From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms

Second-order rewriting systems

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Abstract

Following the homotopic dictum, we define the set of second-order paths, i.e., paths on paths, associated with a second-order rewriting system and equip it with a structure of partial algebra, a structure of category, and a structure of Artinian ordered set. Next, we consider an extension of the signature associated with the second-order rewriting system and we associate each second-order path with a term in the extended signature. This constitutes a second-order Curry-Howard type mapping. After that we prove that a refined quotient of the set of second-order paths by the kernel of the second-order Curry-Howard mapping is equipped with a structure of partial algebra, a structure of 2-category, and a structure of Artinian preordered set. Following this we identify a subquotient of the free term algebra in the extended signature that is isomorphic to the algebraic, categorical, and ordered structures on the quotient of second-order paths. This constitutes a second-order Curry-Howard type isomorphism. Additionally, we prove that these two structures are isomorphic to the free partial algebra on second-order paths in a variety of partial algebras for the extended signature.

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1 Introduction

In the first part of [9] we introduced $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ the set of paths associated with a rewriting system $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$ and equipped it with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, a structure of category, and a structure of Artinian ordered set. We associated each path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ with a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. This constituted a Curry-Howard type mapping [10, 17]. After that we proved that the quotient $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$ is equipped with a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, a structure of category, and a structure of Artinian ordered set. Following this we identified $[\mathrm{PT}_{\Sigma^{\boldsymbol{\mathcal{A}}}}]$, a subquotient of $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, that is isomorphic to the algebraic, categorical, and ordered structures on $[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}}]$. This constituted a Curry-Howard type isomorphism. Additionally, we proved that these two structures are isomorphic to $\mathbf{T}_{\mathcal{E}^{\boldsymbol{\mathcal{A}}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}})$, the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra in $\mathbf{PAlg}(\mathcal{E}^{\boldsymbol{\mathcal{A}}})$, a variety of partial algebras.

What we present here is the second part of the ongoing project presented in [9]. This time, we delve into second-order rewriting systems and second-order paths, exploring how to establish a second-order Curry-Howard type result. This work is the preliminary development of a theory aimed at defining the notions of higher-order many-sorted rewriting systems and higher-order many-sorted categorial algebras and investigating the relationship between them through higher-order many-sorted Curry-Howard isomorphisms. The interested reader can consult all the proofs presented in this article in [9]. Next to each result, the reader will find the corresponding reference. The notation from the first part will be assumed. Recall that our work is framed in the study of syntactic derivation systems in the context of many-sorted algebras. Nevertheless, to facilitate comprehension, in this paper we have opted to present the single-sorted version of our findings. The only prerequisites for reading this work are familiarity with category theory [16, 18], universal algebra [1, 4, 5, 6, 7, 14, 15, 15, 21, 22, 24], the theory of ordered sets [2, 11] and set theory [3, 13]. For historical roots of rewriting theory we refer to [12, 19, 20, 23].

2 First-order translations

For the many-sorted partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, we introduce the concepts of elementary first-order translation and of first-order translation respect to it.

▶ **Definition 1.** *We will denote by* $\mathrm{Etl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *the subset of* $\mathrm{Hom}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X), \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *defined as follows: for every mapping* $T^{(1)} \in \mathrm{Hom}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X), \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$, $T^{(1)} \in \mathrm{Etl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *if and only if one of the following conditions holds*

1. *There is a natural number* $n \in \mathbb{N} - 1$, *an index* $k \in n$, *an* n*-ary operation symbol* $\sigma \in \Sigma_n$, *a family of paths* $(\mathfrak{P}_j)_{j\in k} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}^{k}$ *and a family of paths* $(\mathfrak{P}_l)_{l\in n-(k+1)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}^{n-(k+1)}$ *such that, for every* $P \in \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$
$$T^{(1)}(P) = \sigma^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}_0), \ldots, \mathrm{CH}^{(1)}(\mathfrak{P}_{k-1}), P, \mathrm{CH}^{(1)}(\mathfrak{P}_{k+1}), \ldots, \mathrm{CH}^{(1)}(\mathfrak{P}_{n-1}));$$
2. *There is a path* $\mathfrak{P} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}}$ *such that, for every* $P \in \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$,
$$T^{(1)}(P) = P \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} \mathrm{CH}^{(1)}(\mathfrak{P}) \qquad \text{or} \qquad T^{(1)}(P) = \mathrm{CH}(\mathfrak{P}) \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)} P.$$

We will sometimes add an underlined space to denote where the variable will be placed. In the first case we will say that $T^{(1)}$ *is of type* σ, *while in the second case we will say that* $T^{(1)}$ *is of type* $\circ^0$. *We will call the elements of* $\mathrm{Etl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *the* first-order elementary translations for $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$.

We will denote by $\mathrm{Tl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *the subset of* $\mathrm{Hom}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X), \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *defined as follows: for every mapping* $T^{(1)} \in \mathrm{Hom}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X), \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$, $T^{(1)} \in \mathrm{Tl}_t(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *if and only if there is an* $m \in \mathbb{N} - 1$ *and a family* $(T_j^{(1)})_{j\in m}$ *of first-order elementary translations in* $\mathrm{Etl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))^m$ *for which* $T^{(1)} = T_{m-1}^{(1)} \circ \cdots \circ T_0^{(1)}$. *We will call the elements of* $\mathrm{Tl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *the* first-order translations for $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$. *The notions of prefix, height and type of a first-order translation is defined as in the first part. Besides the mapping* $\mathrm{id}_{\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}$ *will be viewed as an element of* $\mathrm{Tl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$.

The following results will be useful to justify the definition of second-order paths.

▶ **Lemma 2** (Lemma 17.0.3)**.** *Let* $T^{(1)}$ *be a first-order translation in* $\mathrm{Tl}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *and* M, N *path terms in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ *such that* $(\mathrm{ip}^{(1,X)@}(M), \mathrm{ip}^{(1,X)@}(N)) \in \mathrm{Ker}(\mathrm{sc}^{(0,1)}) \cap \mathrm{Ker}(\mathrm{tg}^{(0,1)})$. *Then*

1. $T^{(1)}(M)$ *is a path term in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ *if, and only if,* $T^{(1)}(N)$ *is a path term in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$;
2. *If either* $T^{(1)}(M)$ *or* $T^{(1)}(N)$ *is a path term in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$, *then*
$$(\mathrm{ip}^{(1,X)@}(T^{(1)}(M)), \mathrm{ip}^{(1,X)@}(T^{(1)}(N))) \in \mathrm{Ker}(\mathrm{sc}^{(0,1)}) \cap \mathrm{Ker}(\mathrm{tg}^{(0,1)}).$$

▶ **Proposition 3** (Prop. 17.0.4)**.** *Let* $T^{(1)}$ *be a first-order translation in* $\mathrm{Tl}(\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))$ *and* P, P' *path terms in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$ *such that* $(P, P') \in \Theta^{[1]}$. *If either* $T^{(1)}(P)$ *or* $T^{(1)}(P')$ *is a path term in* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}}$, *then* $(T^{(1)}(P), T^{(1)}(P')) \in \Theta^{[1]}$.

3 Second-order paths on path term classes

In this section we begin by defining the notion of second-order rewriting system.

▶ **Definition 4.** *A* second-order rewriting system *is an ordered quadruple* $(\Sigma, X, \mathcal{A}, \mathcal{A}^{(2)})$, *often abbreviated to* $\boldsymbol{\mathcal{A}}^{(2)}$, *where* $(\Sigma, X, \mathcal{A})$ *is a rewriting system, and, for the signature* $\Sigma^{\boldsymbol{\mathcal{A}}}$, $\mathcal{A}^{(2)}$ *a subset of* $\mathrm{Rwr}(\Sigma^{\boldsymbol{\mathcal{A}}}, X)$, *the set of* second-order rewrite rules with variables in X, *i.e.,*

$$\{([M],[N]) \in [\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]^2 \mid (\mathrm{ip}^{(1,X)@}(M), \mathrm{ip}^{(1,X)@}(N)) \in \mathrm{Ker}(\mathrm{sc}^{(0,1)}) \cap \mathrm{Ker}(\mathrm{tg}^{(0,1)})\},$$

We will call the elements of $\mathrm{Rwr}(\Sigma^{\boldsymbol{\mathcal{A}}}, X)$ second-order rewrite rules of type X *and we will denote them with lowercase Euler fraktur letters with the superscript* (2), *indicating the order, with or without subscripts, e.g.,* $\mathfrak{p}^{(2)}$, $\mathfrak{p}_i^{(2)}$, $\mathfrak{q}^{(2)}$, $\mathfrak{q}_i^{(2)}$, *etc.*

We next define the notion of second-order path in $\boldsymbol{\mathcal{A}}^{(2)}$ from a path term class to another.

▶ **Definition 5.** *Let* $[P]_s, [Q]_s$ *be path term classes in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *and* $m \in \mathbb{N}$. *Then a* second-order m-path in $\boldsymbol{\mathcal{A}}^{(2)}$ from $[P]$ to $[Q]$ *is an ordered triple* $\mathfrak{P}^{(2)} = (([P_i])_{i\in m+1}, (\mathfrak{p}_i^{(2)})_{i\in m}, (T_i^{(1)})_{i\in m})$ *in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]^{m+1} \times (\mathcal{A}^{(2)})^m \times \mathrm{Tl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))^m$, *such that*

1. $[P_0] = [P]$, $[P_m]_s = [Q]$, *and,*
2. *for every* $i \in m$, *if* $\mathfrak{p}_i^{(2)} = ([M_i],[N_i])$, *then* $T_i^{(1)}(M_i) \in [P_i]$ *and* $T_i^{(1)}(N_i) \in [P_{i+1}]$.

