triple*

$$\mathfrak{Q} \circ^0 \mathfrak{P} = ((R_k)_{k\in m+n+1}, (\mathfrak{r}_k)_{k\in m+n}, (V_k)_{k\in m+n}),$$

where

$$R_k = \begin{cases} P_k, & \text{if } k \in m+1; \\ Q_{k-m}, & \text{if } k \in [m+1, m+n+1], \end{cases}$$

$$\mathfrak{r}_k = \begin{cases} \mathfrak{p}_k, & \text{if } k \in m; \\ \mathfrak{q}_{k-m}, & \text{if } k \in [m, m+n], \end{cases}$$

$$V_k = \begin{cases} T_k, & \text{if } k \in m; \\ U_{k-m}, & \text{if } k \in [m, m+n]. \end{cases}$$

When defined, $\mathfrak{Q} \circ^0 \mathfrak{P}$ is a $(m+n)$-path in $\boldsymbol{\mathcal{A}}$ from $\mathrm{sc}^{(0,1)}(\mathfrak{P})$ to $\mathrm{tg}^{(0,1)}(\mathfrak{Q})$. Moreover, when defined, the partial operation of 0-composition is associative and, for every term $P \in \mathrm{T}_\Sigma(X)$, *the* $(1,0)$*-identity path on* P *is, when defined, a neutral element for the operation of* 0*-composition. The above definition gives rise to a category whose objects are terms in* $\mathrm{T}_\Sigma(X)$ *and whose morphisms are paths* $\mathfrak{P}$ *between terms.*

We next define the notion of subpath of a path.

▶ **Definition 7.** *Let* $m \in \mathbb{N}$*, and* $k, l \in m$ *with* $k \leq l$*. Let* $\mathfrak{P}$ *be a* m*-path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *of the form* $\mathfrak{P} = ((P_i)_{i\in m+1}, (\mathfrak{p}_i)_{i\in m}, (T_i)_{i\in m})$*. Then we will denote by* $\mathfrak{P}^{k,l}$ *the ordered triple* $\mathfrak{P}^{k,l} = \big((P_{i+k})_{i\in(l-k)+1}, (\mathfrak{p}_{i+k})_{i\in(l-k)}, (T_{i+k})_{i\in(l-k)}\big)$*. We will call* $\mathfrak{P}^{k,l}$ *the* subpath of $\mathfrak{P}$ beginning at position k and ending at position $l+1$*. In particular, subpaths of the form* $\mathfrak{P}^{0,k}$ *will be called* initial subpaths of $\mathfrak{P}$*, and subpaths of the form* $\mathfrak{P}^{l,m-1}$ *will be called* final subpaths of $\mathfrak{P}$.

We introduce the notion of echelon, a key concept in the development of our theory.

▶ **Definition 8.** *We denote by* $\mathrm{ech}^{(1,\mathcal{A})}$ *the mapping from* $\mathcal{A}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *defined as follows:*

$$\mathrm{ech}^{(1,\mathcal{A})}\begin{cases} \mathcal{A} & \longrightarrow & \mathrm{Pth}_{\boldsymbol{\mathcal{A}}} \\ \mathfrak{p} = (M,N) & \longmapsto & ((M,N), \mathfrak{p}, \mathrm{id}_{\mathrm{T}_\Sigma(X)}) \end{cases}$$

This mapping associates to each rewrite rule $\mathfrak{p} = (M,N)$ *in* $\mathcal{A}$ *the one-step path from* M *to* N *that uses the rewrite rule* $\mathfrak{p}$ *in the identity translation, see Figure 3a. This definition is sound because (1)* $\mathrm{id}_{\mathrm{T}_\Sigma(X)}(M) = M$ *and (2)* $\mathrm{id}_{\mathrm{T}_\Sigma(X)}(N) = N$*. We will call* $\mathrm{ech}^{(1,\mathcal{A})}(\mathfrak{p})$ *the* echelon associated to $\mathfrak{p}$*. Moreover, we will say that a path* $\mathfrak{P} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *is an* echelon *if there exists a rewrite rule* $\mathfrak{p} \in \mathcal{A}$ *such that* $\mathrm{ech}^{(1,\mathcal{A})}(\mathfrak{p}) = \mathfrak{P}$*. Finally, we will say that a path* $\mathfrak{P}$ *is* echelonless *if* $|\mathfrak{P}| \geq 1$ *and none of its one-step subpaths is an echelon.*

From the above it follows that the translations of an echelonless path must be non-identity translations. We next introduce the notion of a head-constant echelonless path.

▶ **Definition 9.** *Let* $\mathfrak{P} = ((P_i)_{i\in m+1}, ((\mathfrak{p}_i)_{i\in m}, (T_i)_{i\in m})$ *be an echelonless path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*. We will say that* $\mathfrak{P}$ *is a* head-constant *echelonless path if* $(T_i)_{i\in m}$*, the family of translations occurring in it, have the same type, i.e., they are associated to the same operation symbol.*

The importance of echelonless paths is that they can only traverse complex terms (terms in $\mathrm{T}_\Sigma(X)$ that are neither variables nor constants) and force homogeneity in this structure. That is, an echelonless path is forced to traverse complex terms associated to a non-constant operation symbol $\sigma \in \Sigma_n$ of arity $n \in \mathbb{N} - \{0\}$.

$$\mathfrak{P}\colon \sigma^{\mathbf{T}_\Sigma(X)}((P_{0,j})_{j\in n}) \xrightarrow{(\mathfrak{p}_0,T_0)} \sigma^{\mathbf{T}_\Sigma(X)}((P_{1,j})_{j\in n}) \xrightarrow{(\mathfrak{p}_1,T_1)} \dots \xrightarrow{(\mathfrak{p}_{m-1},T_{m-1})} \sigma^{\mathbf{T}_\Sigma(X)}((P_{m,j})_{j\in n})$$

Figure 1 An echelonless path.

The following lemma states that every echelonless path is head-constant.

▶ **Lemma 10** (Lemma 6.1.4)**.** *Let* $\mathfrak{P}$ *be an echelonless path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*. Then* $\mathfrak{P}$ *is head-constant.*

This homogeneity allows us to understand the echelonless paths as a sequence of paths, possibly fragmented, acting in parallel in each of the components according to the arity of the operation symbol. Therefore, for an echelonless path, we propose a process of path extraction in each component that returns the family of paths that result from joining the fragments in each of the components. We will refer to it as the *path extraction algorithm.*

$$
\begin{array}{cccccccc}
P_0 = \sigma(& P_{0,0} & ,\cdots, & P_{0,j} & ,\cdots, & P_{0,n-1}) \\
\mathfrak{P} \downarrow & \downarrow \mathfrak{P}_0 \ \ldots & & \downarrow \mathfrak{P}_j \ \ldots & & \downarrow \mathfrak{P}_{n-1} \\
P_m = \sigma(& P_{m,0} & ,\cdots, & P_{m,j} & ,\cdots, & P_{m,n-1})
\end{array}
$$

Figure 2 The path extraction algorithm.

▶ **Lemma 11** (Lemma 6.1.5). *Let $\mathfrak{P} = ((P_i)_{i\in m+1}, (\mathfrak{p}_i)_{i\in m}, (T_i)_{i\in m})$ be an echelonless path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$. Let σ be the unique n-ary operation symbol in Σ_n for which, in virtue of Lemma 10, each of the translations of the family $(T_i)_{i\in m}$ is of type σ. Then there exists a unique pair $((m_j)_{j\in n}, (\mathfrak{P}_j)_{j\in n}) \in \mathbb{N}^n \times \mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}}$ such that, for every $j \in n$, $\mathfrak{P}_j$ is a m_j-path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ and there exists a unique bijective mapping $i\colon \coprod_{j\in n} m_j \longrightarrow m$ such that, for every (j,k) in $\coprod_{j\in n} m_j$, $\mathfrak{p}_{j,k} = \mathfrak{p}_{i(j,k)}$.*

3.1 Algebraic structure on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$

We next define a structure of Σ-algebra in the set $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$. In this regard, Lemma 11 gives us different insights on how a path can be performed. Different strategies can be selected at this point. In our case, we have decided to follow a leftmost innermost strategy.

