Non-divisibility among character degrees *

Mark L. Lewis, Alexander Moretó, and Thomas R. Wolf

Abstract. In this paper, we look at groups for which if $1 < a < b$ are character degrees, then a does not divide b. We say that these groups have the condition no divisibility among degrees (NDAD). We conjecture that the number of character degrees of a group that satisfies NDAD is bounded and we prove this for solvable groups. More precisely, we prove that if a solvable group G has the condition no divisibility among degrees, then G cannot have more than 4 character degrees and the derived length of G cannot exceed 3. We give a group-theoretic characterization of the solvable groups satisfying NDAD with 4 character degrees. We remark that, since the structure of groups with at most 3 character degrees is fairly well-known, these results describe the structure of solvable groups with NDAD.

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1 Introduction

The character table of a finite group G encodes a lot of structural properties of G. However, it is generally a difficult problem to compute the full character table of a group. For this reason, it is interesting to be able to deduce group theoretical properties of G from a small part of the character table. In the last decades, a number of structural properties of a group G have been deduced from the set $\mathrm{cd}(G)$ of irreducible character degrees. For instance, groups whose character degrees are ordered by divisibility where studied by I. M. Isaacs and D. S. Passman in [9] and by R. Gow in [3]. In that situation, they were able to determine a great deal of information about the structure of those groups. In this paper, we look at groups that are in the opposite situation.

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Definition. We say that a group G has the condition **no divisibility among degrees** (NDAD) if for every $a, b \in \mathrm{cd}(G)$ with $1 < a < b$, a does not divide b.

For example, if all the non-linear irreducible characters have prime degree then G has the condition NDAD. This special case of NDAD was also considered by Isaacs and Passman in [9], who proved that in this case $|\mathrm{cd}(G)| \leq 3$. If we consider a solvable group G, then it can be proved that $|\mathrm{cd}(G)| \leq 3$ if we only assume that any two different character degrees in $\mathrm{cd}(G)$ are coprime (see Problem 12.3 of [8]). If G has the condition that any two different degrees are relatively prime, then G has NDAD, and thus, NDAD can be viewed as a weakening of the condition that the degrees are relatively prime. In this paper, we consider the following conjecture.

Conjecture A. There exists an (absolute) constant C such that if G is a finite group that satisfies the condition NDAD, then $|\mathrm{cd}(G)| \leq C$.

We will prove this conjecture for solvable groups. In this case, we can take $C = 4$. Certainly, the conclusion of this conjecture does not hold with $C = 4$ for nonsolvable groups. For instance, consider the character degrees $\mathrm{cd}(A_7) = \{1, 6, 10, 14, 15, 21, 35\}$. We do not know any example of a group G with the condition NDAD and $|\mathrm{cd}(G)| > 7$.

In fact, we can say more in the solvable case. We have found a description of those solvable groups that satisfy NDAD and have more than 3 character degrees. Recall that a p-group P is called a **Camina p-group** if for any $g \in P - P'$ the conjugacy class of g in P is gP'. In terms of notation, we define $P^1 = P$, and inductively, $P^i = [P^{i-1}, P]$. Let $\mathcal{C}$ be the class of groups with the following properties:

(i) $G = C \times K$ where C is abelian.

(ii) $F = \mathbf{F}(K) = D \times Z$ where D is a Sylow p-subgroup of K and Z is central in K.

(iii) D is a Camina p-group of nilpotence class 3.

(iv) K/F is cyclic.

(v) There exist groups A and B so that $F < A < K$ and $F < B < K$ so that:

   (a) $A \cap B = F$.

   (b) The action of K/F on D/D' is Frobenius.

   (c) The action of K/B on D'/D^3 is Frobenius and B centralizes D'/D^3.

(d) The action of K/A on D^3 is Frobenius and A centralizes D^3.

With this definition in mind, we can state our main result.

Theorem B. *Let G be a solvable group that satisfies NDAD. Then either $|\mathrm{cd}(G)| \leq 3$ or G lies in $\mathcal{C}$. Furthermore, if G lies in $\mathcal{C}$, then $\mathrm{cd}(G) = \{1, |K\colon F|, e|K\colon B|, f|K\colon A|\}$ where $e^2 = |P\colon P'|$ and $f^2 = |P\colon P^3|$.*

Since the groups in $\mathcal{C}$ have 4 character degrees, we obtain the following verification of Conjecture A for solvable groups. We know from Theorem 12.15 of [8] that if $|\mathrm{cd}(G)| \leq 3$, then G has derived length at most 3. By noting that the groups in $\mathcal{C}$ have derived length equal to 3, we also obtain a bound on the derived length of solvable groups satisfying NDAD.

Corollary C. *Let G be a solvable group that satisfies NDAD. Then we have $|\mathrm{cd}(G)| \leq 4$ and G has derived length at most 3.*

We have found an example of a group that lies in $\mathcal{C}$. The structure of groups with 3 character degrees is fairly well-known (see [10], [13], and [21]), so Theorem B can be seen as a classification of solvable groups with the condition NDAD. This solves Problem 58 of [1] for solvable groups.

In particular, these results are useful for the problem of determining the sets of integers that can occur as sets of character degrees of solvable groups. For instance, let $\mathcal{S} = \{1, 56, 76, 77, 120, 133, 209\}$. We have that $\mathrm{cd}(J_1) = \mathcal{S}$. By Theorem B, we conclude that if G is a group with $\mathrm{cd}(G) = \mathcal{S}$, then G is not solvable. This seems to bolster the conjecture by Huppert in [5] which states that the character degree set of a nonabelian simple group can only occur as the character degree set for a direct product of that simple group with an abelian group.

Suppose that G is a group lying in $\mathcal{C}$, and note that K/F is cyclic. Since $A \cap B = F$, it follows that $|A\colon F|$ and $|B\colon F|$ must be coprime, and thus, $\mathrm{cd}(G)$ involves at least three different primes. It follows that the set $\{1, p^2, q^2, pq\}$ where p and q are different primes, cannot occur as the set of character degrees of any finite group. (Recall that the character degrees of a nonsolvable group must involve at least three primes.)

We begin in Section 2 by looking at a graph associated with $\mathrm{cd}(G)$. We will prove that $|\mathrm{cd}(G)| \leq 3$ when this graph is disconnected. This result will be used in the proofs of the main theorems. In Section 3, we prove some elementary results about Camina groups. We obtain a number of properties of groups of Fitting height 2 with NDAD in Section 4. In particular, we prove that Theorem B is true under the condition that the group has Fitting height 2. The Fitting height 2 results will be used to prove Theorem B for groups of Fitting height larger than 2 in Section 5. In Section 6, we present a family of examples to show that the set $\mathcal{C}$ is not empty. We conclude the paper with some comments on possible generalizations of the results that we have obtained in Section 7.

Some of the work of the second author was done while he was visiting Kent State University. He thanks the Department of Mathematical Sciences for its hospitality.

2 Groups with disconnected graphs

In studying questions regarding the arithmetic structure of $\mathrm{cd}(G)$, it is often useful to consider the graph $\Delta(G)$, which is called the degree graph. This graph was first introduced in [17]. The vertices of this graph are the primes that divide the members of $\mathrm{cd}(G)$ and there is an edge between to vertices if the product of the two primes divides some member of $\mathrm{cd}(G)$. O. Manz, W. Willems and T. R. Wolf proved in [18] that $\Delta(G)$ has at most three connected components and that if G is solvable then it has at most two connected components. It is well-known that if G is solvable and $\Delta(G)$ is disconnected, then the structure of G is quite limited. This can be seen in [12]. Thus, it is not surprising that we can bound $|\mathrm{cd}(G)|$ for solvable groups with a disconnected degree graph. This is the main result in this section.

Theorem 2.1. *Assume that G is solvable, satisfies NDAD, and $\Delta(G)$ has two connected components. Then $|\mathrm{cd}(G)| = 3$.*

We need some lemmas that will also be useful in the proof of Theorem B. In the sequel, we will use that the NDAD property is inherited by quotients without mentioning it explicitly. First, we study affine semilinear groups. We will rely on results that were proved in [13]. If N is a normal subgroup of a group G, then $\mathrm{Irr}(G|N) = \{\chi \in \mathrm{Irr}(G) | N \not\subseteq \ker(\chi)\}$ and $\mathrm{cd}(G|N) = \{\chi(1) | \chi \in \mathrm{Irr}(G|N)\}$.