That is, at each step $i \in m$, *we consider a second-order rewrite rule* $\mathfrak{p}_i^{(2)}$ *and a first-order translation of sort* $T_i^{(1)}$ *for* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ *and we require that the translation by* $T_i^{(1)}$ *of* M_i *is in* $[P_i]$, *whilst the translation by* $T_i^{(1)}$ *of* N_i *is in* $[P_{i+1}]$. *In this regard, let us recall that, by Lemma 2, the condition on the* $(0,1)$*-source and* $(0,1)$*-target we have imposed in Definition 4 guarantees that the values of the first-order translations will always be path terms. Moreover, according to Proposition 3, ultimately, it does not matter which representative term we use.*

These second-order paths will be variously depicted as $\mathfrak{P}^{(2)}\colon [P] \Longrightarrow [Q]$, *or*

$$\mathfrak{P}^{(2)}\colon [P_0] \xRightarrow{(\mathfrak{p}_0^{(2)}, T_0^{(1)})} [P_1] \xRightarrow{(\mathfrak{p}_1^{(2)}, T_1^{(1)})} \dots [P_{m-2}] \xRightarrow{(\mathfrak{p}_{m-2}^{(2)}, T_{m-2}^{(1)})} [P_{m-1}] \xRightarrow{(\mathfrak{p}_{m-1}^{(2)}, T_{m-1}^{(1)})} [P_m]$$

For every $i \in m$, *we will say that* $[P_{i+1}]$ *is* $(\mathfrak{p}_i^{(2)}, T_i^{(1)})$-directly derivable *or, when no confusion can arise,* directly derivable*from* $[P_i]$. *For every* $i \in m+1$, *the path term class* $[P_i]$ *will be called a* 1-constituent *of the second-order* m*-path* $\mathfrak{P}^{(2)}$. *The path term class* $[P_0]$ *will be called the* $([1],2)$-source *of the second-order* m*-path* $\mathfrak{P}^{(2)}$, *whilst the path term class* $[P_m]$ *will be called the* $([1],2)$-target. *We will say that* $\mathfrak{P}^{(2)}$ *is a* second-order path from $[P_0]$ to $[P_m]$. *The* length *of a second-order* m*-path* $\mathfrak{P}^{(2)}$ *in* $\boldsymbol{\mathcal{A}}^{(2)}$, *denoted by* $|\mathfrak{P}^{(2)}|$, *is* m *and we will say that* $\mathfrak{P}^{(2)}$ *has* m steps. *If* $\mathfrak{P}^{(2)} = 0$, *then we will say that* $\mathfrak{P}^{(2)}$ *is a* $(2,[1])$-identity second-order path. *This happens if, and only if, there exists a path term class* $[P]$ *in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *such that* $\mathfrak{P} = (([P]),\lambda,\lambda)$, *identified to* $([P],\lambda,\lambda)$, *where, by abuse of notation, we have written* (λ,λ) *for the unique element of* $(\mathcal{A}^{(2)})^0 \times \mathrm{Tl}(\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X))^0$. *This path will be called the* $(2,[1])$-identity second-order path on $[P]$. *If* $|\mathfrak{P}^{(2)}| = 1$, *then we will say that* $\mathfrak{P}^{(2)}$ *is a* one-step second-order path. *We will denote by* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *the set of all possible second-order paths in* $\boldsymbol{\mathcal{A}}^{(2)}$. *We define the mappings*

1. $\mathrm{ip}^{(2,X)}$ *the mapping from* X *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *that sends* $x \in X$ *to* $([\eta^{(1,X)}(x)],\lambda,\lambda)$; *by*
2. $\mathrm{ech}^{(2,\mathcal{A})}$ *the mapping from* $\mathcal{A}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *that sends* $\mathfrak{p} \in \mathcal{A}$ *to* $([\eta^{(1,\mathcal{A})}(\mathfrak{p})],\lambda,\lambda)$; *by*
3. $\mathrm{sc}^{([1],2)}$ *the mapping that sends a second-order path to its* $([1],2)$*-source; by*
4. $\mathrm{tg}^{([1],2)}$ *the mapping that sends a second-order path to its* $([1],2)$*-target; and by*
5. $\mathrm{ip}^{(2,[1])\sharp}$ *the mapping that sends a path term class* $[P]$ *to* $([P],\lambda,\lambda)$.

These mappings are depicted in the diagram of Figure 1a.

We next define the partial operation of 1-composition of paths.

▶ **Definition 6.** *Let* $\mathfrak{P}^{(2)}$, $\mathfrak{Q}^{(2)}$ *be second-order paths in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *such that*

$$\mathrm{sc}^{([1],2)}(\mathfrak{Q}^{(2)}) = \mathrm{tg}^{([1],2)}(\mathfrak{P}^{(2)}).$$

Then the 1-composite *of* $\mathfrak{P}^{(2)}$ *and* $\mathfrak{Q}^{(2)}$, *denoted by* $\mathfrak{Q}^{(2)} \circ^1 \mathfrak{P}^{(2)}$, *is the concatenation of the respective sequences. When defined,* $\mathfrak{Q}^{(2)} \circ^1 \mathfrak{P}^{(2)}$ *is a second-order path in* $\boldsymbol{\mathcal{A}}^{(2)}$ *from* $\mathrm{sc}^{([1],2)}(\mathfrak{P}^{(2)})$ *to* $\mathrm{tg}^{([1],2)}(\mathfrak{Q}^{(2)})$. *Moreover, when defined, the partial operation of* 1*-composition is associative and, for every path term class* $[P] \in [\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$, *the* $([2], 1)$*-identity second-order path on* $[P]$ *is, when defined, a neutral element for the operation of* 1*-composition. The above definition gives rise to a category whose objects are path term classes in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$ *and whose morphisms are second-order paths* $\mathfrak{P}^{(2)}$ *between path term classes.*

We next define the notion of subpath of a second-order path.

▶ **Definition 7.** *Let* $m \in \mathbb{N}$, *and* $k, l \in m$ *with* $k \leq l$. *Let* $\mathfrak{P}^{(2)}$ *be a second-order* m*-path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. *We denote by* $\mathfrak{P}^{(2),k,l}$ *the* subpath of $\mathfrak{P}^{(2)}$ beginning at position k and ending at position $l+1$.

We introduce the notion of second-order echelon, a key concept in the development of our theory.

▶ **Definition 8.** *We denote by* $\mathrm{ech}^{(2,\mathcal{A}^{(2)})}$ *the mapping from* $\mathcal{A}^{(2)}$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *defined as follows:*

$$\mathrm{ech}^{(2,\mathcal{A}^{(2)})}\begin{cases} \mathcal{A}^{(2)} & \longrightarrow & \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}} \\ \mathfrak{p}^{(2)} = ([M],[N]) & \longmapsto & (([M],[N]), \mathfrak{p}^{(2)}, \mathrm{id}_{\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}) \end{cases}$$

This mapping associates to each second-order rewrite rule $\mathfrak{p}^{(2)} = ([M],[N])$ *in* $\mathcal{A}^{(2)}$ *the one-step second-order path from* $[M]$ *to* $[N]$ *that uses the second-order rewrite rule* $\mathfrak{p}^{(2)}$ *in the identity translation. This definition is sound because (1)* $\mathrm{id}_{\mathrm{T}_\Sigma(X)}(M) \in [M]$ *and (2)* $\mathrm{id}_{\mathrm{T}_\Sigma(X)}(N) \in [N]$. *We will call* $\mathrm{ech}^{(2,\mathcal{A}^{(2)})}(\mathfrak{p}^{(2)})$ *the* second-order echelon associated to $\mathfrak{p}^{(2)}$. *Moreover, we will say that a second-order path* $\mathfrak{P}^{(2)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *is a* second-order echelon *if there exists a second-order rewrite rule* $\mathfrak{p}^{(2)} \in \mathcal{A}^{(2)}$ *such that* $\mathrm{ech}^{(2,\mathcal{A}^{(2)})}(\mathfrak{p}^{(2)}) = \mathfrak{P}^{(2)}$. *Finally, we will say that a second-order path* $\mathfrak{P}^{(2)}$ *is* echelonless *if* $|\mathfrak{P}^{(2)}| \geq 1$ *and none of its one-step subpaths is a second-order echelon.*

From the above it follows that the first-order translations of an echelonless second-order path must be non-identity translations. We next introduce the notion of a head-constant echelonless second-order path.

▶ **Definition 9.** *Let* $\mathfrak{P}^{(2)} = \left(([P_i])_{i\in m+1}, ((\mathfrak{p}_i^{(2)})_{i\in m}, (T_i^{(1)})_{i\in m}\right)$ *be an echelonless second-order path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. *We will say that* $\mathfrak{P}^{(2)}$ *is a* head-constant *echelonless second-order path if* $(T_i^{(1)})_{i\in m}$, *the family of first-order translations occurring in it, have the same type, i.e., they are associated to the same operation symbol.*

Unlike in the first part of this work, an echelonless second-order path does not necessarily goes across head-constant families of first-order translations. This is shown in Example 41. We next introduce the notion of coherent head-constant echelonless second-order path.

▶ **Definition 10.** *Let* $m \in \mathbb{N} - 1$ $\mathfrak{P}^{(2)}$ *be a head-constant echelonless second-order* m*-path in* $\mathrm{Pth}_{\mathcal{A}^{(2)}}$ *of the form* $\mathfrak{P}^{(2)} = \left(([P_i]_s)_{i\in m+1}, (\mathfrak{p}_i^{(2)})_{i\in m}, (T_i^{(1)})_{i\in m}\right)$ *where, for a unique* $n \in \mathbb{N}$

and a unique n-ary operation symbol $\tau \in \Sigma_n^{\boldsymbol{\mathcal{A}}}$, the family $(T_i^{(1)})_{i\in m}$ is a family of first-order translations of type τ. That is,

$$T_i^{(1)} = \tau^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}_{i,0}), \cdots, \mathrm{CH}^{(1)}(\mathfrak{P}_{i,k_i-1}), T_i^{(1)'}, \mathrm{CH}^{(1)}(\mathfrak{P}_{i,k_i+1}), \cdots, \mathrm{CH}^{(1)}(\mathfrak{P}_{i,n-1})).$$

Assume that, for every $i \in m$, the second-order rewrite rule $\mathfrak{p}_i^{(2)}$ is given by $([M_i], [N_i])$. Hence, for every $i \in m$, $T_i^{(1)}(M_i) \in [P_i]$ and $T_i^{(1)}(N_i) \in [P_{i+1}]$ In particular, for every $i \in m-1$, $[T_i^{(1)}(N_i)] = [T_{i+1}^{(1)}(M_{i+1})]$. That is, the following equality holds

$$\begin{aligned}&[\tau^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}_{i,0}), \cdots, T_i^{(1)'}(N_i), \cdots, \mathrm{CH}^{(1)}(\mathfrak{P}_{i,n-1}))]\\ &\qquad = [\tau^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)}(\mathrm{CH}^{(1)}(\mathfrak{P}_{i+1,0}), \cdots, T_{i+1}^{(1)'}(M_{i+1}), \cdots, \mathrm{CH}^{(1)}(\mathfrak{P}_{i+1,n-1}))].\end{aligned}$$

We will say that $\mathfrak{P}^{(2)}$ is coherent *if, for every $i \in m-1$, from the above equality, we can derive the following n equalities*

$$\begin{aligned}&[\mathrm{CH}^{(1)}(\mathfrak{P}_{i,0})] = [\mathrm{CH}^{(1)}(\mathfrak{P}_{i+1,0})], \quad \cdots \quad [T_i^{(1)'}(N_i)] = [\mathrm{CH}^{(1)}(\mathfrak{P}_{i+1,k_i})],\\ &[\mathrm{CH}^{(1)}(\mathfrak{P}_{i,k_{i+1}})] = [T_{i+1}^{(1)'}(M_{i+1})] \quad \cdots \quad [\mathrm{CH}^{(1)}(\mathfrak{P}_{i,n-1})] = [\mathrm{CH}^{(1)}(\mathfrak{P}_{i+1,n-1})].\end{aligned}$$

Example 41 illustrates two head-constant echelonless second-order paths: one exhibiting coherence, while the other does not. For a coherent head-constant echelonless second-order path, we propose a process of second-order path extraction. We will refer to it as the *secod-order path extraction algorithm.*

▶ **Lemma 11** (Lemma 18.1.9)**.** *Let $\mathfrak{P}^{(2)} = (([P_i])_{i\in m+1}, (\mathfrak{p}_i^{(2)})_{i\in m}, (T_i^{(2)})_{i\in m})$ be coherent head-constant echelonless second-order path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. Let τ be the unique n-ary operation symbol in $\Sigma_n^{\boldsymbol{\mathcal{A}}}$ for which each of the first-order translations of the family $(T_i)_{i\in m}$ is of type τ. Then there exists a unique pair $((m_j)_{j\in n}, (\mathfrak{P}_j^{(2)})_{j\in n}) \in \mathbb{N}^n \times \mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}^{(2)}}$ such that, for every $j \in n$, $\mathfrak{P}_j^{(2)}$ is a second-order m_j-path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ and there exists a unique bijective mapping $i\colon \coprod_{j\in n} m_j \longrightarrow m$ such that, for every (j,k) in $\coprod_{j\in n} m_j$, $\mathfrak{p}_{j,k}^{(2)} = \mathfrak{p}_{i(j,k)}^{(2)}$.*

3.1 Algebraic structure on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$

The next proposition states that every 1-constituent that traverses a second-order path, when interpreted as a path by means of the $\mathrm{ip}^{(1,x)@}$, have the same $(0,1)$-source and $(0,1)$-target.