▶ **Proposition 12** (Prop. 7.0.1). *The set $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ is equipped with a structure of Σ-algebra.*

Proof. Let us denote by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$ the Σ-algebra defined on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ as follows. For every n-ary operation symbol $\sigma \in \Sigma_n$, the operation $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$, from $\mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, assigns to a family of paths $(\mathfrak{P}_j)_{j\in n} \in \mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}}$ where, for every $j \in n$, $\mathfrak{P}_j$ is a m_j-path in $\boldsymbol{\mathcal{A}}$ from $P_{j,0}$ to P_{j,m_j} of the form $\mathfrak{P}_j = \big((P_{j,k})_{k\in m_j+1}, (\mathfrak{p}_{j,k})_{k\in m_j}, (T_{j,k})_{k\in m_j}\big)$,, precisely the m-path in $\boldsymbol{\mathcal{A}}$ given by $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in m}) = ((P_i)_{i\in m+1}, (\mathfrak{p}_i)_{i\in m}, (T_i)_{i\in m},)$, where $m = \sum_{j\in n} m_j$ is the sum of the family of natural numbers $(m_j)_{j\in n}$.

Let us point out the following facts. By construction, the i-th element of m will be the k-th element of the addend m_j, for a unique $j \in n$ and a unique $k \in m_j$. We will write $i = (j,k)$ to denote this dependency.

Returning to the definition of $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$, for $i \in n$ with $i = (j,k)$, we define the 0-constituent at step i of $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$ to be the term

$$P_i = \sigma^{\mathbf{T}_\Sigma(X)}\left(P_{0,m_0}, \cdots, P_{j-1,m_{j-1}}, P_{j,k}, P_{j+1,0}, \cdots, P_{n-1,0}\right).$$

That is, if $i \in m$ and $i = (j,k)$, then we have that, to the left of position j, every subterm is equal to the last term of the corresponding path, and, to the right of position j, every subterm is equal to the initial term of the corresponding path. The j-th subterm of P_i is the k-th term appearing in the path $\mathfrak{P}_j$. In particular, since $0 = (0,0)$, we have that $P_0 = \sigma^{\mathbf{T}_\Sigma(X)}((P_{j,0})_{j\in n})$. Finally, for the case $i = m$, we define $P_m = \sigma^{\mathbf{T}_\Sigma(X)}((P_{j,m_j})_{j\in n})$.

For $i \in m$ with $i = (j,k)$, we define the rewrite rule $\mathfrak{p}_i$ to be equal to $\mathfrak{p}_{j,k}$. That is, the i-th rewrite rule of $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$ is equal to the k-th rewrite rule of the path $\mathfrak{P}_j$.

Finally, for $i \in m$ with $i = (j,k)$, we define the translation at step i of $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$ to be equal to $T_i = \sigma^{\mathbf{T}_\Sigma(X)}\left(P_{0,m_0}, \cdots, P_{j-1,m_{j-1}}, T_{j,k}, P_{j+1,0} \cdots, P_{n-1,0}\right)$. That is, if $i \in m$ and $i = (j,k)$, then we have that, to the left of position j, every subterm is equal to the term of the last 0-constituent of the corresponding path, and, to the right of position j, every subterm is equal to the term of the initial 0-constituent of the corresponding path. The j-th subterm of T_i is the k-th translation appearing in the path $\mathfrak{P}_j$. ◀

It can be shown that $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$ is a path in $\boldsymbol{\mathcal{A}}$ of the form

$$\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})\colon \sigma^{\mathbf{T}_\Sigma(X)}((\mathrm{sc}^{(0,1)}(\mathfrak{P}_j))_{j\in n}) \longrightarrow \sigma^{\mathbf{T}_\Sigma(X)}((\mathrm{tg}^{(0,1)}(\mathfrak{P}_j))_{j\in n}).$$

Moreover, if the family of paths contains at least a non-identity path, then the path $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}((\mathfrak{P}_j)_{j\in n})$ is an echelonless path. Furthermore, the path extraction algorithm applied to it retrieves the original family $(\mathfrak{P}_j)_{j\in n}$. Let us note that the set $\mathrm{ip}^{(1,0)\sharp}[\mathbf{T}_\Sigma(X)]$, of $(1,0)$-identity paths, becomes a Σ-subalgebra of $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$, the mappings $\mathrm{sc}^{(0,1)}$ and $\mathrm{tg}^{(0,1)}$ become Σ-homomorphisms and the mapping $\mathrm{ip}^{(1,0)\sharp}$ is a Σ-homomorphism that can be obtained by the universal property of $\mathbf{T}_\Sigma(X)$ on $\mathrm{ip}^{(1,X)}$, see Figure 3a.

3.2 Order structure on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$

In this subsection we define on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ an Artinian order, which will allow us to justify both proofs by Artinian induction and definitions by Artinian recursion.

▶ **Definition 13.** *We let $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$ denote the binary relation on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ consisting of the ordered pairs $(\mathfrak{Q},\mathfrak{P})\in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}^2$ for which one of the following conditions holds*

1. *$\mathfrak{P}$ and $\mathfrak{Q}$ are $(1,0)$-identity paths of the form $\mathfrak{P}=\mathrm{ip}^{(1,0)\sharp}(P)$, $\mathfrak{Q}=\mathrm{ip}^{(1,0)\sharp}(Q)$, for some terms $P\in \mathrm{T}_\Sigma(X)$ and $Q\in \mathrm{T}_\Sigma(X)$ and the inequality $Q<_{\mathbf{T}_\Sigma(X)} P$ holds, where $\leq_{\mathbf{T}_\Sigma(X)}$ is the subterm preorder on $\mathrm{T}_\Sigma(X)$.*
2. *$\mathfrak{P}$ is a path of length m strictly greater than one containing at least one echelon, and if its first echelon occurs at position $i\in m$, then*
   a. *if $i=0$, then $\mathfrak{Q}$ is equal to $\mathfrak{P}^{0,0}$ or $\mathfrak{P}^{1,m-1}$,*
   b. *if $i>0$, then $\mathfrak{Q}$ is equal to $\mathfrak{P}^{0,i-1}$ or $\mathfrak{P}^{i,m-1}$;*
3. *$\mathfrak{P}$ is an echelonless path and $\mathfrak{Q}$ is one of the paths extracted from $\mathfrak{P}$ in virtue of Lemma 11.*

We will denote by $\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$ the reflexive and transitive closure of $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$, i.e., the preorder on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ generated by $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$.

For the preordered set $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}},\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$ it can be shown that the minimal elements are the identity paths on minimal elements in $\mathrm{T}_\Sigma(X)$, i.e., variables and constants, and the echelons. The most important feature of this relation is that it is antisymmetric and there is not any strictly decreasing ω_0-chain.

▶ **Proposition 14** (Prop. 8.0.12)**.** *$(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}},\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$ is an Artinian ordered set.*

4 The Curry-Howard mapping

In this section we define a new signature, the categorial signature determined by $\boldsymbol{\mathcal{A}}$.