Lemma 2.2. *Let G act faithfully on an elementary abelian p-group V where p is some prime. Let A be maximal among abelian normal subgroups of G, and suppose that A acts irreducibly on V. Then there exists an integer a so that $a|A| \in \mathrm{cd}(GV|V)$ and $a|A|$ divides every degree in $\mathrm{cd}(GV|V)$.*

Proof. We use the notation set up in [13]. In other words, there is a group Γ which contains G, where Γ is the semi-direct product of group H acting on a group F. The group F is cyclic of order $|V| - 1$ and acts transitively on $V - \{1\}$. The group H is cyclic of order $f = |G\colon A|$, and $\mathbf{C}_V(H)$ has order p. Also, we know that $G \cap F = A$. In addition, F acts transitively on the nonprincipal characters of $\mathrm{Irr}(V)$, and for every nonprincipal character $\lambda \in \text{Irr}(V)$, the stabilizer $\mathbf{C}_\Gamma(\lambda)$ is a conjugate of H. Observe that $\mathbf{C}_G(\lambda)$ is cyclic and that λ extends to $V\mathbf{C}_G(\lambda)$. It follows that $\text{cd}(GV|\lambda) = \{|G\colon \mathbf{C}_G(\lambda)|\}$ and all the degrees in $\text{cd}(GV|V)$ have the form $|G\colon \mathbf{C}_G(\lambda)|$ for some character $1 \neq \lambda \in \text{Irr}(V)$.

We know that the action of A on V is Frobenius, so $\text{cd}(AV|V) = \{|A|\}$. It follows that $|A|$ divides every degree in $\text{cd}(GV|V)$. We suppose that the lemma is false, and we work to obtain a contradiction. Thus, there are degrees $m|A|, n|A| \in \text{cd}(GV|V)$ where $m \neq n$, and both $m|A|$ and $n|A|$ are minimal in the partial ordering of $\text{cd}(GV|V)$ determined by divisibility. We can find characters $\nu, \mu \in \text{Irr}(V)$ so that $|G\colon \mathbf{C}_G(\nu)| = n|A|$ and $|G\colon \mathbf{C}_G(\mu)| = m|A|$. Observe that $|G| = f|A|$, so $|\mathbf{C}_G(\nu)| = f/n$ and $|\mathbf{C}_G(\mu)| = f/m$.

Now, $\mathbf{C}_G(\nu) \subseteq \mathbf{C}_\Gamma(\nu)$ and $\mathbf{C}_G(\mu) \subseteq \mathbf{C}_\Gamma(\mu)$. We know that both $\mathbf{C}_\Gamma(\nu)$ and $\mathbf{C}_\Gamma(\mu)$ are conjugate to $H = \langle h \rangle$. It follows that $\mathbf{C}_G(\nu)$ is conjugate to $\langle h^n \rangle$ and $\mathbf{C}_G(\mu)$ is conjugate to $\langle h^m \rangle$. Now, we may apply Lemma 6 (a) of [13] to see that $h^n[h^n, F]$ is the set of Γ-conjugates of h^n and $h^m[h^m, F]$ is the set of Γ-conjugates of h^m. Our previous comments imply that $h^n[h^n, F] \cap G$ and $h^m[h^m, F] \cap G$ are both nonempty sets.

Let d be the greatest common divisor of n and m. Since $n|A|$ and $m|A|$ are minimal among divisibility, it follows that d is less than both n and m. Observe that $[h^n, F]$ and $[h^m, F]$ are both contained in $[h^d, F]$. Thus, there exist elements $x \in h^n[h^d, F] \cap G$ and $y \in h^m[h^d, F] \cap G$. Using the Euclidean algorithm, we can find integers a, b so that $na + mb = d$. We have $x^a y^b \in G$ and $x^a y^b \in h^{na} h^{mb}[h^d, F] = h^d[h^d, F]$, and $h^d[h^d, F] \cap G$ is nonempty. By Lemma 6(a) of [13], h^d has a conjugate $g \in G$, and note that $|G\colon \langle g \rangle| = d|A|$. Also, there exists a character $1 \neq \delta \in \text{Irr}(V)$ which is stabilized by g. As we have seen before, $|G\colon \mathbf{C}_G(\delta)| \in \text{cd}(GV|V)$. Also, $\langle g \rangle \subseteq \mathbf{C}_G(\delta)$. Therefore, $|G\colon \mathbf{C}_G(\delta)|$ divides $d|A|$ which is a proper divisor of both $n|A|$ and $m|A|$. This is our contradiction since $n|A|$ and $m|A|$ were chosen to be minimal in $\text{cd}(GV|V)$. □

Corollary 2.3. *Let G be a group that acts faithfully on an elementary abelian p-group V. Suppose that A is maximal among abelian normal subgroups of G, and that A acts irreducibly on V. If GV satisfies NDAD, then $|\text{cd}(GV)| \leq 3$.*

Proof. We see that $\text{cd}(GV) = \text{cd}(G) \cup \text{cd}(GV|V)$. In Lemma 2.2, we show there is a degree $a|A| \in \text{cd}(GV|V)$ dividing all degrees in $\text{cd}(GV|V)$. Since GV satisfies NDAD, we conclude that $|\text{cd}(GV|V)| = 1$. By Lemma 2.6 of [21], we know that $|G\colon A| \in \text{cd}(G)$, and by Itô's theorem, all degrees in $\text{cd}(G)$ divide $|G\colon A|$. Since G must satisfy NDAD, we conclude that $\text{cd}(G) = \{1, |G\colon A|\}$, and finally, $|\text{cd}(GV)| \leq 2 + 1 = 3$. □

Now, we are ready to proof Theorem 2.1. Recall that it asserts that $|\mathrm{cd}(G)| = 3$ if G is solvable, satisfies NDAD, and has a disconnected graph.

Proof of Theorem 2.1. In [12], the first author classified the solvable groups G where $\Delta(G)$ has two connected components. We will use that classification here.

In the Main Theorem of [12], it was proved that if G is solvable and $\Delta(G)$ has two connected components, then G satisfies the hypotheses of one of six possible examples that are labeled Examples 2.1-2.6 in [12]. We begin by supposing that G satisfies the hypotheses of Example 2.1 of [12]. In this case, G has a nonabelian normal Sylow p-subgroup for some prime p and G/P is abelian. We take F to be the Fitting subgroup of G. By Lemma 3.1 of [12], $|G\colon F| \in \mathrm{cd}(G)$ and $\mathrm{cd}(G)$ consists of powers of p and divisors of $|G\colon F|$. Since G satisfies NDAD, $\mathrm{cd}(G)$ contains at most one nontrivial p-power and no nontrivial, proper divisors of $|G\colon F|$. We conclude that $|\mathrm{cd}(G)| = 3$.

If G satisfies the hypotheses of Examples 2.2 or 2.3 of [12], $\mathrm{cd}(G)$ was computed in Lemmas 3.2 or 3.3 of [12]. In both cases, 2 and 8 lie in $\mathrm{cd}(G)$, so G does not satisfy NDAD.

If G satisfies the hypotheses of Example 2.4 of [12], then G is the semi-direct product of a group H acting on an elementary abelian p-group V for some prime p. Let $Z = \mathbf{C}_H(V)$ and K be the Fitting subgroup of H where K/Z is abelian and V is irreducible under the action of K. In Lemma 3.4 of [12], it is shown that $\mathrm{cd}(G) = \mathrm{cd}(G/Z)$. It is not difficult to see that $G/Z = VH/Z$ satisfies the hypotheses of Corollary 2.3. Applying the conclusion of Corollary 2.3, we have $|\mathrm{cd}(G)| = |\mathrm{cd}(G/Z)| \leq 3$. Since $\Delta(G)$ has two connected components, we must have $|\mathrm{cd}(G)| = 3$.