▶ **Proposition 12** (Prop. 19.2.3)**.** *Let $m \in \mathbb{N}$ and $\mathfrak{P}^{(2)}$ a second-order m-path in $\boldsymbol{\mathcal{A}}^{(2)}$ of the form $\mathfrak{P}^{(2)} = (([P_i])_{i\in m+1}, (\mathfrak{p}_i^{(2)})_{i\in m}, (T_i^{(1)})_{i\in m})$. Then, for every $i, j \in m+1$, we have that*

$$(\mathrm{ip}^{(1,X)@}(P_i), \mathrm{ip}^{(1,X)@}(P_j)) \in \mathrm{Ker}(\mathrm{sc}^{(0,1)}) \cap \mathrm{Ker}(\mathrm{tg}^{(0,1)}).$$

The last proposition justifies the definition of the $(0,2)$-source and the $(0,2)$-target of a second-order path. We will also introduce the $(2,0)$-identity second-order path on a term.

▶ **Definition 13.** *We will denote by*

1. $\mathrm{sc}^{(0,2)}$ *the mapping* $\mathrm{sc}^{(0,[1])} \circ \mathrm{ip}^{([1],X)@} \circ \mathrm{sc}^{([1],2)}$ *from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to* $\mathrm{T}_\Sigma(X)$*; by*
2. $\mathrm{tg}^{(0,2)}$ *the mapping* $\mathrm{tg}^{(0,[1])} \circ \mathrm{ip}^{([1],X)@} \circ \mathrm{tg}^{([1],2)}$ *from* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to* $\mathrm{T}_\Sigma(X)$*; and by*
3. $\mathrm{ip}^{(2,0)\sharp}$ *the mapping* $\mathrm{ip}^{(2,[1])\sharp} \circ \mathrm{CH}^{[1]} \circ \mathrm{ip}^{([1],0)\sharp}$ *from* $\mathrm{T}_\Sigma(X)$ *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$.

These mappings are depicted in the diagram of Figure 1a.

We next define a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra in the set $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$.

▶ **Proposition 14** (Prop. 19.2.10). *The set* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}}$*-algebra.*

Proof. Let us denote by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ the partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra defined on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ as follows. For every n-ary operation symbol $\sigma \in \Sigma_n$, the operation $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$, from $\mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}^{(2)}}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, assigns to a family of second-order paths $(\mathfrak{P}^{(2)}_j)_{j\in n} \in \mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}^{(2)}}$ the second-order path $\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}((\mathfrak{P}^{(2)}_j)_{j\in n})$, defined in a similar way as in the first part. For every rewrite rule $\mathfrak{p} \in \mathcal{A}$, we define $\mathfrak{p}^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$ to be equal to $\mathrm{ech}^{(2,\mathcal{A})}(\mathfrak{p})$, i.e., the $(2,[1])$-identity second-order path on $[\mathfrak{p}^{\mathbf{PT}_{\boldsymbol{\mathcal{A}}}}]$. The 0-source operation symbol sc^0 is interpreted as the unary operation, from $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ to $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, that maps a second-order path $\mathfrak{P}^{(2)}$ in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ to $\mathrm{ip}^{(2,0)\sharp}(\mathrm{sc}^{(0,2)}(\mathfrak{P}^{(2)}))$, the $(2,0)$-identity second-order path on the $(0,2)$-source of $\mathfrak{P}^{(2)}$. The interpretation of the 0-target operation symbol is defined analogously. We will focus our attention on the interpretation of $\circ^0$, the 0-composition operation symbol, as a partial binary operation on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. For two second-order paths of the form $\mathfrak{P}^{(2)} = (([P_i])_{i\in m_1+1}, (\mathfrak{p}^{(2)}_i)_{i\in m_1}, (T^{(1)}_i)_{i\in m_1})$, and $\mathfrak{Q}^{(2)} = (([Q_j])_{j\in m_2+1}, (\mathfrak{q}^{(2)}_j)_{j\in m_2}, (U^{(1)}_j)_{j\in m_2})$, its 0-composition, defined whenever $\mathrm{sc}^{(0,2)}(\mathfrak{Q}^{(2)}) = \mathrm{tg}^{(0,2)}(\mathfrak{P}^{(2)})$, is given by the second-order m-path in $\boldsymbol{\mathcal{A}}^{(2)}$ given by

$$\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)} = (([R_k])_{k\in m+1}, (\mathfrak{r}^{(2)}_k)_{k\in m}(V^{(1)}_k)_{k\in m}),$$

where $m = m_1 + m_2$, and

$$[R_k] = \begin{cases} [Q_0 \circ^{0\mathbf{PT}_{\boldsymbol{\mathcal{A}}}} P_k], & \text{if } k \in m_1+1; \\ [Q_{k-m_1} \circ^{0\mathbf{PT}_{\boldsymbol{\mathcal{A}}}} P_{m_1}], & \text{if } k \in [m_1+1, m+1], \end{cases}$$

$$\mathfrak{r}^{(2)}_k = \begin{cases} \mathfrak{p}^{(2)}_k, & \text{if } k \in m_1; \\ \mathfrak{q}^{(2)}_{k-m_1}, & \text{if } k \in [m_1, m], \end{cases}$$

$$V^{(1)}_k = \begin{cases} \mathrm{CH}^{(1)}(\mathrm{ip}^{(1,X)@}(Q_0)) \circ^{0\mathbf{PT}_{\boldsymbol{\mathcal{A}}}} T^{(1)}_k, & \text{if } k \in m_1; \\ U^{(1)}_{k-m_1} \circ^{0\mathbf{PT}_{\boldsymbol{\mathcal{A}}}} \mathrm{CH}^{(1)}(\mathrm{ip}^{(1,X)@}(P_{m_1})), & \text{if } k \in [m_1, m], \end{cases}$$ ◀

It can be shown that $\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}$ is a path in $\boldsymbol{\mathcal{A}}^{(2)}$ of the form

$$\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)} \colon \mathrm{sc}^{([1],2)}(\mathfrak{Q}^{(2)}) \circ^{0[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]} \mathrm{sc}^{([1],2)}(\mathfrak{P}^{(2)}) \Longrightarrow \mathrm{tg}^{([1],2)}(\mathfrak{Q}^{(2)}) \circ^{0[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]} \mathrm{tg}^{([1],2)}(\mathfrak{P}^{(2)}).$$

Moreover, if $\mathfrak{Q}^{(2)}$ or $\mathfrak{P}^{(2)}$ is a non-$(2,[1])$-identity second-order path, then $\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}$ is an coherent head-constant echelonless second-order path. Furthermore, the second-order path extraction algorithm applied to it retrieves $\mathfrak{P}^{(2)}$ and $\mathfrak{Q}^{(2)}$. Let us note that the set $\mathrm{ip}^{(2,[1])\sharp}[[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]]$, of $(2,[1])$-identity second-order paths, becomes a partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-subalgebra of $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ in the QE-variety $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}})$, the mappings $\mathrm{sc}^{([1],2)}$ and $\mathrm{tg}^{([1],2)}$ become $\Sigma^{\boldsymbol{\mathcal{A}}}$-homomorphisms and the mapping $\mathrm{ip}^{(2,[1])\sharp}$ is a $\Sigma^{\boldsymbol{\mathcal{A}}}$-homomorphism that can be obtained by the universal property of $\mathbf{T}_\Sigma(X)$ on $\mathrm{ip}^{(2,[1])}$, see Figure 1a.

3.2 Order structure on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$

In this subsection we define on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ an Artinian order, which will allow us to justify both proofs by Artinian induction and definitions by Artinian recursion.

▶ **Definition 15.** *We let* $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$ *denote the binary relation on* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *consisting of the ordered pairs* $(\mathfrak{Q}^{(2)}, \mathfrak{P}^{(2)}) \in \mathrm{Pth}^2_{\boldsymbol{\mathcal{A}}^{(2)}}$ *for which one of the following conditions holds*

1. $\mathfrak{P}^{(2)}$ *and* $\mathfrak{Q}^{(2)}$ *are* $(2,[1])$*-identity second-order paths of the form* $\mathfrak{P}^{(2)} = \mathrm{ip}^{(2,[1])\sharp}([P])$ *and* $\mathfrak{Q}^{(2)} = \mathrm{ip}^{(2,[1])\sharp}([Q])$ *for some path term classes* $[P],[Q] \in [\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$*, and* $[Q] <_{[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]} [P]$*, where* $\leq_{[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]}$ *is the Artinian partial order introduced in the first part of this work.*

2. $\mathfrak{P}^{(2)}$ *is a second-order path of length* m *strictly greater than one containing at least one second-order echelon, and if its first second-order echelon occurs at position* $i \in m$*, then*
   a. *if* $i = 0$*, then* $\mathfrak{Q}^{(2)}$ *is equal to* $\mathfrak{P}^{(2),0,0}$ *or* $\mathfrak{P}^{(2),1,m-1}$*,*
   b. *if* $i > 0$*, then* $\mathfrak{Q}^{(2)}$ *is equal to* $\mathfrak{P}^{(2),0,i-1}$ *or* $\mathfrak{P}^{(2),i,m-1}$*;*
3. $\mathfrak{P}^{(2)}$ *is an echelonless second-order path, then*
   a. *if* $\mathfrak{P}^{(2)}$ *is not head-constant, then let* $i \in m$ *be the maximum index for which* $\mathfrak{P}^{(2),0,i}$ *is a head-constant second-order path, then* $\mathfrak{Q}^{(2)}$ *is equal to* $\mathfrak{P}^{(2),0,i}$ *or* $\mathfrak{P}^{(2),i+1,m-1}$*;*
   b. *if* $\mathfrak{P}^{(2)}$ *is head-constant but not coherent, then let* $i \in m$ *be the maximum index for which* $\mathfrak{P}^{(2),0,i}$ *is a coherent second-order path, then* $\mathfrak{Q}^{(2)}$ *is equal to* $\mathfrak{P}^{(2),0,i}$ *or* $\mathfrak{P}^{(2),i+1,m-1}$*;*
   c. *if* $\mathfrak{P}^{(2)}$ *is head-constant and coherent then* $\mathfrak{Q}^{(2)}$ *is one of the second-order paths we can extract from* $\mathfrak{P}^{(2)}$ *in virtue of Lemma 11.*

We will denote by $\leq_{\mathbf{Pth}_{\mathcal{A}^{(2)}}}$ *the reflexive and transitive closure of* $\prec_{\mathbf{Pth}_{\mathcal{A}^{(2)}}}$*, i.e., the preorder on* $\mathrm{Pth}_{\mathcal{A}^{(2)}}$ *generated by* $\prec_{\mathbf{Pth}_{\mathcal{A}^{(2)}}}$.