▶ **Definition 15.** *The* categorial signature determined by $\mathcal{A}$ on Σ, *denoted by $\Sigma^{\boldsymbol{\mathcal{A}}}$, is the signature defined, for every $n\in\mathbb{N}$, as follows:*

$$\Sigma^{\boldsymbol{\mathcal{A}}}_n = \begin{cases} \Sigma_n, & \text{if } n\neq 0,1,2;\\ \Sigma_0 \amalg \mathcal{A}, & \text{if } n=0;\\ \Sigma_1 \amalg \{\mathrm{sc}^0,\mathrm{tg}^0\}, & \text{if } n=1;\\ \Sigma_2 \amalg \{\circ^0\}, & \text{if } n=2.\end{cases}$$

That is, $\Sigma^{\boldsymbol{\mathcal{A}}}$ is the expansion of Σ obtained by adding, (1) as many constants as there are rewrite rules in $\mathcal{A}$, (2) two unary operation symbols sc^0 and tg^0, which will be interpreted as total unary operations, and (3) a binary operation symbol $\circ^0$ which will be interpreted as a partial operation.

Let $\eta^{(1,X)}$ denote the standard insertion of generator from X to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. This extension allows us to view all terms in $\mathbf{T}_{\Sigma}(X)$ as terms in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. Let $\eta^{(1,0)\sharp}$ denote the embedding from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. Furthermore, every rewrite rule in $\mathcal{A}$ can also be seen as a constant in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. Let $\eta^{(1,\mathcal{A})}$ denote the embedding from $\mathcal{A}$ to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, see Figure 3a.

We next show that the set $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ has a natural structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra.

▶ **Proposition 16** (Prop. 9.1.1). *The set $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ is equipped with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra.*

Proof. Let us denote by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$ the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra defined on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ as follows. The operations from Σ are defined as in Proposition 12. Every constant operation symbol $\mathfrak{p} \in \mathcal{A}$ is interpreted as the echelon $\mathrm{ech}^{(1,\mathcal{A})}(\mathfrak{p})$ introduced in Definition 8. The 0-source operation symbol is interpreted as the unary operation that maps a path $\mathfrak{P}$ in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ to the $(1,0)$-identity path $\mathrm{ip}^{(1,0)\sharp}(\mathrm{sc}^{(0,1)}(\mathfrak{P}))$, see Definition 5. The 0-target is interpreted analogously. The 0-composition is the partial operation defined in Proposition 6. ◀

The previous results, will allow us to consider paths in the rewriting system $\boldsymbol{\mathcal{A}}$ as terms relative to $\Sigma^{\boldsymbol{\mathcal{A}}}$ and X. To do this, we will define, by Artinian recursion, a mapping from $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ to $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. In this way, every path in $\boldsymbol{\mathcal{A}}$ will be denoted by a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. Since this mapping reminds us of the classical Curry-Howard correspondence (see [10] and [17]), we have decided to denote it by $\mathrm{CH}^{(1)}$.

▶ **Definition 17.** *The* Curry-Howard mapping *is the mapping* $\mathrm{CH}^{(1)}\colon \mathrm{Pth}_{\boldsymbol{\mathcal{A}}} \longrightarrow \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *defined by Artinian recursion on* $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$ *as follows.*

Base step of the Artinian recursion.

Let $\mathfrak{P}$ be a minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$. Then the path $\mathfrak{P}$ is either (1) an $(1,0)$-identity path or (2) an echelon.

If (1), then $\mathfrak{P} = \mathrm{ip}^{(1,0)\sharp}(P)$ for some term $P \in \mathrm{T}_{\Sigma}(X)$. We define $\mathrm{CH}^{(1)}(\mathfrak{P})$ to be the term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ given by the lift of the term P by $\eta^{(1,0)\sharp}$, i.e., $\mathrm{CH}^{(1)}(\mathfrak{P}) = \eta^{(1,0)\sharp}(P)$.

If (2), if $\mathfrak{P}$ is an echelon associated to $\mathfrak{p} \in \mathcal{A}$, then we define $\mathrm{CH}^{(1)}(\mathfrak{P}) = \mathfrak{p}^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}$.

Inductive step of the Artinian recursion.

Let $\mathfrak{P}$ be a non-minimal element of $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$. We can assume that $\mathfrak{P}$ is a not a $(1,0)$-identity path, since those paths already have an image for the Curry-Howard mapping. Let us suppose that, for every every path $\mathfrak{Q} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, if $\mathfrak{Q} <_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \mathfrak{P}$, then the value of the Curry-Howard mapping at $\mathfrak{Q}$ has already been defined. We have that $\mathfrak{P}$ is either (1) a path of length m strictly greater than one containing at least one echelon or (2) an echelonless path.

If (1), let $i \in m$ be the first index for which the one-step subpath $\mathfrak{P}^{i,i}$ of $\mathfrak{P}$ is an echelon. We consider different cases for i according to the cases presented in Definition 13.

If $i = 0$, we have that the paths $\mathfrak{P}^{0,0}$ and $\mathfrak{P}^{1,m-1}$, s $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$-precede the path $\mathfrak{P}$. In this case, we set $\mathrm{CH}^{(1)}(\mathfrak{P}) = \mathrm{CH}^{(1)}(\mathfrak{P}^{1,m-1}) \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} \mathrm{CH}^{(1)}(\mathfrak{P}^{0,0})$.

If $i \neq 0$, we have that the paths $\mathfrak{P}^{0,i-1}$ and $\mathfrak{P}^{i,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$-precede the path $\mathfrak{P}$. In this case, we set $\mathrm{CH}^{(1)}(\mathfrak{P}) = \mathrm{CH}^{(1)}(\mathfrak{P}^{i,m-1}) \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} \mathrm{CH}^{(1)}(\mathfrak{P}^{0,i-1})$.

If (2), i.e., if $\mathfrak{P}$ is an echelonless path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, then the conditions for the path extraction algorithm, as stated in Lemma 11, are fulfilled. Then, by Lemma 10, there exists a unique n-ary operation symbol $\sigma \in \Sigma_n$ associated to $\mathfrak{P}$. Let $(\mathfrak{P}_j)_{j\in n}$ be the family of paths in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}^{n}$ which we can extract from $\mathfrak{P}$. In this case, we set $\mathrm{CH}^{(1)}(\mathfrak{P}) = \sigma^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}((\mathrm{CH}^{(1)}(\mathfrak{P}_j))_{j\in n})$.

It can be shown that the Curry-Howard mapping is a Σ-homomorphism. However, it is not a $\Sigma^{\boldsymbol{\mathcal{A}}}$-homomorphism. It is enough to consider the case of identity paths, which are idempotent in $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$, but this idempotence is not conserved in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, since the 0-composition operation remains purely syntactic. However, the study of its kernel proves to be interesting. A pair of paths $\mathfrak{P}$ and $\mathfrak{Q}$ is in $\mathrm{Ker}(\mathrm{CH}^{(1)})$ if they basically have the same nature. In fact, if $(\mathfrak{P},\mathfrak{Q})$ is a pair in $\mathrm{Ker}(\mathrm{CH}^{(1)})$ and one of them is a $(1,0)$-identity path, then they are equal. For a pair $(\mathfrak{P},\mathfrak{Q})$ in $\mathrm{Ker}(\mathrm{CH}^{(1)})$, if one of the paths has echelons, then so does the other (and in the same position). If one of the paths has no echelons, then neither does the other, and it is also associated with the same operation symbol $\sigma\in\Sigma_n$. In general, it can be shown that for a pair of paths in $\mathrm{Ker}(\mathrm{CH}^{(1)})$, their length, source and target are equal. The most interesting property of the mapping $\mathrm{CH}^{(1)}$ is that its kernel, $\mathrm{Ker}(\mathrm{CH}^{(1)})$, is a closed $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence on $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$. This proof is achieved by induction on $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}},\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}})$.