Suppose G meets the hypotheses of Example 2.5 in [12]. In this case, G has a normal nonabelian 2-subgroup Q and a Hall 2-complement K so that KQ has index 2 in G. If $Z = \mathbf{C}_K(Q)$, then $|K\colon Z| = 2^a + 1$ for some positive integer K. In Lemma 3.5 of [12], it is shown that $\mathrm{cd}(G)$ consists of 1, $2^a + 1$, and powers of 2. Since G satisfies NDAD, only one nontrivial power of 2 can occur in $\mathrm{cd}(G)$, and thus, $|\mathrm{cd}(G)| = 3$. (This is not proved in [12], but it is not too difficult to show that $\mathrm{cd}(G)$ must contain at least two powers of 2, and so, G cannot satisfy the hypotheses of Example 2.5 of [12].)

Finally, we show that G cannot satisfy the hypotheses of Example 2.6 of [12]. We suppose that G is such an example, and we obtain a contradiction. In particular, G has a normal subgroup $A = T' = [T, D]'$ where A is a nonabelian p-group for some prime p. Take L to be a normal subgroup of G contained in A' so that A'/L is a chief factor for G. We note that G/L also satisfies the hypotheses of Example 2.6 of [12]. If we find a contradiction in G/L, then we will have a contradiction in G, so we assume $L = 1$. This implies that A' is minimal normal in G.

Let F be the Fitting subgroup of G, and E/F the Fitting subgroup of G/F. We know that $A' \cap \mathbf{Z}(F) > 1$, and A' is minimal normal in G, so $A' \subseteq \mathbf{Z}(F)$. In Lemma 3.6 of [12], it is proven that F/A' is the Fitting subgroup of G/A' and G/A' satisfies the hypotheses of Example 2.4 of [12]. Using Lemma 3.4 of [12], both $|G\colon E|$ and $|E\colon F|$ occur in $\mathrm{cd}(G/A')$. Our earlier work shows that $|\mathrm{cd}(G/A')| = 3$, and hence, $\mathrm{cd}(G/A') = \{1, |G\colon E|, |E\colon F|\}$. In addition, G/A' satisfies the hypotheses of Lemma 4.1 (a) of [10]. It follows that $m = |G\colon E|$ is prime. Looking at Lemma 4.1 (a) of [10], we see that we can find a positive integer e so that $|A\colon A'| = q^m$ where $q = p^e$, and $(q^m - 1)/(q - 1)$ divides $|E\colon F|$. We can show $F/A' = \mathbf{C}_{G/A'}(A/A')$. From Lemma 3.6 of [12], we see that m divides no degree in $\mathrm{cd}(G)$ that is divisible by p.

We now consider a nonprincipal character $\varphi \in \mathrm{Irr}(A')$. We take C to be the stabilizer of φ in G, and note that $|G\colon C|$ divides the degree of character in $\mathrm{Irr}(G|\varphi)$. We know that p divides the degree of every character in $\mathrm{Irr}(A|\varphi)$, and so, p divides the degree of every character in $\mathrm{Irr}(G|\varphi)$. Therefore, m divides the degree of no character in $\mathrm{Irr}(G|\varphi)$, and in particular, m does not divide $|G\colon C|$. We conclude that m divides $|C\colon F|$ and $G = CE$. Suppose that φ is fully ramified with respect to A/A'. We know that $|A\colon A'| = q^m = p^{em}$ is a square. Thus, 2 divides em. Using Section 5 of [7], we can specify a character, called the magic character, Ψ of C/F where $\Psi(1) = p^{em/2}$. It follows from the proof of Theorem 5.7 of [7] that $|C\colon F|$ must divide either $p^{em/2} - 1$ or $p^{em/2} + 1$. In either case, we see that m divides $q^m - 1 = (p^{em/2} - 1)(p^{em/2} + 1)$. If m divides $q - 1$, it is not difficult to see that m divides $(q^m - 1)/(q - 1)$. Obviously, if m does not divide $q - 1$, then m must divide $(q^m - 1)/(q - 1)$. This is a contradiction since m and the primes dividing $(q^m - 1)/(q - 1)$ lie in different connected components of $\Delta(G)$. We conclude that φ is not fully ramified with respect to A/A'.

Using [7], φ determines a unique subgroup $A' \subseteq X \subseteq A$ so that all the characters in $\mathrm{Irr}(X|\varphi)$ are extensions of φ and fully ramified with respect to A/X. Since X is uniquely determined by φ and C stabilizes φ, it follows that C normalizes X. From [10], we know that the action of E/F on A/A' is Frobenius. If $C \cap E = F$, then $|E\colon F|$ will divide $|G\colon C|$. Recall that $p|G\colon C|$ divides every degree in $\mathrm{cd}(G|\varphi)$, so this contradicts the fact that G satisfies NDAD. We deduce that $E \cap C > F$. Now, the action of $E \cap C$ on X/A' is Frobenius. Using Glauberman's lemma, we see that φ has a unique C-invariant extension $\hat{\varphi} \in \mathrm{Irr}(X|\varphi)$. We can show that $E \cap C$ permutes the remaining characters in $\mathrm{Irr}(X|\varphi)$ in orbits of size $|E \cap C\colon F|$. Thus,

$\mathrm{cd}(C|\varphi)$ contains at least one degree divisible by $|E \cap C \colon F|$. Applying Clifford's theorem, $\mathrm{cd}(G|\varphi)$ contains a degree divisible by $|G \colon C||E{\cap}C \colon F| = |E \colon E{\cap}C||E{\cap}C \colon F| = |E \colon F|$. Since the degrees in$\mathrm{cd}(G|\varphi)$ are also divisible by p this contradicts NDAD. □

An immediate and useful consequence of Theorem 2.1 is the following.

Corollary 2.4. *If G is a solvable group satisfying NDAD and G has an irreducible character of prime degree, then $|\mathrm{cd}(G)| \leq 3$.*

Proof. Let p be the degree in $\mathrm{cd}(G)$ that is prime. Since G satisfies NDAD, p does not divide any other degree in $\mathrm{cd}(G)$, and so p is an isolated vertex in $\Delta(G)$. Either $\mathrm{cd}(G) = \{1, p\}$ and the result holds, or $\Delta(G)$ has two connected components and the result holds via Theorem 2.1. □

3 Camina Groups

Next, we study the characters of the Camina p-groups. We begin with a character-theoretic characterization.

Lemma 3.1. *A p-group P is a Camina p-group if and only if every nonlinear character in $\mathrm{Irr}(P)$ vanishes off of P'.*

Proof. Let $g \in P - P'$. By the orthogonality relations, we have that

$$|\mathbf{C}_P(g)| = \sum_{\chi \in \mathrm{Irr}(P)} |\chi(g)|^2 = |P : P'| + \sum_{\chi(1)>1} |\chi(g)|^2.$$

If P is a Camina group, then $|\mathbf{C}_P(g)| = |P \colon P'|$, and $0 = \sum_{\chi(1)>1} |\chi(g)|^2$. We determine that $\chi(g) = 0$ for each character $\chi \in \mathrm{Irr}(P)$ with $\chi(1) > 1$. Conversely, if every nonlinear character in $\mathrm{Irr}(P)$ vanishes on g, then $|\mathbf{C}_P(g)| = |P \colon P'|$. This implies that the conjugacy class of g in P has size $|P'|$. Since the conjugacy class of g is contained in gP', we see that g is conjugate to all the elements in gP'. This implies that P is a Camina group. □

We can apply this to Camina p-groups of nilpotence class 2 to obtain the following:

Corollary 3.2. *If P is a Camina p-group of nilpotence class 2, then every nonlinear character in $\mathrm{Irr}(P)$ is fully ramified with respect to P/P'.*

Proof. Consider a nonlinear character $\chi \in \mathrm{Irr}(P)$. By Lemma 3.1, we know that χ vanishes on $P - P'$. Let θ be an irreducible constituent of $\chi_{P'}$. Since P has class 2, P' is central and θ is invariant in P. In Problem 6.3 of [8], one shows that χ and θ are fully ramified with respect to P/P'. □

In our next result, we consider Camina p-groups of nilpotence class 3.