For the preordered set $(\mathrm{Pth}_{\mathcal{A}^{(2)}}, \leq_{\mathbf{Pth}_{\mathcal{A}^{(2)}}})$ it can be shown that the minimal elements are the $(2,[1])$-identity second-order paths on minimal elements in $\mathrm{PT}_{\mathcal{A}}$, i.e., variables, constants, and echelons, and the second-order echelons. The most important feature of this relation is that it is antisymmetric and there is not any strictly decreasing ω_0-chain.

▶ **Proposition 16** (Prop. 20.0.14)**.** $(\mathrm{Pth}_{\mathcal{A}^{(2)}}, \leq_{\mathbf{Pth}_{\mathcal{A}^{(2)}}})$ *is an Artinian ordered set.*

4 The second-order Curry-Howard mapping

In this section we define a new signature, the categorial signature determined by $\boldsymbol{\mathcal{A}}^{(2)}$.

▶ **Definition 17.** *The* categorial signature determined by $\mathcal{A}^{(2)}$ on Σ*, denoted by* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*, is the signature defined, for every* $n \in \mathbb{N}$*, as follows:*

$$\Sigma_n^{\boldsymbol{\mathcal{A}}^{(2)}} = \begin{cases} \Sigma_n, & \text{if } n \neq 0,1,2; \\ \Sigma_0 \amalg \mathcal{A} \amalg \mathcal{A}^{(2)}, & \text{if } n = 0; \\ \Sigma_1 \amalg \{\mathrm{sc}^0, \mathrm{tg}^0, \mathrm{sc}^1, \mathrm{tg}^1\}, & \text{if } n = 1; \\ \Sigma_2 \amalg \{\circ^0, \circ^1\}, & \text{if } n = 2. \end{cases}$$

That is, $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$ *is the expansion of* $\Sigma^{\boldsymbol{\mathcal{A}}}$ *obtained by adding, (1) as many constants as there are second-order rewrite rules in* $\mathcal{A}^{(2)}$*, (2) two unary operation symbols* sc^1 *and* tg^1*, which will be interpreted as total unary operations, and (3) a binary operation symbol* $\circ^1$ *which will be interpreted as a partial operation.*

Let $\eta^{(2,X)}$ denote the standard insertion of generator from X to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. This extension allows us to view all terms in $\mathbf{T}_{\Sigma}(X)$ and terms in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ as terms in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. Let $\eta^{(2,0)\sharp}$ and $\eta^{(2,1)\sharp}$ denote the embedding from, respectively $\mathbf{T}_{\Sigma}(X)$ and $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. Furthermore, every rewrite rule in $\mathcal{A}$ and every second-order rewrite rule in $\mathcal{A}^{(2)}$ can also be seen as a constant in $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. Let $\eta^{(2,\mathcal{A})}$ and $\eta^{(2,\mathcal{A}^{(2)})}$ denote the embedding from, respectively $\mathcal{A}$ and $\mathcal{A}^{(2)}$, to $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, see Figure 1a.

We next show that the set $\mathrm{Pth}_{\mathcal{A}^{(2)}}$ has a natural structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra.

▶ **Proposition 18** (Prop. 21.1.1)**.** *The set* $\mathrm{Pth}_{\mathcal{A}^{(2)}}$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra.*

Proof. Let us denote by $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra defined on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ as follows. The operations from $\Sigma^{\boldsymbol{\mathcal{A}}}$ are defined as in Proposition 14. Every constant operation symbol $\mathfrak{p}^{(2)} \in \mathcal{A}^{(2)}$ is interpreted as the second-order echelon $\mathrm{ech}^{(2,\mathcal{A}^{(2)})}(\mathfrak{p}^{(2)})$ introduced in Definition 8. The 1-source operation symbol is interpreted as the unary operation that maps a second-order path $\mathfrak{P}^{(2)}$ in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ to the $(2,[1])$-identity second-order path $\mathrm{ip}^{(2,[1])\sharp}(\mathrm{sc}^{([1],2)}(\mathfrak{P}^{(2)}))$, see Definition 5. The 1-target is interpreted analogously. The 1-composition is the partial operation defined in Proposition 6. ◀

The previous results, will allow us to consider paths in the second-order rewriting system $\boldsymbol{\mathcal{A}}^{(2)}$ as terms relative to $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$ and X. To do this, we will define, by Artinian recursion, a mapping from $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ to $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. In this way, every second-order path in $\boldsymbol{\mathcal{A}}^{(2)}$ will be denoted by a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. To stress this situation we will refer to this mapping as the second-order Curry-Howard mapping and we will denote it by $\mathrm{CH}^{(2)}$.

▶ **Definition 19.** *The* second-order Curry-Howard mapping *is* $\mathrm{CH}^{(2)}\colon \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}} \longrightarrow \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *defined by Artinian recursion on* $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$ *as follows.*

Base step of the Artinian recursion.

Let $\mathfrak{P}^{(2)}$ *be a minimal element of* $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$. *Then,* $\mathfrak{P}^{(2)}$ *is either (1) an* $(2,[1])$*-identity second-order path or (2) a second-order echelon.*

If (1), then $\mathfrak{P}^{(2)} = \mathrm{ip}^{(2,[1])\sharp}([P])$ *for some path term class* $[P]$ *in* $[\mathrm{PT}_{\boldsymbol{\mathcal{A}}}]$. *we define* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})$ *to be the term in* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *given by the lift of the original Curry-Howard mapping applied to the interpretation as a path of its defining path term class, i.e.,* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \eta^{(2,1)\sharp}(\mathrm{CH}^{(1)}(\mathrm{ip}^{([1],X)@}([P])))$.

If (2), if $\mathfrak{P}^{(2)}$ *is a second-order echelon associated to a second-order rewrite rule* $\mathfrak{p}^{(2)}$ *then we define* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \mathfrak{p}^{(2)\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}$.

Inductive step of the Artinian recursion.

Let $\mathfrak{P}^{(2)}$ *be a non-minimal element of* $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}})$ *We can assume that* $\mathfrak{P}^{(2)}$ *is not a* $(2,[1])$*-identity second-order path, since those second-order paths already have an image for the second-order Curry-Howard mapping. Let us suppose that, for every second-order path* $\mathfrak{Q}^{(2)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, *if* $\mathfrak{Q}^{(2)} <_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}$, *then the value of the second-order Curry-Howard mapping at* $\mathfrak{Q}^{(2)}$ *has already been defined.*

We have that $\mathfrak{P}^{(2)}$ *is either (1) a second-order path of length* m *strictly greater than one containing at least one second-order echelon or (2) an echelonless second-order* m*-path.*

If (1), let $i \in m$ *be the first index for which the one-step subpath* $\mathfrak{P}^{(2),i,i}$ *of* $\mathfrak{P}^{(2)}$ *is a second-order echelon. We consider different cases for* i *according to Definition 15.*

If $i = 0$, *we have that the second-order paths* $\mathfrak{P}^{(2),0,0}$ *and* $\mathfrak{P}^{(2),1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$*-precede the second-order path* $\mathfrak{P}^{(2)}$. *We set* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),1,m-1}) \circ^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),0,0})$.

If $i \neq 0$, *we have that the second-order paths* $\mathfrak{P}^{(2),0,i-1}$ *and* $\mathfrak{P}^{(2),i,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$*-precede the second-order path* $\mathfrak{P}^{(2)}$. *We set* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),i,m-1}) \circ^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),0,i-1})$.

If (2), i.e., if $\mathfrak{P}^{(2)}$ *is an echelonless second-order path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. *It could be the case that (2.1)* $\mathfrak{P}^{(2)}$ *is not head-constant. Then let* $i \in m$ *be the maximum index for which the subpath* $\mathfrak{P}^{(2),0,i}$ *of* $\mathfrak{P}^{(2)}$ *is a head-constant, echelonless second-order path. Note that the second-order pairs* $\mathfrak{P}^{(2),0,i}$ *and* $\mathfrak{P}^{(2),i+1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$*-precede the second-order path* $\mathfrak{P}^{(2)}$. *We set* $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),i+1,m-1}) \circ^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),0,i})$.

Therefore we are left with the case of $\mathfrak{P}^{(2)}$ *being a head-constant echelonless second-order path. It could be the case that (2.2)* $\mathfrak{P}^{(2)}$ *is not coherent. Then let* $i \in m$ *be the maximum index for which the subpath* $\mathfrak{P}^{(2),0,i}$ *of* $\mathfrak{P}^{(2)}$ *is a coherent head-constant echelonless second-*

order path. Note that the pairs $\mathfrak{P}^{(2),0,i}$ and $\mathfrak{P}^{(2),i+1,m-1}$ $\prec_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}$-precede the second-order path $\mathfrak{P}^{(2)}$. We set $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),i+1,m-1}) \circ^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2),0,i})$.

Therefore we are left with the case (2.3) of $\mathfrak{P}^{(2)}$ being a coherent head-constant echelonless second-order path. Under this setting, the conditions for the second-order extraction algorithm, that is, Lemma 11 are fulfilled. Then there exists a unique $n \in \mathbb{N}$ and a unique n-ary operation symbol $\tau \in \Sigma^{\boldsymbol{\mathcal{A}}}_n$ associated to $\mathfrak{P}^{(2)}$. Let $(\mathfrak{P}^{(2)}_j)_{j\in n}$ be the family of second-order paths in $\mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}^{(2)}}$ which we can extract from $\mathfrak{P}^{(2)}$. We set $\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}) = \tau^{\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}((\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}_j))_{j\in n})$.