▶ **Proposition 18** (Prop. 10.1.1). $\mathrm{Ker}(\mathrm{CH}^{(1)})$ *is a closed* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-congruence on the partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra* $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$.

4.1 The quotient of paths

The last results opens up a new object of study, the quotient of paths by $\mathrm{Ker}(\mathrm{CH}^{(1)})$. For a path $\mathfrak{P}\in\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$, we will let $[\mathfrak{P}]$ stand for $[\mathfrak{P}]_{\mathrm{Ker}(\mathrm{CH}^{(1)})}$, the $\mathrm{Ker}(\mathrm{CH}^{(1)})$-equivalence class of $\mathfrak{P}$, and we will call it the *path class of* $\mathfrak{P}$. Moreover, the quotient $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}/\mathrm{Ker}(\mathrm{CH}^{(1)})$ will simply be denoted by $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$. In this subsection we investigate the algebraic, categorial and order structures that we can define on $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$.

As an immediate consequence of the fact that $\mathrm{Ker}(\mathrm{CH}^{(1)})$ is a closed $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence we have that $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ inherits a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra.

▶ **Proposition 19** (Prop. 11.1.1). *The set* $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra, that we denote by* $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$.

The set $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ is equipped with a structure of categorial Σ-algebra.

▶ **Proposition 20** (Prop. 11.2.9). *The set* $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ *is equipped with a structure of categorial* Σ*-algebra, that we denote by* $[\mathsf{Pth}_{\boldsymbol{\mathcal{A}}}]$.

Proof. We begin by showing that the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$ satisfies the defining equations of a category. Following Definition 1, the interpretation of operations from Σ need to be functors. In particular we prove that, for every $n\in\mathbb{N}$, $\sigma\in\Sigma_n$ and every family $([\mathfrak{P}_j])_{j\in n}$ in $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]^n$, the following equalities holds

$$\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}((\mathrm{sc}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}([\mathfrak{P}_j]))_{j\in n}) = \mathrm{sc}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(([\mathfrak{P}_j])_{j\in n}));$$
$$\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}((\mathrm{tg}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}([\mathfrak{P}_j]))_{j\in n}) = \mathrm{tg}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(([\mathfrak{P}_j])_{j\in n})).$$

Furthermore, for every $n\in\mathbb{N}$, $\sigma\in\Sigma_n$ and $([\mathfrak{P}_j])_{j\in n}$, $([\mathfrak{Q}_j])_{j\in n}$ families in $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]^n$, such that, for every $j\in n$, $\mathrm{sc}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}([\mathfrak{Q}_j]) = \mathrm{tg}^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}([\mathfrak{P}_j])$. Then the following equality holds

$$\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(([\mathfrak{Q}_j]\circ^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}[\mathfrak{P}_j])_{j\in n}) = \sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(([\mathfrak{Q}_j])_{j\in n})\circ^{0[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}\sigma^{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}(([\mathfrak{P}_j])_{j\in n}).$$ ◀

Furthermore, the set $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ is equipped with an Artinian order.

▶ **Definition 21.** *Let* $\leq_{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]}$ *be the binary relation defined on* $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ *containing every pair* $([\mathfrak{Q}],[\mathfrak{P}])$ *in* $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]^2$ *for which there exists a pair of representatives* $\mathfrak{Q}'\in[\mathfrak{Q}]$ *and* $\mathfrak{P}'\in[\mathfrak{P}]$ *satisfying that* $\mathfrak{Q}'\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}\mathfrak{P}'$.

▶ **Proposition 22** (Prop. 11.3.8). $([\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}],\leq_{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]})$ *is an Artinian ordered set.*

5 Path terms

Following ideas of Burmeister and Schmidt [4, 5, 6, 21, 22, 8], we consider, for a signature Γ and a partial Γ-algebra $\mathbf{A}$, its free Γ-completion, denoted by $\mathbf{F}_{\Gamma}(\mathbf{A})$. This is constructed using A, the domain of $\mathbf{A}$, as a set of generators for the free Γ-algebra $\mathrm{T}_{\Gamma}(A)$, in which we interpret the operations of Γ as the operations on $\mathbf{A}$ whenever they are defined and as purely syntactic operations on a term algebra in case they are not defined. The free completion is the best possible solution to the problem of having fully defined operations in a partial algebra. Therefore, this total Γ-algebra has the following universal property; for every partial Γ-algebra $\mathbf{B}$ and every Γ-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$, there exists a unique Γ-homomorphism, f^{fc}, the free completion of f, from $\mathbf{F}_{\Gamma}(\mathbf{A})$ to $\mathbf{B}$ extending the Γ-homomorphism f as usual, i.e., satisfying that $f^{\mathrm{fc}} \circ \eta^{\mathbf{A}} = f$, where $\eta^{\mathbf{A}}$, from $\mathbf{A}$ to $\mathbf{F}_{\Gamma}(\mathbf{A})$, is the standard insertion of generators.

With the aforementioned ideas we consider the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$.

▶ **Definition 23.** *Consider the mapping* $\mathrm{ip}^{(1,X)}$ *from the set of variables* X *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*, introduced in Definition 5. If we consider* $\mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$*, the discrete* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra on* X*, i.e., no operation in* $\Sigma^{\boldsymbol{\mathcal{A}}}$ *is defined, the application* $\mathrm{ip}^{(1,X)}$ *becomes a* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-homomorphism of the form* $\mathrm{ip}^{(1,X)}\colon \mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X) \longrightarrow \mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$*. By the universal property of the free completion, there exists a unique* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-homomorphism* $(\eta^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \circ \mathrm{ip}^{(1,X)})^{\mathrm{fc}}$*, simply denoted* $\mathrm{ip}^{(1,X)@}$*, from* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$*, the free completion of the discrete* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra* $\mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$*, to* $\mathbf{F}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$*, the free* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-completion of the path algebra* $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$*, such that* $\mathrm{ip}^{(1,X)@} \circ \eta^{(1,X)} = \eta^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \circ \mathrm{ip}^{(1,X)}$*.*

At this point we begin to study the $\Sigma^{\boldsymbol{\mathcal{A}}}$-homomorphism $\mathrm{ip}^{(1,X)@}$. The following proposition is fundamental for the rest of this work. It states that $\mathrm{ip}^{(1,X)@}$ acting on the value of $\mathrm{CH}^{(1)}$ at a path $\mathfrak{P}$ is always another path, not necessarily equal to the input $\mathfrak{P}$, but which belongs to the equivalence class $[\mathfrak{P}]$.

▶ **Proposition 24** (Prop. 12.1.4)**.** *The mapping* $\mathrm{ip}^{(1,X)@} \circ \mathrm{CH}^{(1)}\colon \mathrm{Pth}_{\boldsymbol{\mathcal{A}}} \longrightarrow \mathrm{F}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$ *sends every path* $\mathfrak{P}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *to a path in the class* $[\mathfrak{P}]$*.*

It can be shown that the element $\mathrm{ip}^{(1,X)@}(\mathrm{CH}^{(1)}(\mathfrak{P}))$ is a normalised version of $\mathfrak{P}$, since the derivations follow a leftmost innermost derivation strategy, reflecting the definition of the operations in $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}$.

We next define a binary relation on $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ with the objective of matching different terms that, by $\mathrm{ip}^{(1,X)@}$, are sent to paths in the same equivalence class relative to $\mathrm{Ker}(\mathrm{CH}^{(1)})$.