Lemma 3.3. *Let P be a Camina p-group of nilpotence class 3. Then all the nonlinear characters of P whose kernels contains $[P', P]$ are fully ramified with respect to P/P' and all the characters of $\mathrm{Irr}(P|[P', P])$ are fully ramified with respect to $P/[P', P]$.*

Proof. The first statement of the lemma follows from Corollary 3.2. It was proved in Theorem 5.2 of [14] that if P is a Camina p-group of class 3, then $|P : P'| = p^{4n}$ and $|P' : [P', P]| = p^{2n}$ for some positive integer n and that the conjugacy class of any element $x \in P' - [P', P]$ is $x[P', P]$. For $x \in P' - [P', P]$, we have

$$\left|\frac{P}{[P', P]}\right| = |\mathbf{C}_P(x)| = \sum_{\chi \in \mathrm{Irr}(P)} |\chi(x)|^2 = \sum_{\chi \in \mathrm{Irr}(\frac{P}{[P',P]})} |\chi(x)|^2 + \sum_{\chi \in \mathrm{Irr}(P|[P',P])} |\chi(x)|^2$$

For the characters in $\mathrm{Irr}(P/[P', P])$, we may replace x by the coset $x[P', P]$. Since $x[P', P]$ is central in $P/[P', P]$, the second orthogonality relation tells us that

$$\sum_{\chi \in \mathrm{Irr}(\frac{P}{[P',P]})} |\chi(x)|^2 = \left|\frac{P}{[P', P]}\right|.$$

The rest of the equation implies that

$$\sum_{\chi \in \mathrm{Irr}(P|[P',P])} |\chi(x)|^2 = 0.$$

Therefore, for every character $\chi \in \mathrm{Irr}(P|[P', P])$, we obtain $\chi(x) = 0$. We know from Lemma 3.1 that $\chi(y) = 0$ for $y \in P - P'$. It follows that χ vanishes on $P - [P', P]$. If γ is an irreducible constituent of $\chi_{[P',P]}$, then γ is G-invariant since $[P', P]$ is central in G. Again, we may apply Problem 6.3 of [8] to see χ is fully ramified with respect to $P/[P', P]$. □

In [2], Dark and Scoppola proved that a Camina p-group has nilpotence class at most 3. Thus, the previous lemmas cover all of the possible cases for a Camina p-group.

4 Groups of Fitting height 2

In this section we prove some lemmas concerning the structure of groups of Fitting height 2 with the NDAD property. In particular, we will prove Theorem B in the Fitting height 2 case.

We begin with the following lemma, that will be used several times.

Lemma 4.1. *Suppose that $\alpha, \beta \in \mathrm{Irr}(N)$ with N normal in G and G satisfies NDAD. Assume that α extend to $I_G(\alpha)$ and β extends to $I_G(\beta)$.*

(i) *Then either (a) $I_G(\alpha)/N$ is abelian or (b) $I_G(\alpha) = G$ and $\alpha(1) = 1$. In both cases, $|G : I_G(\alpha)|\alpha(1) \in \mathrm{cd}(G)$.*

(ii) *Assume that α is linear and $I_G(\alpha) < G$ and that $\beta(1) > 1$ or $I_G(\beta) < G$. If $I_G(\alpha\beta) = I_G(\alpha) \cap I_G(\beta)$, then $I_G(\alpha) = I_G(\beta)$.*

Proof. (i) Let $\tilde{\alpha} \in \mathrm{Irr}(I_G(\alpha))$ be an extension of α. By Clifford's Theorem, $(\tilde{\alpha})^G \in \mathrm{Irr}(G)$ and $|G : I_G(\alpha)|\alpha(1) \in \mathrm{cd}(G)$. If $\tau \in \mathrm{Irr}(I_G(\alpha)/N)$, then $\tau\tilde{\alpha}\mathrm{Irr}(I_G(\alpha)|\alpha)$ and $(\tau\tilde{\alpha})^G \in \mathrm{Irr}(G)$. Thus $|G : I_G(\alpha)|\alpha(1)\tau(1) \in \mathrm{cd}(G)$. It follows from NDAD that $\tau(1) = 1$ for all $\tau \in \mathrm{Irr}(I_G(\alpha)/N)$ or that $|G : I_G(\alpha)|\alpha(1) = 1$. In the first case, $I_G(\alpha)/N$ is abelian; while in the second case α is linear and G-invariant, proving (i).

(ii) Since α and β both extend to $I_G(\alpha\beta)$ and α is linear, also $\alpha\beta$ extends to $I_G(\alpha\beta)$. By (i) and hypotheses, $|G : I_G(\alpha)|, |G : I_G(\beta)|\beta(1)$ and $|G : I_G(\alpha\beta)|\beta(1)$ are all character degrees of G. Since $I_G(\alpha\beta) \subseteq I_G(\alpha)$, it follows from NDAD that $I_G(\alpha\beta) = I_G(\alpha)$. Likewise, $I_G(\alpha\beta) = I_G(\beta)$. This proves (ii). □

If G is a nonabelian, nilpotent group satisfying NDAD, it is not difficult to see that $\mathrm{cd}(G) = \{1, p^a\}$ for some positive integer a and a prime p. We now show that if G is a solvable group satisfying NDAD that has a nonabelian nilpotent quotient, then G either is itself nilpotent, or $\mathrm{cd}(G) = \{1, p\}$.

Lemma 4.2. *Let G be a solvable group satisfying NDAD. Assume $G/\mathbf{O}^p(G)$ is nonabelian for some prime p. Then $\mathrm{cd}(G) = \mathrm{cd}(G/\mathbf{O}^p(G)) = \{1, p^a\}$ for some positive integer a and either G is nilpotent or $\mathrm{cd}(G) = \{1, p\}$.*

Proof. Let $N = \mathbf{O}^p(G)$. The condition NDAD is inherited by quotients, so as we mentioned earlier, $\mathrm{cd}(G/N) = \{1, p^a\}$ for some positive integer a. Consider a character $\chi \in \mathrm{Irr}(G)$. If p does not divide $\chi(1)$, then $\chi_N \in \mathrm{Irr}(N)$, and $\chi(1)p^a \in \mathrm{cd}(G)$ via Gallagher's theorem. Using NDAD, we see that $\chi(1) = 1$. It follows that p divides every nontrivial degree in $\mathrm{cd}(G)$.

By a theorem of Thompson (see Corollary 12.2 of [8]), N is a normal p-complement of G.

We need to show that $\chi(1)$ is a power of p. Assume not and let $\theta \in \mathrm{Irr}(N)$ be a constituent of χ_N. Then θ is non-linear since G/N is a p-group. By Lemma 4.1 proposition, $I_G(\alpha)/N$ is abelian and $|G : I_G(\theta)|\theta(1) \in \mathrm{cd}(G)$. By Problem 2.9 of [8] applied to G/N, $|G : I_G(\theta)| \geq p^a$ and so p^a is a proper divisor of $|G : I_G(\theta)|\theta(1)$, contradicting NDAD.

We now apply Lemma 1.6 of [21]. That result states that if $|\mathrm{cd}(G)| = 2$ and G is not nilpotent, then either the nontrivial degree is a prime or else all Sylow subgroups of G are abelian. Since the Sylow p-subgroup of G is not abelian, it follows that either $a = 1$ or G is nilpotent. ☐

Let G be a group. We define $F_0(G) = 1$ and $F_1(G) = \mathbf{F}(G)$ the Fitting subgroup of G. Inductively, we define $F_i(G)$ by $F_i(G)/F_{i-1}(G) = \mathbf{F}(G/F_{i-1}(G))$. This yields a series of characteristic subgroups $F_0(G) \subseteq F_1(G) \subseteq \dots$ which is called the Fitting series of G. When G is solvable, it is not difficult to see that there is some integer n so that $G = F_n(G)$. The smallest such n is called the Fitting height of G.

We now apply Lemma 4.2 to the Fitting height 2 case. Notice that this result gives the character degrees of G when G satisfies NDAD, has Fitting height 2, and has an abelian Fitting subgroup.