It can be shown that the second-order Curry-Howard mapping is a Σ-homomorphism. However, it is not a $\Sigma^{\boldsymbol{\mathcal{A}}}$-homomorphism, much less a $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-homomorphism. It is enough to consider the case of identity second-order paths. However, as it was the case in the first part of this work, the study of its kernel proves to be interesting. In general, it can be shown that for a pair of second-order paths in $\mathrm{Ker}(\mathrm{CH}^{(2)})$, their length, $([1],2)$-source and $([1],2)$-target are equal. The most interesting property of the mapping $\mathrm{CH}^{(2)}$ is that its kernel, $\mathrm{Ker}(\mathrm{CH}^{(2)})$, is a closed $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-congruence on $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. This proof is achieved by induction on $(\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}, \leq_{\mathbf{Pth}^{(2)}_{\boldsymbol{\mathcal{A}}}})$.

▶ **Proposition 20** (Prop. 22.1.1)**.** *$\mathrm{Ker}(\mathrm{CH}^{(2)})$ is a closed $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-congruence on the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$.*

4.1 The quotient of second-order paths

The last results opens up a new object of study, the quotient of second-order paths by $\mathrm{Ker}(\mathrm{CH}^{(2)})$. However, in contrast to the first part, this is not the ultimate interesting object of study, since the corresponding quotient is not a 2-category. This is due to the fact that the 0-composition does not behave accordingly. To surpass this problem, we consider a new congruence.

▶ **Definition 21.** *We define $\Upsilon^{(1)}$ to be the relation on $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ consisting exactly of the following pairs of second-order paths*

1. *For every second-order path $\mathfrak{P}^{(2)}$, $(\mathfrak{P}^{(2)}, \mathfrak{P}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathrm{sc}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)})) \in \Upsilon^{(1)}$;*
2. *For every second-order path $\mathfrak{P}^{(2)}$, $(\mathfrak{P}^{(2)}, \mathrm{tg}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}) \in \Upsilon^{(1)}$;*
3. *For every second-order paths $\mathfrak{P}^{(2)}, \mathfrak{Q}^{(2)}, \mathfrak{R}^{(2)}$ in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ with $\mathrm{sc}^{(0,2)}(\mathfrak{R}^{(2)}) = \mathrm{tg}^{(0,2)}(\mathfrak{Q}^{(2)})$ and $\mathrm{sc}^{(0,2)}(\mathfrak{Q}^{(2)}) = \mathrm{tg}^{(0,2)}(\mathfrak{R}^{(2)})$, then*
$$(\mathfrak{R}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} (\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}), (\mathfrak{R}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{Q}^{(2)}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}) \in \Upsilon^{(1)};$$
4. *For every $n \in \mathbb{N}$ and every n-ary operation symbol $\sigma \in \Sigma_n$, for every two families of second-order paths $(\mathfrak{P}^{(2)}_j)_{j\in n}$ and $(\mathfrak{Q}^{(2)}_j)_{j\in n}$ in $\mathrm{Pth}^n_{\boldsymbol{\mathcal{A}}^{(2)}}$ satisfying that, for every $j \in n$, $\mathrm{sc}^{(0,2)}(\mathfrak{Q}^{(2)}_j) = \mathrm{tg}^{(0,2)}(\mathfrak{P}^{(2)}_j)$, then*

$$(\sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}((\mathfrak{Q}^{(2)}_j \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}_j)_{j\in n}), \sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}((\mathfrak{Q}^{(2)}_j)_{j\in n}) \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \sigma^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}((\mathfrak{P}^{(2)}_j)_{j\in n})) \in \Upsilon^{(1)}.$$

Finally, we denote by $\Upsilon^{[1]}$ the smallest $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-congruence on $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ containing $\Upsilon^{(1)}$.

In this subsection we study $\mathrm{Ker}(\mathrm{CH}^{(2)}) \vee \Upsilon^{[1]}$, the supremum of the two $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-congruences, $\mathrm{Ker}(\mathrm{CH}^{(2)})$ and $\Upsilon^{[1]}$, defined on $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$. For simplicity, the quotient will be denoted by $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$. Following this simplification the equivalence class of a second-order path $\mathfrak{P}^{(2)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, will simply be denoted by $[\![\mathfrak{P}^{(2)}]\!]$. We next investigate the algebraic, categorial and order structures that we can define on $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$.

▶ **Proposition 22** (Prop. 25.1.2). *The set* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra, that we denote by* $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$.

The set $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ is equipped with a structure of 2-categorial Σ-algebra.

▶ **Proposition 23** (Prop. 25.2.20). *The set* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is equipped with a structure of* 2*-categorial* Σ*-algebra, that we denote by* $[\![\mathsf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *.*

Furthermore, the set $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ is equipped with an Artinian preorder.

▶ **Definition 24.** *Let* $\leq_{[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]}$ *be the binary relation defined on* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *containing every pair* $([\![\mathfrak{Q}^{(2)}]\!],[\![\mathfrak{P}^{(2)}]\!])$ *in* $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]^2$ *satisfying that there exists a natural number* $m\in\mathbb{N}-\{0\}$*, and a family of second-order paths* $(\mathfrak{R}^{(2)}_k)_{k\in m+1}$ *in* $\mathrm{Pth}^{m+1}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *such that* $[\![\mathfrak{R}^{(2)}_0]\!]=[\![\mathfrak{Q}^{(2)}]\!]$ $[\![\mathfrak{R}^{(2)}_m]\!]=[\![\mathfrak{P}^{(2)}]\!]$ *and, for every* $k\in m$, $[\![\mathfrak{R}^{(2)}_k]\!]=[\![\mathfrak{R}^{(2)}_{k+1}]\!]$ *or* $\mathfrak{R}^{(2)}\leq_{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}\mathfrak{R}^{(2)}_{k+1}$.

▶ **Proposition 25** (Prop. 25.3.2). $([\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!],\leq_{[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]})$ *is an Artinian preordered set.*

5 Second-order path terms

Following ideas of Burmeister and Schmidt [4, 5, 6, 21, 22, 8], we consider, for a signature Γ and a partial Γ-algebra $\mathbf{A}$, its free Γ-completion, denoted by $\mathbf{F}_\Gamma(\mathbf{A})$. This total Γ-algebra has the following universal property; for every partial Γ-algebra $\mathbf{B}$ and every Γ-homomorphism f from $\mathbf{A}$ to $\mathbf{B}$, there exists a unique Γ-homomorphism, f^{fc}, the free completion of f, from $\mathbf{F}_\Gamma(\mathbf{A})$ to $\mathbf{B}$ satisfying that $f^{\mathrm{fc}}\circ\eta^{\mathbf{A}}=f$, where $\eta^{\mathbf{A}}$, from $\mathbf{A}$ to $\mathbf{F}_\Gamma(\mathbf{A})$, is the standard insertion of generators.

With the aforementioned ideas we consider the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$.

▶ **Definition 26.** *Consider the mapping* $\mathrm{ip}^{(2,X)}$ *from the set of variables* X *to* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$*, introduced in Definition 5. If we consider* $\mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$*, the discrete* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra on* X*, i.e., no operation in* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$ *is defined, the application* $\mathrm{ip}^{(2,X)}$ *becomes a* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-homomorphism of the form* $\mathrm{ip}^{(2,X)}\colon\mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)\longrightarrow\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$*. By the universal property of the free completion, there exists a unique* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-homomorphism* $(\eta^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}\circ\mathrm{ip}^{(2,X)})^{\mathrm{fc}}$*, simply denoted* $\mathrm{ip}^{(2,X)@}$*, from* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$*, the free completion of the discrete* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra* $\mathbf{D}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$*, to* $\mathbf{F}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$*, the free* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-completion of the second-order path algebra* $\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$*, such that* $\mathrm{ip}^{(2,X)@}\circ\eta^{(2,X)}=\eta^{\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}\circ\mathrm{ip}^{(2,X)}$*.*

At this point we begin to study the $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-homomorphism $\mathrm{ip}^{(2,X)@}$. The following proposition states that $\mathrm{ip}^{(2,X)@}$ acting on the value of $\mathrm{CH}^{(2)}$ at a second-order path $\mathfrak{P}^{(2)}$ is always another second-order path, not necessarily equal to the input $\mathfrak{P}^{(2)}$, but which has the same image under the second-order Curry-Howard mapping. Moreover, it preserves the $\Upsilon^{(1)}$ relation.

▶ **Proposition 27** (Prop. 26.1.6). *The mapping* $\mathrm{ip}^{(2,X)@}\circ\mathrm{CH}^{(2)}\colon\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}\longrightarrow\mathrm{F}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$ *sends every second-order path* $\mathfrak{P}^{(2)}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *to a second-order path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$*. Moreover,* $\mathrm{CH}^{(2)}(\mathrm{ip}^{(2,X)@}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})))=\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})$*. Furthermore, if* $\mathfrak{P}^{(2)},\mathfrak{Q}^{(2)}$ *are second-order paths and* $(\mathfrak{P}^{(2)},\mathfrak{Q}^{(2)})\in\Upsilon^{(1)}$*, then* $(\mathrm{ip}^{(2,X)@}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})),\mathrm{ip}^{(2,X)@}(\mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)})))\in\Upsilon^{(1)}$*.*

We next define two binary relations on $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ with the objective of matching different terms that, by $\mathrm{ip}^{(2,X)@}$, are sent to second-order paths in the same $\mathrm{Ker}(\mathrm{CH}^{(2)})\vee\Upsilon^{[1]}$-class.

▶ **Definition 28.** *We let* $\Theta^{(2)}$ *stand for the binary relation on* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *consisting exactly of the following pairs of terms:*

- *For every second-order path* $\mathfrak{P}^{(2)}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$,

$$(\mathrm{CH}^{(2)}(\mathrm{sc}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)})), \mathrm{sc}^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}))) \in \Theta^{(2)};$$
$$(\mathrm{CH}^{(2)}(\mathrm{tg}^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)})), \mathrm{tg}^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}))) \in \Theta^{(2)};$$
$$(\mathrm{CH}^{(2)}(\mathrm{sc}^{1\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)})), \mathrm{sc}^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}))) \in \Theta^{(2)};$$
$$(\mathrm{CH}^{(2)}(\mathrm{tg}^{1\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathfrak{P}^{(2)})), \mathrm{tg}^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)}(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}))) \in \Theta^{(2)};$$

- *For every pair of second-order paths* $\mathfrak{Q}^{(2)}, \mathfrak{P}^{(2)}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, *if* $\mathrm{sc}^{(0,2)}(\mathfrak{Q}^{(2)}) = \mathrm{tg}^{(0,2)}(\mathfrak{P}^{(2)})$,

$$(\mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)} \circ^{0\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}), \mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)}) \circ^{0\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})) \in \Theta^{(2)}.$$

- *For every pair of second-order paths* $\mathfrak{Q}^{(2)}, \mathfrak{P}^{(2)}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, *if* $\mathrm{sc}^{([1],2)}(\mathfrak{Q}^{(2)}) = \mathrm{tg}^{([1],2)}(\mathfrak{P}^{(2)})$,

$$(\mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)} \circ^{1\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} \mathfrak{P}^{(2)}), \mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)}) \circ^{1\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)} \mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})) \in \Theta^{(2)}.$$

Finally, we denote by $\Theta^{[2]}$ *the smallest* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-congruence on* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *containing* $\Theta^{(2)}$.