▶ **Definition 25.** *We let* $\Theta^{(1)}$ *stand for the binary relation on* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *consisting exactly of the following pairs of terms:*

- *For every path* $\mathfrak{P}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*,* $(\mathrm{CH}^{(1)}(\mathrm{sc}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}(\mathfrak{P})), \mathrm{sc}^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}))) \in \Theta^{(1)}$*;*
- *For every path* $\mathfrak{P}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*,* $(\mathrm{CH}^{(1)}(\mathrm{tg}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}(\mathfrak{P})), \mathrm{tg}^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}))) \in \Theta^{(1)}$*;*
- *For every pair of paths* $\mathfrak{Q}, \mathfrak{P}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$*, if* $\mathrm{sc}^{(0,1)}(\mathfrak{Q}) = \mathrm{tg}^{(0,1)}(\mathfrak{P})$*,*

$$(\mathrm{CH}^{(1)}(\mathfrak{Q} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \mathfrak{P}), \mathrm{CH}^{(1)}(\mathfrak{Q}) \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} \mathrm{CH}^{(1)}(\mathfrak{P})) \in \Theta^{(1)}.$$

Finally, we denote by $\Theta^{[1]}$ *the smallest* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-congruence on* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *containing* $\Theta^{(1)}$*.*

We next provide two lemmas to understand the usefulness of the $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence $\Theta^{[1]}$. The first lemma proves that if a term is such that its image under $\mathrm{ip}^{(1,X)@}$ is a path, then it is related, with respect to the $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence $\Theta^{[1]}$ with a term in $\mathrm{CH}^{(1)}[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$. Actually, we prove that such a term is related with its image under the action of $\mathrm{CH}^{(1)} \circ \mathrm{ip}^{(1,X)@}$.

▶ **Lemma 26** (Lemma 13.1.7). *Let P be a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. If $\mathrm{ip}^{(1,X)@}(P)$ is a path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ then $(P, \mathrm{CH}^{(1)}(\mathrm{ip}^{(1,X)@}(P))) \in \Theta^{[1]}$.*

The next lemma proves that if two terms are $\Theta^{[1]}$-related and one of them, when mapped under $\mathrm{ip}^{(1,X)@}$, returns a path, then the other term has a similar behaviour. Moreover, if this situation happens, then these two paths will have the same image under $\mathrm{CH}^{(1)}$.

▶ **Lemma 27** (Lemma 13.1.8). *Let $P, Q \in \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ be such that $(P, Q) \in \Theta^{[1]}$, then*

- $\mathrm{ip}^{(1,X)@}(P) \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *if, and only if,* $\mathrm{ip}^{(1,X)@}(Q) \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$;
- *If* $\mathrm{ip}^{(1,X)@}(P)$ *or* $\mathrm{ip}^{(1,X)@}(Q)$ *is a path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *then*

$$\mathrm{CH}^{(1)}(\mathrm{ip}^{(1,X)@}(P)) = \mathrm{CH}^{(1)}(\mathrm{ip}^{(1,X)@}(Q)).$$

We next introduce the notion of path term.

▶ **Definition 28.** *We let $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ stand for $[\mathrm{CH}^{(1)}[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]]^{\Theta^{[1]}} = \bigcup_{\mathfrak{P} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}} [\mathrm{CH}^{(1)}(\mathfrak{P})]_{\Theta^{[1]}}$, the $\Theta^{[1]}$-saturation of the subset $\mathrm{CH}^{(1)}[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ of $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. We call $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ the set of* path terms.

It can be shown that a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ is a path term if, and only if, it can be interpreted as a path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ by means of $\mathrm{ip}^{(1,X)@}$. Following this, some already known mappings have nice restrictions, corestrictions or birestrictions to the set of path terms. Indeed, the embeddings $\eta^{(1,X)}$ and $\eta^{(1,\mathcal{A})}$ from, respectively, X and $\mathcal{A}$ to $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ corestrict to $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$. Also, the embedding $\eta^{(1,0)\sharp}$, from $\mathrm{T}_{\Sigma}(X)$ to $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, corestricts to $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$. Furthermore, the restriction of $\mathrm{ip}^{(1,X)@}$ to the set of path terms corestricts to $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$. Finally, the Curry-Howard mapping, also corestricts to $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$. When possible, we will use these refinements instead of the original mappings, see Figure 3b.

5.1 Algebraic structure on $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$

We next show that $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ is equipped with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra.

▶ **Proposition 29** (Prop. 14.1.1). *The set $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ is equipped with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, which is a $\Sigma^{\boldsymbol{\mathcal{A}}}$-subalgebra of $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$.*

Proof. Let us denote by $\mathbf{PT}_{\boldsymbol{\mathcal{A}}}$ the partial Σ-algebra defined on $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ as follows. All the operations from $\Sigma^{\boldsymbol{\mathcal{A}}}$ have the same interpretation as in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, except the operation of 0-composition. For two path terms $P, Q \in \mathrm{PT}_{\boldsymbol{\mathcal{A}}}$, the 0-composition $Q \circ^0 P$ is defined if, and only if, $\mathrm{sc}^{(0,1)}(\mathrm{ip}^{(1,X)@}(Q)) = \mathrm{tg}^{(0,1)}(\mathrm{ip}^{(1,X)@}(P))$. In the positive case, the 0-composition operation is interpreted as in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. ◀

5.2 Order structure on $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$

We next define an Artinian order on $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$. The following definition is sound because the subterms of path terms are also path terms.

▶ **Definition 30.** *Let $\leq_{\mathbf{PT}_{\boldsymbol{\mathcal{A}}}}$ be the binary relation on $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ containing every pair (Q, P) in $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}^2$ such that $Q \leq_{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} P$. Thus, $Q \leq_{\mathbf{PT}_{\boldsymbol{\mathcal{A}}}} P$ if, and only if, Q is a subterm of P.*

▶ **Proposition 31** (Prop. 14.2.3). *$(\mathrm{PT}_{\boldsymbol{\mathcal{A}}}, \leq_{\mathbf{PT}_{\boldsymbol{\mathcal{A}}}})$ is an Artinian ordered set.*

5.3 The quotient of path terms

In this subsection we define the set of path term classes as the quotient of $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ by the restriction of $\Theta^{[1]}$ to it. From this point on, to simplify the notation, for a path term $P \in \mathrm{PT}_{\boldsymbol{\mathcal{A}}}$, we will let $[P]$ stand for $[P]_{\Theta^{[1]}}$, the $\Theta^{[1]}$-equivalence class of P, and we will call it the *path term class of* P.

▶ **Definition 32.** *We denote by* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *the image of* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ *under* $\mathrm{pr}^{\Theta^{[1]}}$*, the canonical projection from* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *to* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)/\Theta^{[1]}$*, i.e.,* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}] = \mathrm{pr}^{\Theta^{[1]}}[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$*. We call it the set of* path term classes*. Let us note that* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *is a subset of the quotient* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)/\Theta^{[1]}$*, i.e., that* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *is a subquotient of* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$*. Actually, we have that* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}] = \mathrm{PT}_{\boldsymbol{\mathcal{A}}}/\Theta^{[1]} \upharpoonright \mathrm{PT}_{\boldsymbol{\mathcal{A}}}$*.*

The projection, from $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *to* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)/\Theta^{[1]}$*, birestricts to* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ *and* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$*.*

We investigate the algebraic, categorial and order structures that we can define on $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$. As an immediate consequence of the definition, the set of path term classes inherits a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra.

▶ **Proposition 33** (Prop. 14.4.1)**.** *The set* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra, that we denote by* $[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]$*.*

The set $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ is equipped with a structure of categorial Σ-algebra.

▶ **Proposition 34** (Prop. 14.5.10)**.** *The set* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *is equipped with a structure of categorial* Σ*-algebra, that we denote by* $[\mathsf{PT}_{\boldsymbol{\mathcal{A}}}]$*.*

Finally, we define an Artinian order on $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$.