Corollary 4.3. *Suppose that G is a solvable group with Fitting height 2 that satisfies NDAD. Let F be the Fitting subgroup of G. Then G/F is cyclic and $\mathrm{cd}(G/F') = \{1, |G\colon F|\}$.*

Proof. Let A/F be an abelian normal subgroup of G/F. By Lemma 18.1 of [19], $|A : F| = \sigma(1)$ for some character $\sigma \in \mathrm{Irr}(A)$. By Clifford's theorem, $|A : F|$ divides $\chi(1)$ for some character $\chi \in \mathrm{Irr}(G)$.

If G/F is abelian, we now have $|G : F|$ divides $\chi(1)$ for some character $\chi \in \mathrm{Irr}(G)$. By Itô's theorem, every degree in $\mathrm{cd}(G/F')$ divides $|G : F|$, whence NDAD implies that $\mathrm{cd}(G/F') = \{1, |G : F|\}$. We apply Lemma 1.6 of [21] to see that G/F is cyclic.

Assuming now that G/F is nonabelian but nilpotent, it follows from Lemma 4.2 that $\mathrm{cd}(G) = \{1, p\}$ for some prime p. By the first paragraph, the abelian normal subgroups of G/F have order p. This is a contradiction since every nonabelian p-group has a normal abelian subgroup of order divisible by p^2. ☐

An immediate consequence of this result is the following.

Corollary 4.4. *Let G be a solvable group of Fitting height $l > 1$ that satisfies NDAD. Then G/F_{l-1} is cyclic and $|G : F_{l-1}| \in \mathrm{cd}(G)$, where $F_{l-1} = F_{l-1}(G)$.*

The next result gives us the structure of the Fitting subgroup for a solvable group that satisfies NDAD.

Lemma 4.5. *Let G be a solvable group that satisfies NDAD. Let F be the Fitting subgroup of G. Then $F = P \times Z$ where P is the Sylow p-subgroup of F for some prime p and Z is abelian. Furthermore, if G is not nilpotent, then F is abelian or $Z \subseteq \mathbf{Z}(G)$. Also if G/F is cyclic and F is not abelian, then $\mathrm{cd}(G) = \mathrm{cd}(G/Z)$.*

Proof. If G is nilpotent the result is clear, so we may assume that $G > F$.

By Gaschütz's Theorem (see III.4.2, III.4.3 and III.4.5 of [4]), $F/\Phi(G)$ is a completely reducible and faithful G/F-module (possibly of mixed characteristic). Furthermore, $F/\Phi(G)$ is complemented in $G/\Phi(G)$ by $H/\Phi(G)$ for some $H \leq G$. Now each $\gamma \in \mathrm{Irr}(F/\Phi(G))$ is linear and since $F/\Phi(G)$ is complemented, we have that γ extends to $I_G(\gamma)$ (by Problem 6.18 of [8], for instance).

Since $G > F$, we may fix a non-central chief factor F/M. We let p be the prime divisor of $|F/M|$. If $r \neq p$ is another prime, we may choose N normal in G such that F/N is isomorphic to a Sylow r-subgroup of F. Now,

$$F/M \cap N \cong M/M \cap N \times N/M \cap N.$$

If $\delta \in \mathrm{Irr}(F/M)$ is not principal, then $\delta_N \in \mathrm{Irr}(N)$ and $F \subseteq I_G(\delta_N) = I_G(\delta) < G$. By Lemma 4.1, F/N is abelian and so is the Sylow r-subgroup of F. Thus the Hall p'subgroup of F is abelian and $F = P \times Z$, where P is the Sylow p-subgroup of F and Z is abelian.

Since $F \subseteq I_G(\mu)$ for all $\mu \in \mathrm{Irr}(Z)$, it follows from Lemma 4.1 that P is abelian or all the irreducible characters of Z are G-invariant. Thus $Z \subseteq \mathbf{Z}(G)$ or F is abelian.

Assume now that F is not abelian and G/F is cyclic. Then $Z \subseteq \mathbf{Z}(G)$ and $1_P \times \delta$ extends to G for all $\delta \in \mathrm{Irr}(Z)$, whence δ extends to G and $\mathrm{cd}(G|\delta) = \mathrm{cd}(G/Z)$. Thus $\mathrm{cd}(G) = \mathrm{cd}(G/Z)$. □

The next pair of technical lemmas will be useful, among other things, to prove that $|\mathrm{cd}(G)| \leq 3$ when G is a solvable group of Fitting height 2 whose a Fitting subgroup has nilpotence class 2.

Lemma 4.6. *Let a group H act faithfully and coprimely on a p-group P for some prime p. Write $C = \mathbf{C}_P(H)$ and $D = [P, H]$. Assume that the* *action of H on DP'/P' is Frobenius. Suppose that there is an H-invariant subgroup M in $\mathbf{Z}(P)$ so that $P/M = CM/M \times DM/M$, and suppose that $\mathbf{C}_H(M) > 1$. Then C centralizes D, so C is normal in P. Furthermore, if $C \cap M = 1$, then $P = C \times D$.*

Proof. Since the action of H on P is coprime, $P = CD$, and since C normalizes itself, C will be normal in P once we have shown that C centralizes D. Suppose that C does not centralize D, then there is some element $c \in C$ so that $[D, c]$ is not trivial. We define a map $f_c : D \to [D, c]$ by $f_c(d) = [d, c]$ for all $d \in D$. Since $P/M = CM/M \times D/M$, we know that $[D, c] \subseteq [D, C] \subseteq M \subseteq \mathbf{Z}(P)$, and it follows that f_c is a surjective homomorphism. For each element $x \in H$ and $d \in D$, we have

$$f_c(d^x) = [d^x, c] = [d^x, c^x] = [d, c]^x = (f_c(d))^x,$$

so f_c commutes with the action of H. Let $B = \mathbf{C}_D(c)$, and note that B is the kernel of f_c so B will be normalized by H. Because M is abelian and $f_c(D) = [D, C] \subseteq M$, it follows that $P' \subseteq B$.

Let $Y = \mathbf{C}_H(M)$. Because the action of H on D/P' is Frobenius, the action of Y on D/B is Frobenius. On the other hand, Y centralizes $[D, c] \subseteq M$. This is a contradiction since f_c is an isomorphism from D/B to $[D, c]$ that commutes with the action of Y, and we have proved that C centralizes D.

Suppose that $C \cap M = 1$. We know that $C \cap D \subseteq CM \cap D \subseteq M$, and thus, $C \cap D = C \cap M = 1$. We also have $P = CD$ where both C and D are normal in P, so it follows that $P = C \times D$. □

Lemma 4.7. *Let G be a solvable group of Fitting height 2 that satisfies NDAD. Suppose P is a normal Sylow p-subgroup of G of nilpotence class 2. Take H to be a Hall p-complement of G, and set $C = \mathbf{C}_P(H)$. Then C is a normal subgroup of G, and furthermore, if $\mathrm{cd}(G)$ contains no nontrivial p-powers, then C is central in G and we have the decompositions: $P = [P, H] \times C$ and $G = [P, H]H \times C$.*

Proof. Let F be the Fitting subgroup of G. By Lemma 4.5, $F \cap H$ is central in G, and $\mathrm{cd}(G) = \mathrm{cd}(G/F \cap H)$. Notice that if the conclusions hold for $G/(F \cap H)$, then they will hold for G. Thus, without loss of generality, we may assume that $F \cap H = 1$. This implies that $P = F$ and H acts faithfully on P. Also, $H \cong G/F$ which is cyclic via Corollary 4.3.

We know that H normalizes C, so to prove that C is normal in G, it suffices to prove that C is normal in P. Let $D = [P, H]$. By Corollary 4.3, $\mathrm{cd}(G/P') = \{1, |G\colon P|\} = \{1, |H|\}$, so it is not difficult to see that the action of H on P/CP' is Frobenius. By Fitting's lemma, we know that

$P/P' = CP'/P' \times DP'/P'$, and thus, the action of H on $DP'/P' \cong P/CP'$ is Frobenius.