▶ **Definition 29.** *We let* $\Psi^{(1)}$ *stand for the binary relation on* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *consisting exactly of the following pairs of terms:*

- *For every pair of second-order paths* $\mathfrak{P}^{(2)}, \mathfrak{Q}^{(2)}$ *in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$, *if* $(\mathfrak{P}^{(2)}, \mathfrak{Q}^{(2)}) \in \Upsilon^{(1)}$ *then*

$$(\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)}), \mathrm{CH}^{(2)}(\mathfrak{Q}^{(2)})) \in \Psi^{(1)}.$$

Finally, we denote by $\Psi^{[1]}$ *the smallest* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-congruence on* $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *containing* $\Psi^{(1)}$.

We next consider the congruence $\Theta^{[2]} \vee \Psi^{[1]}$ on $\mathbf{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$, i.e., the supremum of the $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-congruences $\Theta^{[2]}$ and $\Psi^{[1]}$ that we will denote by $\Theta^{[\![2]\!]}$. To simplify the presentation, the $\Theta^{[\![2]\!]}$-equivalence class of a term P in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ will be denoted by $[\![P]\!]$. We next provide two lemmas to understand the usefulness of the $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence $\Theta^{[\![2]\!]}$.

▶ **Lemma 30** (Lemma 29.0.4). *Let* P *be a term in* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. *If* $\mathrm{ip}^{(2,X)@}(P)$ *is a second-order path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *then* $(P, \mathrm{CH}^{(2)}(\mathrm{ip}^{(2,X)@}(P))) \in \Theta^{[\![2]\!]}$.

▶ **Lemma 31** (Lemma 29.0.5). *Let* $P, Q \in \mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *be such that* $(P, Q) \in \Theta^{[\![2]\!]}$, *then*

- $\mathrm{ip}^{(2,X)@}(P) \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *if, and only if,* $\mathrm{ip}^{(2,X)@}(Q) \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$;
- *If* $\mathrm{ip}^{(2,X)@}(P)$ *or* $\mathrm{ip}^{(2,X)@}(Q)$ *is a second-order path in* $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *then*

$$[\![\mathrm{ip}^{(2,X)@}(P)]\!] = [\![\mathrm{ip}^{(2,X)@}(Q)]\!].$$

We next introduce the notion of second-order path term.

▶ **Definition 32.** *We let* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *stand for* $[\mathrm{CH}^{(2)}[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]]^{\Theta^{[\![2]\!]}} = \bigcup_{\mathfrak{P}^{(2)} \in \mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}} [\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})]_{\Theta^{[\![2]\!]}}$, *the* $\Theta^{[\![2]\!]}$*-saturation of the subset* $\mathrm{CH}^{(2)}[\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]$ *of* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$. *We call* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *the set of* second-order path terms.

It can be shown that a term in $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ is a second-order path term if, and only if, it can be interpreted as a second-order path in $\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}$ by means of $\mathrm{ip}^{(2,X)@}$. Following this, all the known mappings from or to $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$ have nice restrictions, corestrictions or birestrictions to the set of path terms. When possible, we will use these refinements instead of the original mappings, see Figure 1b.

5.1 The quotient of second-order path terms

In this subsection we define the set of second-order path term classes as the quotient of $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ by the restriction of $\Theta^{[\![2]\!]}$ to it.

▶ **Definition 33.** *We denote by* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *the image of* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *under* $\mathrm{pr}^{\Theta^{[\![2]\!]}}$*, the canonical projection from* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *to* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)/\Theta^{[\![2]\!]}$*, i.e.,* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!] = \mathrm{pr}^{\Theta^{[\![2]\!]}}[\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]$*. We call it the set of* second-order path term classes*. Let us note that* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is a subset of the quotient* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)/\Theta^{[\![2]\!]}$*, i.e., that* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is a subquotient of* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$*. Actually, we have that* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!] = \mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}/\Theta^{[\![2]\!]}{\upharpoonright}\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$*.*

The projection, from $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)$ *to* $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}}(X)/\Theta^{[\![2]\!]}$*, birestricts to* $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ *and* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$*.*

We investigate the algebraic, categorial and order structures that we can define on $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$. As an immediate consequence of the definition, the set of second-order path term classes inherits a structure of partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra.

▶ **Proposition 34** (Prop. 30.4.1)**.** *The set* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is equipped with a structure of partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebra, that we denote by* $[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$*.*

The set $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ is equipped with a structure of 2-categorial Σ-algebra.

▶ **Proposition 35** (Prop. 30.5.20)**.** *The set* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *is equipped with a structure of* 2*-categorial* Σ*-algebra, that we denote by* $[\![\mathsf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$*.*

Finally, we define an Artinian preorder on $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$.

▶ **Definition 36.** *We let* $\leq_{[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]}$ *stand for the binary relation on* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *which consists of those ordered pairs* $([\![Q]\!],[\![P]\!])$ *in* $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]^2$ *satisfying that*

$$[\![\mathrm{ip}^{(2,X)@}(Q)]\!] \leq_{[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]} [\![\mathrm{ip}^{(2,X)@}(P)]\!].$$

▶ **Proposition 37** (Prop. 30.6.3)**.** $([\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!], \leq_{[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]})$ *is an Artinian preordered set.*

6 Second-order isomorphisms

In this section we are in position to prove the main results of the paper, that the algebraic, 2-categorial and preorder structures that we have defined on second-order path classes and on second-order path terms are isomorphic.

▶ **Theorem 38** (Th. 31.1.1, 31.2.1, 31.3.3)**.** *The partial* $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$*-algebras* $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *and* $[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *are isomorphic. The* 2*-categorial* Σ*-algebras,* $[\![\mathsf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *and* $[\![\mathsf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ *are isomorphic. The Artinian preordered sets* $([\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!], \leq_{[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]})$ *and* $([\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!], \leq_{[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]})$ *are isomorphic.*

Proof. We let $\mathrm{ip}^{([\![2]\!],X)@}$ stand for the mapping from $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ to $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ that maps a second-order path term class $[\![P]\!]$ in $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ to the second-order path class $[\![\mathrm{ip}^{(2,X)@}(P)]\!]$ in $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$. This mapping is well-defined because two second-order path terms P, Q in $\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}$ for which $[\![Q]\!] = [\![P]\!]$ satisfy that $[\![\mathrm{ip}^{(2,X)@}(Q)]\!] = [\![\mathrm{ip}^{(2,X)@}(P)]\!]$. We let $\mathrm{CH}^{[\![2]\!]}$ stand for the mapping from $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ to $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ that maps a second-order path class $[\![\mathfrak{P}^{(2)}]\!]$ in $[\![\mathrm{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ to the second-order path term class $[\![\mathrm{CH}^{(2)}(\mathfrak{P}^{(2)})]\!]$ in $[\![\mathrm{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$.

This two mappings constitute a pair of inverse $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-isomorphisms, a pair of inverse 2-functors, i.e., of categorial Σ-isomorphisms. Finally, we show that the mappings also form a pair of inverse order-preserving mappings, see Figure 1b. ◀

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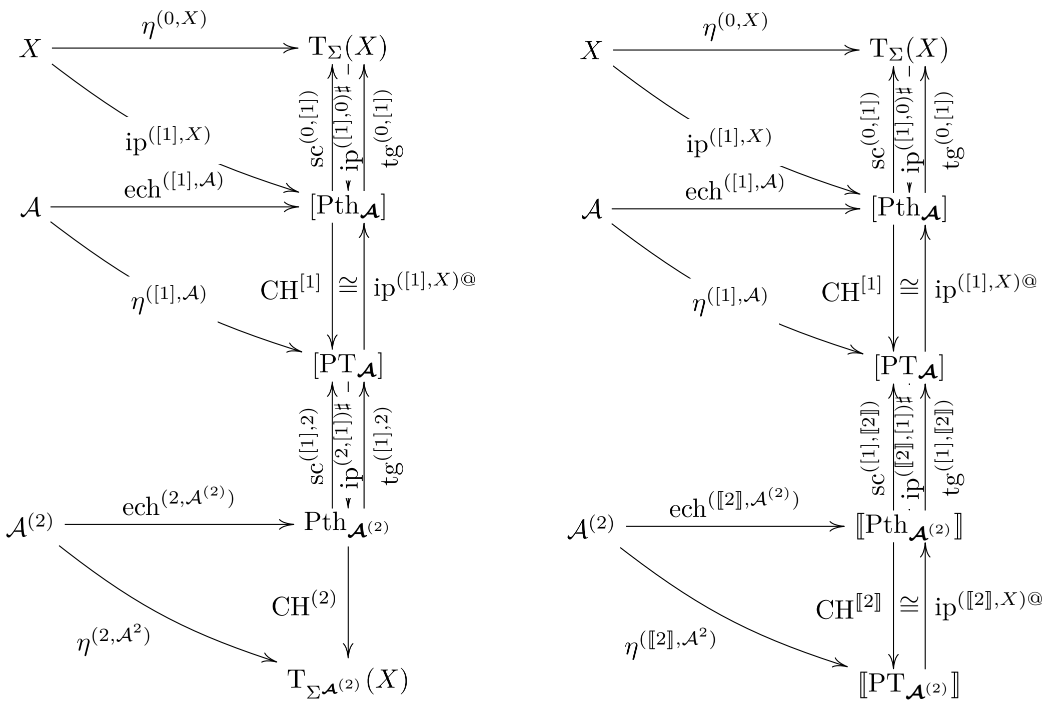
A Diagrams

The following figure collects all the mappings considered in this work.



(a) Mappings at layers 0, 1 & 2.

(b) Quotient mappings at layers 0, 1 & 2.

Figure 1 Mappings considered in this work.

B Freedom

For a specification $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}$ associated to the second-order rewriting system $\boldsymbol{\mathcal{A}}^{(2)}$, whose defining equations $\mathcal{E}^{\boldsymbol{\mathcal{A}}^{(2)}}$ are QE-equations, we define a QE-variety of partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras $\mathcal{V}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$.