▶ **Definition 35.** *We let* $\leq_{[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]}$ *stand for the binary relation on* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *which consists of those ordered pairs* $([Q],[P])$ *in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]^2$ *for which there exists a pair of representatives* $Q' \in [Q]$ *and* $P' \in [P]$ *satisfying that* $\mathrm{ip}^{(1,X)@}(Q') \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}} \mathrm{ip}^{(1,X)@}(P')$*.*

▶ **Proposition 36** (Prop. 14.6.2)**.** $([\mathrm{PT}_{\boldsymbol{\mathcal{A}}}], \leq_{[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]})$ *is an Artinian ordered set.*

6 First-order isomorphisms

In this section we are in position to prove the main results of the paper, that the algebraic, categorial and order structures that we have defined on path classes and on path terms are isomorphic. The isomorphisms are constructed using refinements of the Curry-Howard mapping and the free completion of the identity path mapping.

▶ **Theorem 37** (Th. 15.1.1, 15.2.1, 15.3.3)**.** *The partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebras* $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$ *and* $[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]$ *are isomorphic. The categorial* Σ*-algebras,* $[\mathsf{Pth}_{\boldsymbol{\mathcal{A}}}]$ *and* $[\mathsf{PT}_{\boldsymbol{\mathcal{A}}}]$ *are isomorphic. The Artinian ordered sets* $([\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}], \leq_{[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]})$ *and* $([\mathrm{PT}_{\boldsymbol{\mathcal{A}}}], \leq_{[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]})$ *are isomorphic.*

Proof. We let $\mathrm{ip}^{([1],X)@}$ stand for the mapping from $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ to $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ that maps a path term class $[P]$ in $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ to the path class $[\mathrm{ip}^{(1,X)@}(P)]$ in $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$. This mapping is well-defined because two path terms P, Q in $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ such that $[Q] = [P]$ satisfy that $[\mathrm{ip}^{(1,X)@}(Q)] = [\mathrm{ip}^{(1,X)@}(P)]$. We let $\mathrm{CH}^{[1]}$ stand for the mapping from $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ to $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ that maps a path class $[\mathfrak{P}]$ in $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ to the path term class $[\mathrm{CH}^{(1)}(\mathfrak{P})]$ in $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$.

This two mappings constitute a pair of inverse $\Sigma^{\boldsymbol{\mathcal{A}}}$-isomorphisms, a pair of inverse functors, i.e., of categorial Σ-isomorphisms. Finally, we show that the mappings also form a pair of inverse order-preserving mappings, see Figure 3b. ◀

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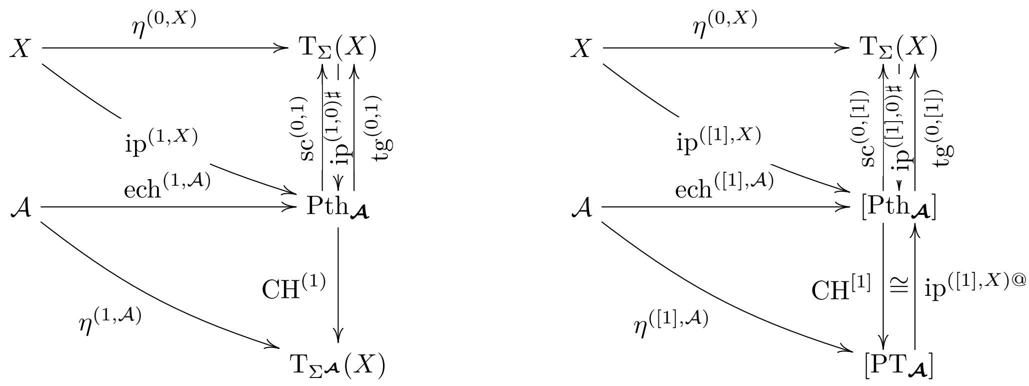
A Diagrams

The following figure collects all the mappings considered in this work.



(a) Mappings at layers 0 & 1.

(b) Quotient mappings at layers 0 & 1.

Figure 3 Mappings considered in this work.

B Freedom

For a specification $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}$ associated to the rewriting system $\boldsymbol{\mathcal{A}}$, whose defining equations $\mathcal{E}^{\boldsymbol{\mathcal{A}}}$ are QE-equations, we define a QE-variety of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras $\mathcal{V}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$.

▶ **Definition 38.** *For the rewriting system $\boldsymbol{\mathcal{A}}$, we will denote by $(\Sigma^{\boldsymbol{\mathcal{A}}}, V, \mathcal{E}^{\boldsymbol{\mathcal{A}}})$, written $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}$ for short, the specification in which $\Sigma^{\boldsymbol{\mathcal{A}}}$ is the signature introduced in Definition 15, V a fixed set with a countable infinity of variables, and $\mathcal{E}^{\boldsymbol{\mathcal{A}}}$ the subset of $\mathrm{QE}(\Sigma^{\boldsymbol{\mathcal{A}}})_V$, consisting of the following equations:*

For every $n \in \mathbb{N}$, every n-ary operation symbol $\sigma \in \Sigma_n$, and every family of variables $(x_j)_{j\in n} \in V^n$, the operation σ applied to the family $(x_j)_{j\in n}$ is always defined. Formally,

$$\sigma((x_j)_{j\in n}) \stackrel{\mathrm{e}}{=} \sigma((x_j)_{j\in n}). \tag{A0}$$

For every variable $x \in V$, the 0-source and 0-target of x is always defined. Formally,

$$\mathrm{sc}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{tg}^0(x). \tag{A1}$$

For every every variable $x \in V$, we have the following equations:

$$\begin{aligned} \mathrm{sc}^0(\mathrm{sc}^0(x)) &\stackrel{\mathrm{e}}{=} \mathrm{sc}^0(x); & \mathrm{sc}^0(\mathrm{tg}^0(x)) &\stackrel{\mathrm{e}}{=} \mathrm{tg}^0(x);\\ \mathrm{tg}^0(\mathrm{sc}^0(x)) &\stackrel{\mathrm{e}}{=} \mathrm{sc}^0(x); & \mathrm{tg}^0(\mathrm{tg}^0(x)) &\stackrel{\mathrm{e}}{=} \mathrm{tg}^0(x). \end{aligned} \tag{A2}$$

In other words, sc^0 and tg^0 are right zeros. In particular, sc^0 and tg^0 are idempotent.

For every pair of variables $x, y \in V$, $x \circ^0 y$ is defined if and only if the 0-target of y is equal to the 0-source of x. Formally,

$$\begin{aligned} x \circ^0 y \stackrel{\mathrm{e}}{=} x \circ^0 y &\;\rightarrow\; \mathrm{sc}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{tg}^0(y);\\ \mathrm{sc}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{tg}^0(y) &\;\rightarrow\; x \circ^0 y \stackrel{\mathrm{e}}{=} x \circ^0 y. \end{aligned} \tag{A3}$$

For every pair of variables $x, y \in V$, if $x \circ^0 y$ is defined, then the 0-source of $x \circ^0 y$ is that of y and the 0-target of $x \circ^0 y$ is that of x. Formally,

$$x \circ^0 y \stackrel{e}{=} x \circ^0 y \quad \to \quad \mathrm{sc}^0(x \circ^0 y) \stackrel{e}{=} \mathrm{sc}^0(y);$$
$$x \circ^0 y \stackrel{e}{=} x \circ^0 y \quad \to \quad \mathrm{tg}^0(x \circ^0 y) \stackrel{e}{=} \mathrm{tg}^0(x). \tag{A4}$$