Let $Y = \mathbf{C}_H(P')$, and our goal is to prove that $Y > 1$. We suppose that $Y = 1$. This implies that H acts faithfully $P'/\Phi(P')$. Thus, there is a character $\lambda \in \mathrm{Irr}(P')$ with $\mathbf{C}_H(\lambda) = 1$. Pick a character $\gamma \in \mathrm{Irr}(P|\lambda)$, and note that $\gamma_{P'} = \gamma(1)\lambda$ since P' is central in P. We have $\mathbf{C}_H(\gamma) \subseteq \mathbf{C}_H(\lambda) = 1$. We deduce that $\mathbf{C}_H(\gamma) = 1$, and thus, $\gamma(1)|H| \in \mathrm{cd}(G)$. Because $\gamma(1) > 1$ and $|H| \in \mathrm{cd}(G)$, we may use NDAD to see that this cannot occur, and thus, $Y > 1$. We may now apply Lemma 4.6 to see that C is normal in P.

Assume now that $\mathrm{cd}(G)$ contains no nontrivial powers of p. We will prove that this implies $C \cap P' = 1$, and using Lemma 4.6, we will have $P = C \times [P, H]$. Since H centralizes C, this implies that $G = C \times [P, H]H$. Suppose that $C \cap P' > 1$, and pick a nonprincipal character $\delta \in \mathrm{Irr}(C \cap P')$. We can find a character $\gamma \in \mathrm{Irr}(C|\delta)$, and let $\gamma^* \in \mathrm{Irr}(P)$ be the preimage under the Glauberman correspondence of γ (see Chapter 13 of [8]). If $\gamma^*(1) = 1$, then $P' \subseteq \ker(\gamma^*)$ and $C \cap P' \subseteq \ker(\gamma)$. Since this does not occur, $\gamma^*(1) > 1$. Now, γ^* is G-invariant, and using Corollary 6.28 of [8] this implies that γ^* extends to G which would yield a nontrivial p-power in $\mathrm{cd}(G)$. This contradicts the assumption that $\mathrm{cd}(G)$ has no nontrivial p-powers, so we must have $C \cap P' = 1$. □

Next, we prove $|\mathrm{cd}(G)| = 3$ when G is a solvable group of Fitting height 2 whose Fitting subgroup has nilpotence class 2. In the process, we obtain additional structural information of G.

Lemma 4.8. *Let G be a solvable group of Fitting height 2 that satisfies NDAD. Let P be a normal Sylow p-subgroup of G that has nilpotence class 2 where p is a prime. Let H be a Hall p-complement of G, $Z = \mathbf{C}_H(P)$, $Y = \mathbf{C}_H(P')$, $C = \mathbf{C}_P(H)$ and $E = \mathbf{C}_P(Y)$. Then $E = C \times [P', H]$, all the non-linear characters of $\mathrm{Irr}(P)$ are fully ramified with respect to P/E and have Y as their stabilizer in H, $\mathrm{cd}(P) = \{1, e\}$, and $\mathrm{cd}(G) = \{1, |H : Z|, e|H : Y|\}$. In particular, $|\mathrm{cd}(G)| = 3$, $Y > Z$, and $P' = [P, H]'$. Furthermore, if $Y < H$, then $e^2 = |P : E|$, and C is abelian.*

Proof. Let F be the Fitting subgroup of G. We know from Lemma 4.5 that $F = P \times Z$ where $Z \subseteq \mathbf{Z}(G)$. (Note that $Z = F \cap H \subseteq \mathbf{C}_H(P)$. On the other hand, $\mathbf{C}_H(P)$ is a normal, abelian subgroup of G, so $\mathbf{C}_H(P) \subseteq F \cap H = Z$. So the definition of Z here is consistent with the Z in the statement of the lemma.) Also, we have via Corollary 4.3 G/F is cyclic and $|G : F| \in \mathrm{cd}(G)$. Let H be a Hall p-complement for G. Note that $Z \subseteq Y$. It is not difficult to see that H/Y acts faithfully on the modules $P'/\Phi(P')$ and $\mathrm{Irr}(P'/\Phi(P'))$. We can find a character $1 \neq \lambda \in \mathrm{Irr}(P'/\Phi(P'))$ so that $\mathbf{C}_H(\lambda) = Y$. Using Glauberman's correspondence, we can find a character $\gamma \in \mathrm{Irr}(P|\lambda)$ so that

γ is stabilized by Y. Since $P' \subseteq \mathbf{Z}(P)$, we see that $\gamma_{P'} = \gamma(1)\lambda$, and thus, $\mathbf{C}_H(\gamma) \subseteq \mathbf{C}_H(\lambda) = Y$. We deduce that $Y = \mathbf{C}_H(\gamma)$. We see that PY is the stabilizer of γ in G, and γ extends to PY. This implies that $|H : Y|\gamma(1) \in \mathrm{cd}(G)$. Since $\gamma(1) > 1$, it follows that $|H : Z|$ does not divide $|H : Y|$, and hence, $Y > Z$.

Consider any nonlinear character $\delta \in \mathrm{Irr}(P)$. Observe that $P\mathbf{C}_H(\delta)$ is the stabilizer of δ in G, and δ extends to $P\mathbf{C}_H(\delta)$. This implies that $\delta(1)|H : \mathbf{C}_H(\delta)| \in \mathrm{cd}(G)$ and $\mathbf{C}_H(\delta)$ is abelian. We know that $\delta_{P'} = \delta(1)\mu$ for some character $1 \neq \mu \in \mathrm{Irr}(P')$, so $\mathbf{C}_H(\delta) \subseteq \mathbf{C}_H(\mu)$. Using the Glauberman correspondence as before, we can find a character $\sigma \in \mathrm{Irr}(P|\mu)$ so that $\mathbf{C}_H(\sigma) = \mathbf{C}_H(\mu)$. As before, $P\mathbf{C}_H(\sigma)$ is the stabilizer of σ in G and σ extends to $P\mathbf{C}_H(\sigma)$. It follows that $\sigma(1)|H : \mathbf{C}_H(\sigma)| \in \mathrm{cd}(G)$. By [7], there is a subgroup A with $P' \subseteq A \subseteq P$, so that every character in $\mathrm{Irr}(A|\mu)$ is an extension of μ and is fully ramified with respect to P/A. We conclude that $\delta(1) = \sigma(1)$. This implies that $\delta(1)|H : \mathbf{C}_H(\delta)|$ divides $\sigma(1)|H : \mathbf{C}_H(\sigma)|$. By NDAD, we have $\mathbf{C}_H(\delta) = \mathbf{C}_H(\sigma) = \mathbf{C}_H(\mu)$. Since $Y \subseteq \mathbf{C}_H(\mu)$, it follows that δ has only extensions on Y.

We have shown that $\mathrm{cd}(YP|P') = \mathrm{cd}(P) - \{1\}$. On the other hand, it is not difficult to see that $\mathrm{cd}(YP/P') = \{1, |Y : Z|\}$ since $\mathrm{cd}(G/P') = \{1, |G : F|\} = \{1, |H : Z|\}$. This implies that $\Delta(YP)$ has two connected components. We may apply the results of [12] to YP. We know from [12] that $P' \subseteq E$ and every nonlinear character in $\mathrm{Irr}(P)$ is fully ramified with respect to P/E. Since H normalizes both P and Y, it follows that H normalizes E. As E is normal in P, this is enough to show that E is normal in G. Observe that Y stabilizes every nonlinear character in $\mathrm{Irr}(P)$, so we may apply Theorem 19.3 of [19] to deduce that $P' = [P, H]'$.

By Lemma 4.7, we know that C is normal in G. Since $\mathrm{cd}(G/P') = \{1, |G : F|\}$, any coset of P/P' centralized by Y must be centralized by H. We determine that $E = CP'$. From Fitting's lemma, $P' = (C \cap P') \times [P', H]$, and thus, $E = C \times [P', H]$. Because P' is central in P, we see that $[P', H]$ is normal in P and in G. If $[P', H] = 1$, then $Y = H$, and the assertions of the lemma follow. We may assume that $[P', H] > 1$, and we choose a nonprincipal character $\alpha \in \mathrm{Irr}([P', H])$. Let $T = \mathbf{C}_H(\alpha)$, and let $\hat{\alpha}$ be the unique irreducible constituent of $(1_C \times \alpha)^P$. Observe that PT is the stabilizer of $\hat{\alpha}$ in G and that $\hat{\alpha}$ extends to PT. It follows that $e|H : T| \in \mathrm{cd}(G)$ where $e^2 = |P : E|$ so that $e = \hat{\alpha}(1)$.