▶ **Definition 39.** *For the second-order rewriting system $\boldsymbol{\mathcal{A}}^{(2)}$, we will denote by $(\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}, V, \mathcal{E}^{\boldsymbol{\mathcal{A}}^{(2)}})$, written $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}$ for short, the specification in which $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$ is the signature introduced in Definition 17, V a fixed set with a countable infinity of variables, and $\mathcal{E}^{\boldsymbol{\mathcal{A}}^{(2)}}$ the subset of* $\mathrm{QE}(\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}})_V$, *consisting of the following equations:*

For every $n \in \mathbb{N}$, every n-ary operation symbol $\sigma \in \Sigma_n$, and every family of variables $(x_j)_{j\in n} \in V^n$, the operation σ applied to the family $(x_j)_{j\in n}$ is always defined. Formally,

$$\sigma((x_j)_{j\in n}) \stackrel{\mathrm{e}}{=} \sigma((x_j)_{j\in n}). \tag{A0}$$

For every variable $x \in V$, the 0-source and 0-target of x is always defined. Formally,

$$\mathrm{sc}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^0(x) \stackrel{\mathrm{e}}{=} \mathrm{tg}^0(x). \tag{A1}$$

For every every variable $x \in V$, we have the following equations:

$$\mathrm{sc}^0(\mathrm{sc}^0(x)) \stackrel{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{sc}^0(\mathrm{tg}^0(x)) \stackrel{\mathrm{e}}{=} \mathrm{tg}^0(x);$$

$$\mathrm{tg}^0(\mathrm{sc}^0(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^0(\mathrm{tg}^0(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^0(x). \tag{A2}$$

In other words, sc^0 *and* tg^0 *are right zeros. In particular,* sc^0 *and* tg^0 *are idempotent.*

For every pair of variables $x, y \in V$*,* $x \circ^0 y$ *is defined if and only if the* 0*-target of* y *is equal to the* 0*-source of* x*. Formally,*

$$x \circ^0 y \overset{\mathrm{e}}{=} x \circ^0 y \;\;\rightarrow\;\; \mathrm{sc}^0(x) \overset{\mathrm{e}}{=} \mathrm{tg}^0(y);$$
$$\mathrm{sc}^0(x) \overset{\mathrm{e}}{=} \mathrm{tg}^0(y) \;\;\rightarrow\;\; x \circ^0 y \overset{\mathrm{e}}{=} x \circ^0 y. \tag{A3}$$

For every pair of variables $x, y \in V$*, if* $x \circ^0 y$ *is defined, then the* 0*-source of* $x \circ^0 y$ *is that of* y *and the* 0*-target of* $x \circ^0 y$ *is that of* x*. Formally,*

$$x \circ^0 y \overset{\mathrm{e}}{=} x \circ^0 y \;\;\rightarrow\;\; \mathrm{sc}^0(x \circ^0 y) \overset{\mathrm{e}}{=} \mathrm{sc}^0(y);$$
$$x \circ^0 y \overset{\mathrm{e}}{=} x \circ^0 y \;\;\rightarrow\;\; \mathrm{tg}^0(x \circ^0 y) \overset{\mathrm{e}}{=} \mathrm{tg}^0(x). \tag{A4}$$

For every variable $x \in V$*, the compositions* $x \circ^0 \mathrm{sc}^0(x)$ *and* $\mathrm{tg}^0(x) \circ^0 x$ *are always defined and are equal to* x*, i.e.,* $\mathrm{sc}^0(x)$ *is a right unit element for the* 0*-composition with* x *and* $\mathrm{tg}^0(x)$ *is a left unit element for the* 0*-composition with* x*. Formally,*

$$x \circ^0 \mathrm{sc}^0(x) \overset{\mathrm{e}}{=} x; \qquad \mathrm{tg}^0(x) \circ^0 x \overset{\mathrm{e}}{=} x. \tag{A5}$$

For every triple of variables $x, y, z \in V$*, if the* 0*-compositions* $x \circ^0 y$ *and* $y \circ^0 z$ *are defined, then the* 0*-compositions* $x \circ^0 (y \circ^0 z)$ *and* $(x \circ^0 y) \circ^0 z$ *are defined and they are equal, i.e., the* 0*-composition, when defined, is associative. Formally,*

$$(x \circ^0 y \overset{\mathrm{e}}{=} x \circ^0 y) \wedge (y \circ^0 z \overset{\mathrm{e}}{=} y \circ^0 z) \;\;\rightarrow\;\; (x \circ^0 y) \circ^0 z \overset{\mathrm{e}}{=} x \circ^0 (y \circ^0 z). \tag{A6}$$

For every $n \in \mathbb{N}$*, every* n*-ary operation symbol* $\sigma \in \Sigma_n$*, and every family of variables* $(x_j)_{j\in n} \in V^n$*, the* 0*-source of* $\sigma((x_j)_{j\in n})$ *is equal to* σ *applied to the family* $((\mathrm{sc}^0(x_j))_{j\in n})$*, and the* 0*-target of* $\sigma((x_j)_{j\in n})$ *is equal to* σ *applied to the family* $((\mathrm{tg}^0(x_j))_{j\in n})$*. Formally,*

$$\mathrm{sc}^0(\sigma((x_j)_{j\in n})) \overset{\mathrm{e}}{=} \sigma((\mathrm{sc}^0(x_j))_{j\in n}); \qquad \mathrm{tg}^0(\sigma((x_j)_{j\in n})) \overset{\mathrm{e}}{=} \sigma((\mathrm{tg}^0(x_j))_{j\in n}). \tag{A7}$$

For every $n \in \mathbb{N}$*, every* n*-ary operation symbol* $\sigma \in \Sigma_n$*, and every pair of families of variables* $(x_j)_{j\in n}, (y_j)_{j\in n} \in V^n$*, if, for every* $j \in n$*, the* 0*-compositions* $x_j \circ^0 y_j$ *are defined, then the* 0*-composition* $\sigma((x_j)_{j\in n}) \circ^0 \sigma((y_j)_{j\in n})$ *is defined and it is equal to* σ *applied to the family* $(x_j \circ^0 y_j)_{j\in n}$*. Formally,*

$$\textstyle\bigwedge_{j\in n}(x_j \circ^0 y_j \overset{\mathrm{e}}{=} x_j \circ^0 y_j) \;\;\rightarrow\;\; \sigma((x_j \circ^0 y_j)_{j\in n}) \overset{\mathrm{e}}{=} \sigma((x_j)_{j\in n}) \circ^0 \sigma((y_j)_{j\in n}) \tag{A8}$$

For every rewrite rule $\mathfrak{p} \in \mathcal{A}$*,* $\mathfrak{p}$ *is always defined. Formally,*

$$\mathfrak{p} \overset{\mathrm{e}}{=} \mathfrak{p}. \tag{A9}$$

For every variable $x \in V$*, the* 1*-source and* 1*-target of* x *is always defined. Formally,*

$$\mathrm{sc}^1(x) \overset{\mathrm{e}}{=} \mathrm{sc}^1(x); \qquad \mathrm{tg}^1(x) \overset{\mathrm{e}}{=} \mathrm{tg}^1(x). \tag{B1}$$

For every every variable $x \in V$*, we have the following equations:*

$$\mathrm{sc}^1(\mathrm{sc}^1(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^1(x); \qquad \mathrm{sc}^1(\mathrm{tg}^1(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^1(x);$$
$$\mathrm{tg}^1(\mathrm{sc}^1(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^1(x); \qquad \mathrm{tg}^1(\mathrm{tg}^1(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^1(x). \tag{B2}$$

In other words, sc^1 *and* tg^1 *are right zeros. In particular,* sc^1 *and* tg^1 *are idempotent.*

For every pair of variables $x, y \in V$*,* $x \circ^1 y$ *is defined if and only if the* 1*-target of* y *is equal to the* 1*-source of* x*. Formally,*

$$x \circ^1 y \overset{\mathrm{e}}{=} x \circ^1 y \;\rightarrow\; \mathrm{sc}^1(x) \overset{\mathrm{e}}{=} \mathrm{tg}^1(y);$$
$$\mathrm{sc}^1(x) \overset{\mathrm{e}}{=} \mathrm{tg}^1(y) \;\rightarrow\; x \circ^1 y \overset{\mathrm{e}}{=} x \circ^1 y. \tag{B3}$$

For every pair of variables $x, y \in V$*, if* $x \circ^1 y$ *is defined, then the* 1*-source of* $x \circ^1 y$ *is that of* y *and the* 1*-target of* $x \circ^1 y$ *is that of* x*. Formally,*

$$x \circ^1 y \overset{\mathrm{e}}{=} x \circ^1 y \;\rightarrow\; \mathrm{sc}^1(x \circ^1 y) \overset{\mathrm{e}}{=} \mathrm{sc}^1(y);$$
$$x \circ^1 y \overset{\mathrm{e}}{=} x \circ^1 y \;\rightarrow\; \mathrm{tg}^1(x \circ^1 y) \overset{\mathrm{e}}{=} \mathrm{tg}^1(x). \tag{B4}$$

For every variable $x \in V$*, the compositions* $x \circ^1 \mathrm{sc}^1(x)$ *and* $\mathrm{tg}^1(x) \circ^1 x$ *are always defined and are equal to* x*, i.e.,* $\mathrm{sc}^1(x)$ *is a right unit element for the* 1*-composition with* x *and* $\mathrm{tg}^1(x)$ *is a left unit element for the* 1*-composition with* x*. Formally,*

$$x \circ^1 \mathrm{sc}^1(x) \overset{\mathrm{e}}{=} x; \qquad \mathrm{tg}^1(x) \circ^1 x \overset{\mathrm{e}}{=} x. \tag{B5}$$

For every triple of variables $x, y, z \in V$*, if the* 1*-compositions* $x \circ^1 y$ *and* $y \circ^1 z$ *are defined, then the* 1*-compositions* $x \circ^1 (y \circ^1 z)$ *and* $(x \circ^1 y) \circ^1 z$ *are defined and they are equal, i.e., the* 1*-composition, when defined, is associative. Formally,*

$$(x \circ^1 y \overset{\mathrm{e}}{=} x \circ^1 y) \wedge (y \circ^1 z \overset{\mathrm{e}}{=} y \circ^1 z) \;\rightarrow\; (x \circ^1 y) \circ^1 z \overset{\mathrm{e}}{=} x \circ^1 (y \circ^1 z). \tag{B6}$$

For every $n \in \mathbb{N}$*, every* n*-ary operation symbol* $\sigma \in \Sigma_n$*, and every family of variables* $(x_j)_{j \in n} \in V^n$*, the* 1*-source of* $\sigma((x_j)_{j \in n})$ *is equal to* σ *applied to the family* $((\mathrm{sc}^1(x_j))_{j \in n})$*, and the* 1*-target of* $\sigma((x_j)_{j \in n})$ *is equal to* σ *applied to the family* $((\mathrm{tg}^1(x_j))_{j \in n})$*. Formally,*

$$\mathrm{sc}^1(\sigma((x_j)_{j \in n})) \overset{\mathrm{e}}{=} \sigma((\mathrm{sc}^1(x_j))_{j \in n}); \qquad \mathrm{tg}^1(\sigma((x_j)_{j \in n})) \overset{\mathrm{e}}{=} \sigma((\mathrm{tg}^1(x_j))_{j \in n}). \tag{B7}$$