For every variable $x \in V$, the compositions $x \circ^0 \mathrm{sc}^0(x)$ and $\mathrm{tg}^0(x) \circ^0 x$ are always defined and are equal to x, i.e., $\mathrm{sc}^0(x)$ is a right unit element for the 0-composition with x and $\mathrm{tg}^0(x)$ is a left unit element for the 0-composition with x. Formally,

$$x \circ^0 \mathrm{sc}^0(x) \stackrel{e}{=} x; \qquad \mathrm{tg}^0(x) \circ^0 x \stackrel{e}{=} x. \tag{A5}$$

For every triple of variables $x, y, z \in V$, if the 0-compositions $x \circ^0 y$ and $y \circ^0 z$ are defined, then the 0-compositions $x \circ^0 (y \circ^0 z)$ and $(x \circ^0 y) \circ^0 z$ are defined and they are equal, i.e., the 0-composition, when defined, is associative. Formally,

$$(x \circ^0 y \stackrel{e}{=} x \circ^0 y) \wedge (y \circ^0 z \stackrel{e}{=} y \circ^0 z) \quad \to \quad (x \circ^0 y) \circ^0 z \stackrel{e}{=} x \circ^0 (y \circ^0 z). \tag{A6}$$

A model of axioms A1–A6 is a category.

For every $n \in \mathbb{N}$, every n-ary operation symbol $\sigma \in \Sigma_n$, and every family of variables $(x_j)_{j \in n} \in V^n$, the 0-source of $\sigma((x_j)_{j \in n})$ is equal to σ applied to the family $((\mathrm{sc}^0(x_j))_{j \in n})$, and the 0-target of $\sigma((x_j)_{j \in n})$ is equal to σ applied to the family $((\mathrm{tg}^0(x_j))_{j \in n})$. Formally,

$$\mathrm{sc}^0(\sigma((x_j)_{j \in n})) \stackrel{e}{=} \sigma((\mathrm{sc}^0(x_j))_{j \in n}); \qquad \mathrm{tg}^0(\sigma((x_j)_{j \in n})) \stackrel{e}{=} \sigma((\mathrm{tg}^0(x_j))_{j \in n}). \tag{A7}$$

For every $n \in \mathbb{N}$, every n-ary operation symbol $\sigma \in \Sigma_n$, and every pair of families of variables $(x_j)_{j \in n}, (y_j)_{j \in n} \in V^n$, if, for every $j \in n$, the 0-compositions $x_j \circ^0 y_j$ are defined, then the 0-composition $\sigma((x_j)_{j \in n}) \circ^0 \sigma((y_j)_{j \in n})$ is defined and it is equal to σ applied to the family $(x_j \circ^0 y_j)_{j \in n}$. Formally,

$$\bigwedge_{j \in n}(x_j \circ^0 y_j \stackrel{e}{=} x_j \circ^0 y_j) \quad \to \quad \sigma((x_j \circ^0 y_j)_{j \in n}) \stackrel{e}{=} \sigma((x_j)_{j \in n}) \circ^0 \sigma((y_j)_{j \in n}) \tag{A8}$$

For every rewrite rule $\mathfrak{p} \in \mathcal{A}$, $\mathfrak{p}$ is always defined. Formally,

$$\mathfrak{p} \stackrel{e}{=} \mathfrak{p}. \tag{A9}$$

We will let $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$ stand for the category canonically associated to the QE*-variety $\mathcal{V}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$ determined by the specification $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}$.*

Another fundamental result of this work is that the two partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$, which is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra in the category $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$, and $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$ are isomorphic.

▶ **Theorem 39** (Th. 16.2.9). *The partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras $[\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$ are isomorphic. As a consequence of Theorem 37, the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras $[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$ are isomorphic.*

C An example

For the sake of illustration, here is an example of the notions defined in this work.

▶ **Example 40.** Consider the signature Σ containing a constant operation symbol $\top$ and a binary operation σ, i.e., $\Sigma_0 = \{\top\}$, $\Sigma_2 = \{\sigma\}$, and $\Sigma_n = \varnothing$, for $n \neq 0, 2$. Let $X = \{x, y\}$ be a set of variables and let $\mathcal{A}$ be the subset of $\mathrm{T}_\Sigma(X)^2$ given by

$$\mathcal{A} = \{\mathfrak{p} = (x, y), \mathfrak{q} = (\sigma(y, y), \top), \mathfrak{r} = (\sigma(\top, y), x)\}$$

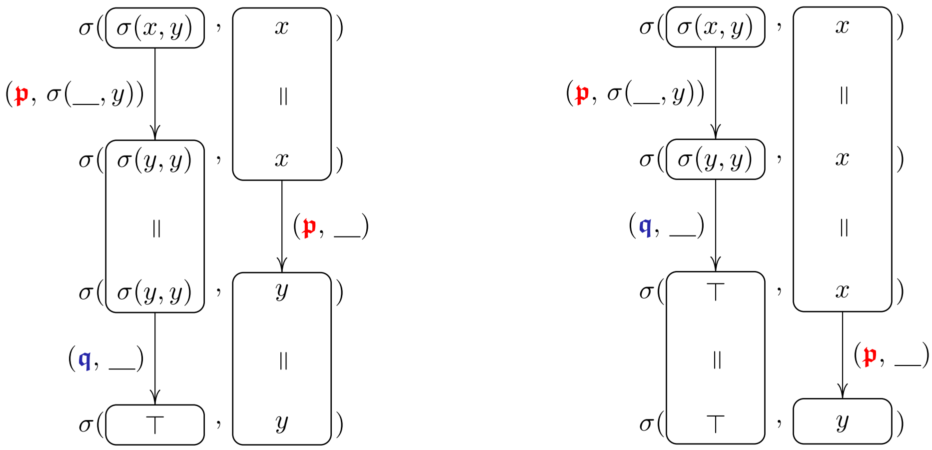
Let $\mathfrak{P}$ be the path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ defined as the following sequence of steps

$$\mathfrak{P}\colon \sigma(\sigma(x,y),x) \xrightarrow{(\mathfrak{p},\sigma(\sigma(\_\_,y),x))} \sigma(\sigma(y,y),x) \xrightarrow{(\mathfrak{p},\sigma(\sigma(y,y),\_\_))} \sigma(\sigma(y,y),y) \xrightarrow{(\mathfrak{q},\sigma(\_\_,y))} \sigma(\top,y) \xrightarrow{(\mathfrak{r},\_\_)} x$$

This is a path in $\boldsymbol{\mathcal{A}}$ of length 5 from $\sigma(\sigma(x,y),x)$ to x. The final one-step subpath $\mathfrak{P}^{3,3}$, from $\sigma(\top,y)$ to x, is equal to $\mathrm{ech}^{(1,\mathcal{A})}(\mathfrak{r})$, the echelon associated with $\mathfrak{r}$. The initial subpath $\mathfrak{P}^{(0,3)}$ is an echelonless path in $\boldsymbol{\mathcal{A}}$ (none of its translations is the identity translation) of length 4 from $\sigma(\sigma(x,y),x)$ to $\sigma(\top,y)$. According to Lemma 10, the initial subpath $\mathfrak{P}^{0,3}$ is head-constant. Note that all the translations of $\mathfrak{P}^{0,3}$ are of type σ. According to Lemma 11, the path extraction algorithm applied to it retrieves two paths in $\mathcal{A}$, that we call $\mathfrak{Q}$ and $\mathfrak{R}$. See Figure 4a.

$$\mathfrak{Q}\colon \sigma(x,y) \xrightarrow{(\mathfrak{p},\sigma(\_\_,y))} \sigma(y,y) \xrightarrow{(\mathfrak{q},\_\_)} \top$$
$$\mathfrak{R}\colon x \xrightarrow{(\mathfrak{p},\_\_)} y$$

Following Proposition 12, we can consider the path $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}(\mathfrak{Q},\mathfrak{R})$. Note that this path is not equal to $\mathfrak{P}^{0,3}$. In it, the transformation follows a leftmost innermost derivation strategy. It is also an echelonless path associated to the operation symbol σ and the path extraction algorithm applied to it retrieves exactly $\mathfrak{Q}$ and $\mathfrak{R}$, see Figure 4b.