For any character $\beta \in \mathrm{Irr}(C)$, we see that $T = \mathbf{C}_H(\beta \times \alpha)$. We take $\hat{\beta}_\alpha$ to be the unique irreducible constituent of $(\beta \times \alpha)^P$. Observe that $\hat{\beta}_\alpha(1) = e\beta(1)$ and PT is the stabilizer of $\hat{\beta}_\alpha$ in G. This implies that $e\beta(1)|H : T| \in \mathrm{cd}(G)$. Applying NDAD, we deduce that $\beta(1) = 1$, and thus, C is abelian. This implies that E is abelian, and in particular, $\mathrm{cd}(P) = \{1, e\}$. Recall that

$Y = \mathbf{C}_H(P')$ and that there is a character $\gamma \in \mathrm{Irr}(P)$ so that $|H\colon Y|\gamma(1) = |H\colon Y|e \in \mathrm{cd}(G)$. Furthermore, we showed for every nonlinear character $\delta \in \mathrm{Irr}(P)$ that $Y \subseteq \mathbf{C}_H(\delta)$ and that δ extends to $\mathbf{C}_H(\delta)$. This implies that $e|H\colon \mathbf{C}_H(\delta)| \in \mathrm{cd}(G)$ and $e|H\colon \mathbf{C}_H(\delta)|$ divides $e|H\colon Y|$. By NDAD, this implies $\mathbf{C}_H(\delta) = Y$, and we conclude that $\mathrm{cd}(G) = \{1, |H\colon Z|, e|H\colon Y|\}$ which is the desired result. □

We now come to the main result of this section. The following theorem proves Theorem B when G has Fitting height 2.

Theorem 4.9. *Let G be a solvable group with Fitting height 2 that satisfies NDAD. Then G satisfies the conclusion of Theorem B.*

Proof. Let F be the Fitting subgroup of G. We know from Lemma 4.5 that $F = P \times Z$ where P is a normal Sylow p-subgroup of G for some prime p and $Z \subseteq \mathbf{Z}(G)$. Also, $\mathrm{cd}(G/Z) = \mathrm{cd}(G)$. We may assume that $Z = 1$, and $F = P$. We have via Corollary 4.3 G/P is cyclic and $\mathrm{cd}(G/P') = \{1, |G\colon P|\}$. We assume that $|\mathrm{cd}(G)| > 3$, and we prove that G lies in $\mathcal{C}$. Take H to be a Hall p-complement of G. Observe that H acts faithfully on P. If P has nilpotence class 2, then Lemma 4.8 would imply that $|\mathrm{cd}(G)| \leq 3$ which is a contradiction, so P must have nilpotence class at least 3. This implies that $P^3 > 1$.

Define $C = \mathbf{C}_P(H)$ and $D = [P, H]$. Since H acts coprimely on P, we have $P = CD$. Obviously, D and H will normalize $DP'H$. In addition, C normalizes DP' and centralizes H, so $DP'H$ is a normal subgroup of G. We may apply Lemma 4.8 to G/P^3 to see that $P'/P^3 = (P/P^3)' = (DP^3/P^3)' = D'P^3/P^3$, and $P' = D'P^3$. Also, from that lemma, we determine that $Y > 1$ and Y is the stabilizer of all the characters in $\mathrm{Irr}(P/P^3|P'/P^3)$, where $Y = \mathbf{C}_H(P'/P^3)$.

For the time being, we assume that P has nilpotence class 3, and we will prove that G lies in $\mathcal{C}$. We will then use the results about the nilpotence class 3 case to prove that P must have nilpotence class 3. Therefore, until we say differently, assume that P has nilpotence class 3. That is, we assume that $P^4 = [P^3, P] = 1$.

We take a nonprincipal character $\mu \in \mathrm{Irr}(P^3)$. Consider a character $\nu \in \mathrm{Irr}(CP'|\mu)$, and observe that $\nu_{P^3} = \nu(1)\mu$, so $\mathbf{C}_H(\nu) \subseteq \mathbf{C}_H(\mu)$. Assume that $\mathbf{C}_H(\nu) > 1$. Let T be the stabilizer of ν in P, and using [7], we find a subgroup S so $CP' \subseteq S \subseteq T$ that is uniquely determined by ν so that every character in $\mathrm{Irr}(S|\nu)$ is an extension of ν and is fully ramified with respect to T/S. In particular, all the characters in $\mathrm{cd}(P|\nu)$ have degree $|P : T|n\nu(1)$ where $n^2 = |T\colon S|$. Since $\mathbf{C}_H(\nu)$ stabilizes ν and ν uniquely determines T and S, it follows that $\mathbf{C}_H(\nu)$ normalizes T and S. Now, the action of H

on P/CP' is Frobenius, so the action of $\mathbf{C}_H(\nu)$ on S/CP' is Frobenius or $S = CP'$. Thus, there is a unique $\mathbf{C}_H(\nu)$-invariant character $\sigma \in \mathrm{Irr}(S|\nu)$. In particular, for any element $h \in \mathbf{C}_H(\nu)$, the character σ is the unique character in $\mathrm{Irr}(S|\nu)$ that is invariant under the action of h.

Assume now that $S > CP'$. By Gallagher's theorem, any other character $\kappa \in \mathrm{Irr}(S|\nu)$ has the form $\lambda\sigma$ for some character $1 \neq \lambda \in \mathrm{Irr}(S/CP')$. It follows that $\mathbf{C}_H(\lambda) \cap \mathbf{C}_H(\nu) = 1$, and thus, $\mathbf{C}_H(\kappa) = \mathbf{C}_H(\sigma) \cap \mathbf{C}_H(\lambda) = 1$. Take $\hat{\kappa}$ to be the unique irreducible constituent of κ^T. This implies that $\hat{\kappa}^P$ is irreducible, and $\mathbf{C}_H(\hat{\kappa}^P) = \mathbf{C}_H(\hat{\kappa}) = \mathbf{C}_H(\kappa) = 1$. Thus, $|G : P|\hat{\kappa}^P(1) \in \mathrm{cd}(G)$ which contradicts NDAD since $\hat{\kappa}^P(1) > 1$ and $|G\colon P| = |H| \in \mathrm{cd}(G)$. It follows that $S = CP'$ and $\mathrm{Irr}(P|\nu) = \{\hat{\sigma}^P\}$ where $\hat{\sigma}$ is the unique irreducible constituent of σ^T. We conclude that $\mathbf{C}_H(\hat{\sigma}^P) = \mathbf{C}_H(\nu)$.

Define $X = \mathbf{C}_H(P^3)$, and recall $Y = \mathbf{C}_H(P'/P^3)$. We now work to show that $X \cap Y = 1$, and to find a contradiction we assume that $X \cap Y > 1$. Consider a nonlinear character $\gamma \in \mathrm{Irr}(P)$. If P^3 is in the kernel of γ, then Y (and hence, $X \cap Y$) stabilizes γ. Suppose that P^3 is not in the kernel of γ. We take ν to be an irreducible constituent of $\gamma_{CP'}$, and we take μ to be the unique irreducible constituent of ν_{P^3}. In fact, $\gamma_{P^3} = \gamma(1)\mu$, and thus, $\mu \neq 1_{P^3}$. Observe that X centralizes P^3, Y centralizes P'/P^3, and both X and Y centralize C, so $X \cap Y$ centralizes CP'. This implies that $X \cap Y$ stabilizes ν. Using the previous paragraph, we see that $X \cap Y$ stabilizes γ. Thus, $X \cap Y$ stabilizes all nonlinear characters in $\mathrm{Irr}(P)$. By Theorem 19.3 of [19], this implies that P has nilpotence class 2 which is a contradiction. Therefore, $X \cap Y = 1$.

