Character tables and Sylow 2-generation

Joint work with Gabriel Navarro, Noelia Rizo and Mandi Schaeffer Fry

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London Algebra Colloquium

Introduction: Objective

In this talk, all groups will be finite.

G finite group, $P \in \mathrm{Syl}_2(G)$. Assume $P = \langle x, y \rangle$ (includes P cyclic, dihedral, semidihedral, generalized quaternion, etc.).

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Write $f(2^n, 2)$ to be the number of isomorphism classes of 2-generated groups of order 2^n. Then

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Aim: Understand the character theory of groups possessing a 2-generated Sylow 2-subgroup.

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- In general, $\mathrm{Lin}(G) = \mathrm{Hom}(G, \mathbb{C}^\times) \subseteq \mathrm{Irr}(G)$.
- $\rho\colon \mathsf{S}_3 \to \mathrm{GL}_2(\mathbb{C})$ given by $(1\ 2\ 3) \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ and $(2\ 3) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is an irreducible representation of degree 2.

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Write $\mathrm{Irr}(G) = \{\chi_i\}_{i=1}^k$ and $\{g_j\}_{j=1}^k$ for G-conjugacy class representatives, then

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For example, $X(\mathrm{S}_3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$.

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For a prime p, x and y have the same p'-part iff $\chi(x) \equiv \chi(y) \mod (p)$ (in general, modulo any ideal of the ring of algebraic integers containing p).

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✗ The exponent of the group: $X(p_+^{1+2}) = X(p_-^{1+2})$, for p odd.

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Property or invariant	$\mathrm{X}(G)$
$\|P\|$	✓
$\mathbf{N}_G(P) = G$	✓
$\mathbf{N}_G(P) = P$	✓ (using CFSG) [Navarro-Tiep-Turull, '07] and [Schaeffer Fry, '19]
$\mathbf{N}_G(P)$ p-nilpotent	✓ (using CFSG) [Schaeffer Fry-Taylor, '18] and [Navarro-Tiep-V., '19]
$\|\mathbf{N}_G(P)\|$	☞ ?

Above $P \leqslant \mathbf{N}_G(P) = \{g \in G \mid P^g = P\} \leqslant G$.

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Theorem (Itô-Michler, 1986)

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$.

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- This theorem was one of the first applications of the CFSG.

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Write $G^0 = \{g \in G \mid p \nmid o(g)\} \subseteq G$ for the p-regular elements of G.

$\mathrm{Irr}(B_0(G)) = \{\chi \in \mathrm{Irr}(G) \mid \sum_{g \in G^0} \chi(g) \neq 0\} \ni \mathbf{1}_G$ is the principal block of G.

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($\Rightarrow$) Recently shown by Malle and Navarro (2021).

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• From this result, we cannot tell whether P is cyclic or not by just looking at $X(G)$. Are there ways to do so? Ideally in terms of $\mathrm{Irr}(B_0(G))$.

Galois action on characters and 1-generation of P

Let $\mathcal{G} = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/|G|})/\mathbb{Q})$. Then $\mathcal{G}$ acts on $\mathrm{Irr}(G)$ (and on $\mathrm{Irr}(B_0(G))$).

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• By (Navarro-Tiep, 2019 and Malle, 2020) $P \in \mathrm{Syl}_2(G)$ cyclic depends on the action of specific $\sigma_{2,e}$'s on $\mathrm{Irr}_{2'}(B_0(G))$. ✓ ✓

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Recall $\sigma_{2,1} \in \mathcal{G}$ fixes odd roots of unity and $\sigma_{2,1}(\omega) = \omega^3$ for every 2-power root of unity ω. Write $\sigma_1 = \sigma_{2,1}$ and $\mathrm{Irr}_{2'}(B_0(G))^{\sigma_1}$ for σ_1-fixed elements.

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Theorem A (Rizo-Schaeffer Fry-V., 2020)

G, $P \in \mathrm{Syl}_2(G)$ and $B_0 = B_0(G)$.

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- Theorem A *conjecturally* extends to general blocks and defect groups.

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$(\Leftarrow)$ Assume $P \in \mathrm{Syl}_2(G)$ is cyclic, then $G = M \rtimes P$ and

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($\Rightarrow$) Requires the use of the CFSG.

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cylic	1	1	1	1	1	$\dots$	1
abelian	1	2	3	5	7	$\dots$	$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$
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Recall $\sigma_1 \in \mathcal{K}_2$ sends 2-power roots of unity to their cube, and

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• For the general block version of Theorem B, we would like to know if the following problem has a positive answer.

Problem: Suppose that B is a 2-block of G with defect $P \trianglelefteq G$, such that P is elementary abelian. Is it true that $|\mathrm{Irr}(B)| = 4$ if, and only if, $D = \mathrm{C}_2 \times \mathrm{C}_2$?

An example. Does G have a 2-generated Sylow 2-subgroup?

$$X(G) = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -i & i & 1 & -1 & 1 & -1 & -i & i & -i & i & 1 & -1 \\
1 & -1 & i & -i & 1 & -1 & 1 & -1 & i & -i & i & -i & 1 & -1 \\
4 & 4 & -2 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
4 & 4 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\
4 & -4 & 2i & -2i & 1 & -1 & 0 & 0 & 0 & 0 & -i & i & -1 & 1 \\
4 & -4 & -2i & 2i & 1 & -1 & 0 & 0 & 0 & 0 & i & -i & -1 & 1 \\
5 & 5 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 \\
5 & 5 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 \\
5 & -5 & i & -i & -1 & 1 & 1 & -1 & -i & i & i & -i & 0 & 0 \\
5 & -5 & i & i & -1 & 1 & 1 & -1 & i & -i & -i & i & 0 & 0 \\
6 & 6 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & 0 & 0 & 0 & 1 & 1 \\
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1 & -1 & i & -i & 1 & -1 & 1 & -1 & i & -i & i & -i & 1 & -1 \\
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4 & -4 & 2i & -2i & 1 & -1 & 0 & 0 & 0 & 0 & -i & i & -1 & 1 \\
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5 & 5 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 \\
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5 & -5 & i & -i & -1 & 1 & 1 & -1 & -i & i & i & -i & 0 & 0 \\
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$$\sigma_1(i) = i^3 = -i$$

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Yes, it does! $G = \mathrm{A}_5 \rtimes \mathrm{C}_4$.

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- Theorem A and B follow from the Galois refinement of the Alperin-McKay conjecture proposed by Navarro in 2004.

Thanks for your attention!