For every $n \in \mathbb{N}$*, every* n*-ary operation symbol* $\sigma \in \Sigma_n$*, and every pair of families of variables* $(x_j)_{j \in n}, (y_j)_{j \in n} \in V^n$*, if, for every* $j \in n$*, the* 1*-compositions* $x_j \circ^1 y_j$ *are defined, then the* 1*-composition* $\sigma((x_j)_{j \in n}) \circ^1 \sigma((y_j)_{j \in n})$ *is defined and it is equal to* σ *applied to the family* $(x_j \circ^1 y_j)_{j \in n}$*. Formally,*

$$\textstyle\bigwedge_{j \in n}(x_j \circ^1 y_j \overset{\mathrm{e}}{=} x_j \circ^1 y_j) \;\rightarrow\; \sigma((x_j \circ^1 y_j)_{j \in n}) \overset{\mathrm{e}}{=} \sigma((x_j)_{j \in n}) \circ^1 \sigma((y_j)_{j \in n}) \tag{B8}$$

For every second-order rewrite rule $\mathfrak{p}^{(2)} \in \mathcal{A}^{(2)}$*,* $\mathfrak{p}^{(2)}$ *is always defined. Formally,*

$$\mathfrak{p}^{(2)} \overset{\mathrm{e}}{=} \mathfrak{p}^{(2)}. \tag{B9}$$

For every variable $x \in V$*, the elements* $\mathrm{sc}^1(\mathrm{sc}^0(x))$*,* $\mathrm{sc}^0(\mathrm{sc}^1(x))$ *and* $\mathrm{sc}^0(\mathrm{tg}^1(x))$ *are always defined and are equal to* $\mathrm{sc}^0(x)$*. Analogously, the elements* $\mathrm{tg}^1(\mathrm{tg}^0(x))$*,* $\mathrm{tg}^0(\mathrm{tg}^1(x))$ *and* $\mathrm{tg}^0(\mathrm{sc}^1(x))$ *are always defined and are equal to* $\mathrm{tg}^0(x)$*. Formally,*

$$\mathrm{sc}^1(\mathrm{sc}^0(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^1(\mathrm{tg}^0(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^0(x);$$
$$\mathrm{sc}^0(\mathrm{sc}^1(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^0(\mathrm{tg}^1(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^0(x);$$
$$\mathrm{sc}^0(\mathrm{tg}^1(x)) \overset{\mathrm{e}}{=} \mathrm{sc}^0(x); \qquad \mathrm{tg}^0(\mathrm{sc}^1(x)) \overset{\mathrm{e}}{=} \mathrm{tg}^0(x). \tag{AB1}$$

For every pair of variables $x, y \in V$, if $x \circ^0 y$ is defined, then the 1-source of $x \circ^0 y$ is the 0-composition of the 1-source of x with the 1-source of y and the 1-target of $x \circ^0 y$ is the 0-composition of the 1-target of x with the 1-target of y. Formally,

$$x \circ^0 y \stackrel{e}{=} x \circ^0 y \;\to\; \mathrm{sc}^1(x \circ^0 y) \stackrel{e}{=} \mathrm{sc}^1(x) \circ^0 \mathrm{sc}^1(y);$$
$$x \circ^0 y \stackrel{e}{=} x \circ^0 y \;\to\; \mathrm{tg}^1(x \circ^0 y) \stackrel{e}{=} \mathrm{tg}^1(x) \circ^0 \mathrm{tg}^1(y). \qquad \text{(AB2)}$$

For every four variables $x, y, z, t \in V$, if the 0-compositions $x \circ^0 y$ and $z \circ^0 t$ are defined and the 1-compositions $x \circ^1 z$ and $y \circ^1 t$ are defined, the 1-composition of $x \circ^0 y$ with $z \circ^0 t$ is equal to the 0-composition of $x \circ^1 z$ with $y \circ^1 t$. Formally,

$$(x \circ^0 y \stackrel{e}{=} x \circ^0 y) \wedge (z \circ^0 t \stackrel{e}{=} z \circ^0 t) \wedge (x \circ^1 z \stackrel{e}{=} x \circ^1 z) \wedge (y \circ^1 t \stackrel{e}{=} y \circ^1 t)$$
$$\to\; (x \circ^0 y) \circ^1 (z \circ^0 t) \stackrel{e}{=} (x \circ^1 z) \circ^0 (y \circ^1 t). \qquad \text{(AB3)}$$

A model of axioms A1–A6, B1–B6, and AB1–AB3 is a 2-category.

We will let $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$ stand for the category canonically associated to the QE-variety $\mathcal{V}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$ determined by the specification $\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}$.

Another fundamental result of this work is that the two partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$, which is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra in the category $\mathbf{PAlg}(\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}})$, and $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ are isomorphic.

▶ **Theorem 40** (Th. 32.2.9). *The partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras $[\![\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$ are isomorphic. As a consequence of Theorem 38, the partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras $[\![\mathbf{PT}_{\boldsymbol{\mathcal{A}}^{(2)}}]\!]$ and $\mathbf{T}_{\boldsymbol{\mathcal{E}}^{\boldsymbol{\mathcal{A}}^{(2)}}}(\mathbf{Pth}_{\boldsymbol{\mathcal{A}}^{(2)}})$ are isomorphic.*

C An example

For the sake of illustration, here is an example of an echelonless second-order path that is not head-constant and a head-constant echelonless second-order path that is not coherent. Finally, we present a coherent head-constant echelonless second-order path.

▶ **Example 41.** Let Σ be the signature containing a unique binary operation symbol σ. For $X = \{x, y, z\}$, we consider $\mathbf{T}_\Sigma(X)$, the free Σ-algebra on X. Consider the set of rewrite rules

$$\mathcal{A} = \{\mathfrak{p} = (x, y), \mathfrak{q} = (y, z), \mathfrak{r} = (x, z)\}$$

and the set of second-order rewrite rules

$$\mathcal{A}^{(2)} = \{\mathfrak{p}^{(2)} = ([\mathfrak{q} \circ^0 \mathfrak{p}], [\mathfrak{r}])\}$$

Let us note that $\mathfrak{p}^{(2)}$ is a valid second-order rewrite rule because

$$\mathrm{sc}^{(0,1)}(\mathrm{ip}^{(1,X)@}(\mathfrak{q} \circ^0 \mathfrak{p})) = \mathrm{sc}^{(0,1)}(\mathrm{ip}^{(1,X)@}(\mathfrak{r})) = x;$$
$$\mathrm{tg}^{(0,1)}(\mathrm{ip}^{(1,X)@}(\mathfrak{q} \circ^0 \mathfrak{p})) = \mathrm{tg}^{(0,1)}(\mathrm{ip}^{(1,X)@}(\mathfrak{r})) = z.$$

Consider the second-order path

$$\mathfrak{P}^{(2)} \colon [\sigma(\mathfrak{q} \circ^0 \mathfrak{p}, \mathfrak{q} \circ^0 \mathfrak{p})] \xRightarrow{(\mathfrak{p}^{(2)}, \sigma(\underline{}, \mathfrak{q} \circ^0 \mathfrak{p}))} [\sigma(\mathfrak{r}, \mathfrak{q} \circ^0 \mathfrak{p})]$$
$$\xRightarrow{(\mathfrak{p}^{(2)}, \sigma(\mathfrak{r}, z) \circ^0 \sigma(x, \underline{}))} [\sigma(\mathfrak{r}, \mathfrak{r})]$$

One can easily verify that $\mathfrak{P}^{(2)}$ is a second-order path from $[\sigma(\mathfrak{q}\circ^0\mathfrak{p},\mathfrak{q}\circ^0\mathfrak{p})]$ to $[\sigma(\mathfrak{r},\mathfrak{r})]$ of length 2. Let us note that none of the one-step subpaths of $\mathfrak{P}^{(2)}$ is a second-order echelon. Thus $\mathfrak{P}^{(2)}$ is an echelonless second-order path. In contrast to what happens in the first part of this work, the first-order translations $\sigma(\mathfrak{r},\underline{\quad})$ and $\sigma(\mathfrak{r},z)\circ^0\sigma(x,\underline{\quad})$ are non-identity translations but they are not associated to the same operation symbol.

Now consider the second-order path

$$\mathfrak{Q}^{(2)}\colon [\sigma(\mathfrak{q}\circ^0\mathfrak{p},\mathfrak{q}\circ^0\mathfrak{p})] \xRightarrow{(\mathfrak{p}^{(2)},\sigma(\underline{\quad},z)\circ^0\sigma(x,\mathfrak{q}\circ^0\mathfrak{p}))} [\sigma(\mathfrak{r},\mathfrak{q}\circ^0\mathfrak{p})] \xRightarrow{(\mathfrak{p}^{(2)},\sigma(z,z)\circ^0\sigma(\mathfrak{r},\underline{\quad})))} [\sigma(\mathfrak{r},\mathfrak{r})]$$

One can easily verify that $\mathfrak{Q}^{(2)}$ is a head-constant echelonless second-order path from $[\sigma(\mathfrak{q}\circ^0\mathfrak{p},\mathfrak{q}\circ^0\mathfrak{p})]$ to $[\sigma(\mathfrak{r},\mathfrak{r})]$ of length 2 associated to $\circ^0$, the 0-composition operation symbol. However, this second-order path is not coherent.

This is so, because from the equality

$$([\sigma(\mathfrak{r},z)])\circ^{0[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]}([\sigma(x,\mathfrak{q}\circ^0\mathfrak{p}]) = ([\sigma(z,z)])\circ^{0[\mathbf{PT}_{\boldsymbol{\mathcal{A}}}]}([\sigma(\mathfrak{r},\mathfrak{q}\circ^0\mathfrak{p}]),$$

we cannot infer that $[\sigma(\mathfrak{r},z)]=[\sigma(z,z)]$. In this regard, let us note that the class on the left represents paths of length 1, while the class on the right represents paths of length 0.

Now consider the second-order path

$$\mathfrak{R}^{(2)}\colon [\sigma(\mathfrak{q}\circ^0\mathfrak{p},\mathfrak{q}\circ^0\mathfrak{p})] \xRightarrow{(\mathfrak{p}^{(2)},\sigma(\underline{\quad},z)\circ^0\sigma(x,\mathfrak{q}\circ^0\mathfrak{p}))} [\sigma(\mathfrak{r},\mathfrak{q}\circ^0\mathfrak{p})] \xRightarrow{(\mathfrak{p}^{(2)},\sigma(\mathfrak{r},z)\circ^0\sigma(x,\underline{\quad}))} [\sigma(\mathfrak{r},\mathfrak{r})]$$

This is also a head-constant echelonless second-order path associated to the 0-composition operation symbol. However, this second-order path is coherent.