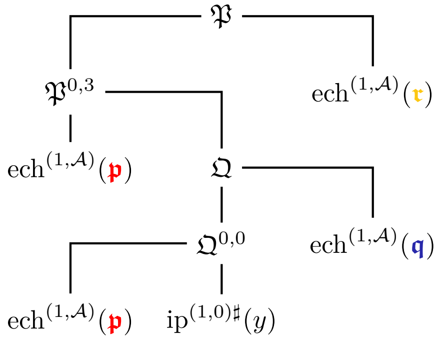


(a) The subpath $\mathfrak{P}^{0,3}$. **(b)** The path $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}(\mathfrak{Q},\mathfrak{R})$.

Figure 4 The path extraction algorithm at use.

Note that $\mathfrak{R}$ is equal to the echelon associated with $\mathfrak{p}$, whilst $\mathfrak{Q}$ is composed of a an echelonless path, namely $\mathfrak{Q}^{0,0}$, followed by the echelon associated with $\mathfrak{q}$. Following Definition 13, the paths in $\boldsymbol{\mathcal{A}}$ that are under $\mathfrak{P}$ with respect to the order $\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$ are depicted in the Hasse diagram of Figure 5a. This process of decomposition ultimately halts, according to Proposition 14, until we reach echelons or identity paths on variables or constants, the minimal elements of the order $\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$.

Next we consider the extended signature $\Sigma^{\boldsymbol{\mathcal{A}}}$, enlarging Σ by adding as constant operation symbols as many rewrite rules as there are in $\mathcal{A}$, two unary operation symbols of source and target, and a new binary operation symbol of composition, i.e., $\Sigma^{\boldsymbol{\mathcal{A}}}_0 = \{\top,\mathfrak{p},\mathfrak{q},\mathfrak{r}\}$, $\Sigma^{\boldsymbol{\mathcal{A}}}_1 = \{\mathrm{sc}^0,\mathrm{tg}^0\}$, $\Sigma_2 = \{\sigma,\circ^0\}$, and $\Sigma^{\boldsymbol{\mathcal{A}}}_n = \varnothing$, for $n \neq 0,1,2$. Following Definition 17 we can define $\mathrm{CH}^{(1)}(\mathfrak{P})$, the image of the Cury-Howard mapping on $\mathfrak{P}$ by recursion on $\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}}}$, as seen in Figure 5b.



$$
\begin{aligned}
\mathrm{CH}^{(1)}(\mathfrak{P}) &= \mathrm{CH}^{(1)}(\mathfrak{P}^{3,3}) \circ^0 \mathrm{CH}^{(1)}(\mathfrak{P}^{0,3})\\
&= \mathfrak{r} \circ^0 \sigma(\mathrm{CH}^{(1)}(\mathfrak{Q}), \mathrm{CH}^{(1)}(\mathfrak{R}))\\
&= \mathfrak{r} \circ^0 \sigma(\mathrm{CH}^{(1)}(\mathfrak{Q}^{1,1}) \circ^0 \mathrm{CH}^{(1)}(\mathfrak{Q}^{0,0}), \mathfrak{p})\\
&= \mathfrak{r} \circ^0 \sigma(\mathfrak{q} \circ^0 \sigma(\mathfrak{p}, y), \mathfrak{p}).
\end{aligned}
$$

(a) Paths under $\mathfrak{P}$ with respect to $\leq_{\mathbf{Pth}_{\mathcal{A}}}$. **(b)** Recursive definition of $\mathrm{CH}^{(1)}(\mathfrak{P})$.

Figure 5 The Curry-Howard mapping.

The Curry-Howard mapping at $\mathfrak{P}$, i.e., $\mathrm{CH}^{(1)}(\mathfrak{P}) = \mathfrak{r} \circ^0 \sigma(\mathfrak{q} \circ^0 \sigma(\mathfrak{p}, y), \mathfrak{p})$ is a term in $\mathrm{T}_{\Sigma^{\mathcal{A}}}(X)$ that contains all the interesting derivation processes occurring in $\mathfrak{P}$.

Now, consider the $\Sigma^{\mathcal{A}}$-homomorphism $\mathrm{ip}^{(1,X)@}$, from $\mathbf{T}_{\Sigma^{\mathcal{A}}}(X)$ to $\mathbf{F}_{\Sigma^{\mathcal{A}}}(\mathbf{Pth}_{\mathcal{A}})$. Note that $\mathrm{ip}^{(1,X)@}$ interprets the operations as in $\mathbf{Pth}_{\mathcal{A}}$ when possible, whilst leaving purely syntactic terms when this is not possible. Since $\mathrm{CH}^{(1)}(\mathfrak{P})$ is, by definition, a path term, the image of it under $\mathrm{ip}^{(1,X)@}$ is, according to Proposition 24 a path, not necessarily equal to $\mathfrak{P}$ but in the same $\mathrm{Ker}(\mathrm{CH}^{(1)})$-class. In fact, $\mathrm{ip}^{(1,X)@}(\mathrm{CH}^{(1)}(\mathfrak{P}))$ denoted by $\mathfrak{P}^{@}$ for simplicity, is given by the path

$$
\mathfrak{P}^{@}\colon \sigma(\sigma(x,y),x) \xrightarrow{(\mathfrak{p},\,\sigma(\sigma(\_\_,y),x))} \sigma(\sigma(y,y),x) \xrightarrow{(\mathfrak{q},\,\sigma(\_\_,x))} \sigma(\top,x) \xrightarrow{(\mathfrak{p},\,\sigma(\top,\_\_))} \sigma(\top,y) \xrightarrow{(\mathfrak{r},\,\_\_)} x
$$

As said above, the paths $\mathfrak{P}$ and $\mathfrak{P}^{@}$ have the same image under the Curry-Howard mapping. One can see that in $\mathfrak{P}^{@}$ all transformations follow a leftmost innermost derivation strategy.

Nevertheless, $\mathrm{CH}^{(1)}(\mathfrak{P})$ is not the unique term to denote paths in $[\mathfrak{P}]$. According to Definition 25, the following term is $\Theta^{[1]}$-related with $\mathrm{CH}^{(1)}(\mathfrak{P})$.

$$
\mathfrak{r} \circ^0 \sigma(\mathfrak{q}, y) \circ^0 \sigma(\sigma(y,y), \mathfrak{p}) \circ^0 \sigma(\sigma(\mathfrak{p}, y), x)
$$

Thus, following Lemma 27, the above term is a path term, i.e., an alternative term description of the path class $[\mathfrak{P}]$. In fact, when mapped to a path in $\mathrm{Pth}_{\mathcal{A}}$ under the action of $\mathrm{ip}^{(1,X)@}$, the above term retrieves precisely the original path $\mathfrak{P}$.

The isomorphism $\mathrm{CH}^{[1]}$, introduced in Theorem 37, maps the path class $[\mathfrak{P}]$ to the path term class $[\mathrm{CH}^{(1)}(\mathfrak{P})]$, whilst its inverse, i.e., $\mathrm{ip}^{([1],X)@}$ will map $[\mathrm{CH}^{(1)}(\mathfrak{P})]$ to $[\mathfrak{P}]$.