Now, H/X acts faithfully on P^3, and H is cyclic, so we can find a character $\mu \in \mathrm{Irr}(P^3)$ so that $\mathbf{C}_H(\mu) = X$. Using the Glauberman correspondence, we can find a character $\nu \in \mathrm{Irr}(CP'|\mu)$ so that ν is X-invariant. Note that $\nu_{P^3} = \nu(1)\mu$, and thus, $X \subseteq \mathbf{C}_H(\nu) \subseteq \mathbf{C}_H(\mu) = X$. We saw earlier that ν^P has a unique irreducible constituent γ and $\mathbf{C}_H(\gamma) = X$. Observe that PX is the stabilizer of γ in G and that γ extends to PX. Thus, $\gamma(1)|H\colon X| \in \mathrm{cd}(G)$. Since $\gamma(1) > 1$ and $|H| \in \mathrm{cd}(G)$, we conclude that $X > 1$. Recall that $Y > 1$, and since $X \cap Y = 1$, this implies that $X < H$ and $Y < H$.

By Lemma 4.8, we see that $\mathrm{cd}(G/P^3) = \{1, |H|, e|H\colon Y|\}$ where $e^2 = |P\colon P'|$ since $Y < H$. This contains no powers of p, so we may apply Lemma 4.7 to P/P^3 to see that $P/P^3 = CP^3/P^3 \times DP^3/P^3$. Now, P^3 is central in P, and $X > 1$. Using Lemma 4.6, we determine that C centralizes D and C is normal in P. Recall that $P = CD$ and $P' = D'P^3$. This implies that $P^3 = [P', P] = [D'P^3, P] = [D', P] = [D', CD] = [D', D]$, since P^3 is central in P and C centralizes D. We determine that $P' = D'P^3 = D'[D', D] = D'$ and $DP^3 = D[D', D] = D$.

We are now finally ready to show that D is a Camina group. In Lemma 4.8, we showed that all the the nonlinear characters in $\mathrm{Irr}(P/P^3)$ are fully ramified with respect to P/CP'. This implies that all the nonlinear characters in $\mathrm{Irr}(P/CP^3)$ are fully ramified with respect to $P/CP' \cong \frac{P/CP^3}{CP'/CP^3}$. Since $D/P^3 \cong P/CP^3$, it follows that all nonlinear characters in $\mathrm{Irr}(D/P^3)$ are fully ramified with respect to D/P'. This implies that all the nonlinear characters in $\mathrm{Irr}(D/P^3)$ vanish on $D - D'$, and D/P^3 is a Camina p-group. By Corollary 2.3 of [14], D/D' is elementary abelian. Since H centralizes C and $P = CD$, we see that DH is a normal subgroup of G. Also, it is not difficult to see that DH/D' is a Frobenius group with $\mathrm{cd}(DH/D') = \{1, |H|\}$. For any nonlinear character $\delta \in \mathrm{Irr}(D)$, if $\delta(1)|H| \in \mathrm{cd}(DH)$, then there is a degree in $\mathrm{cd}(G)$ which is divisible by p and $|H|$ which contradicts NDAD since $|H| \in \mathrm{cd}(G)$. By Theorem 12.4 of [8], this implies that δ vanishes on $D - D'$. Therefore, D is a Camina p-group. By Lemma 3.3, all the characters in $\mathrm{Irr}(D/[D', D]|D'/[D', D])$ are fully ramified with respect to D/D' and all the characters in $\mathrm{Irr}(D|[D', D])$ are fully ramified with respect to $D/[D', D]$, so $\mathrm{cd}(D) = \{1, e, f\}$.

By Lemma 3.3, we know that all nonprincipal characters in $\mathrm{Irr}(P^3)$ are fully ramified with respect to D/P^3. Also, H stabilizes all the characters in $\mathrm{Irr}(P^3/[P^3, H])$. This implies that H stabilizes all the characters in $\mathrm{Irr}(D/[P^3, H]|P^3/[P^3, H])$. Since Y stabilizes all the nonlinear characters in $\mathrm{Irr}(D/P^3)$, we see that Y stabilizes all the nonlinear characters in $\mathrm{Irr}(D/[P^3, H])$. As we have noted before, Theorem 19.3 of [19] implies that $D/[P^3, H]$ has nilpotence class 2, and thus, $[D', D] \subseteq [P^3, H] \subseteq P^3 = [D', D]$. We conclude that $P^3 = [P^3, H]$, and Fitting's lemma implies that $C \cap P^3 = 1$. We may now use Lemma 4.6one more time to obtain $P = C \times D$. Since $C' \subseteq P' \subseteq D$ and $C \cap D = 1$, it follows that C is abelian. We obtain $P = C \times D$ and $G = C \times DH$. Note that D will have nilpotence class 3. It is not difficult to determine that $\mathrm{cd}(G) = \{1, |H|, e|H : Y|, f|H : X|\}$. We take $K = DH$ and $A = DX$ and $B = DY$. We have now shown that G has the desired properties when P has nilpotence class 3.

We now assume that G is a counterexample with $|G|$ minimal. It is not difficult to see that P has nilpotence class 4 where $P^4 = [P^3, P]$ is a minimal normal subgroup of G. This implies that P^4 is central in P. The previous work shows that $P/P^4 = CP^4/P^4 \times DP^4/P^4$, $P^3 \subseteq DP^4$, and $C' \subseteq P^4$. Now, $H/\mathbf{C}_H(P^4)$ acts faithfully on P^4, so we can find a character $\delta \in \mathrm{Irr}(P^4)$ with $\mathbf{C}_H(\delta) = \mathbf{C}_H(P^4)$. Working as before, we can find a character $\eta \in \mathrm{Irr}(P|\delta)$ so that $\mathbf{C}_H(\eta) = \mathbf{C}_H(\delta)$. This implies that $\eta(1)|H : \mathbf{C}_H(P^4)| \in \mathrm{cd}(G)$. Since $|H| \in \mathrm{cd}(G)$, we may use NDAD to see $\mathbf{C}_H(P^4) > 1$. By Lemma 4.6, C is normal in P and centralizes D. We know that either $P^4 \subseteq D$ or $P^4 \cap D = 1$. If $P^4 \cap D = 1$, then $P = C \times D$ where C has nilpotence class at most 2 and D has nilpotence class at most 3 which contradicts P having nilpotence class 4. We obtain $P^4 \subseteq D$, and hence, $P' = D'$. Note that D will have nilpotence class 4. We know that D/P^3 is a Camina p-group, as before D/D' is elementary abelian, and DH/D' is a Frobenius group. As before, we deduce that all the nonlinear characters in $\mathrm{Irr}(D)$ vanish on $D - D'$. Therefore, D is a Camina p-group. This is a contradiction, since the Main theorem of [2] shows that a Camina p-group has nilpotence class at most 3. □

5 Groups of Fitting height larger than 2

The goal of this section is to prove Theorem B for solvable groups that have Fitting height greater than 2. We will see that this case reduces to looking at GV where G is a G of Fitting height 2, V is an irreducible, faithful G-module, and GV satisfies NDAD. Our first lemma looks at GV without assuming that GV satisfies NDAD, but it does assume that G acts transitively on $V - \{0\}$.

Lemma 5.1. *Suppose that G is a solvable group acting faithfully on a vector space V of order q^n. Assume that G acts transitively on the non-zero vectors of V. Let F be the Fitting subgroup of G. Then one of the following occurs:*

(i) *G is nilpotent;*

(ii) *There exists a character $\sigma \in \mathrm{Irr}(G)$ such that $1 \neq \sigma(1)$ is a proper divisor of $q^n - 1$; or*

(iii) *G/F and F are cyclic of orders dividing n and $q^n - 1$ (respectively) and $\mathrm{cd}(GV|V) = \{q^n - 1\}$. Furthermore, if $\mathrm{cd}(G) = \{1, |G : F|\}$, then either (1) $|G : F|$ properly divides $q^n - 1$ or (2) $|G : F| = r$ and $|F| = q^n - 1$ where r is a prime divisor of n that does not divide $q^n - 1$.*

Proof. If $\lambda \in \mathrm{Irr}(V)$, then λ extends to $I_{GV}(\lambda) = VI_G(\lambda)$ since V is complemented in GV (see Problem 6.18 of [8]). It follows that $|G : I_G(\lambda)| = |GV : I_{GV}(\lambda)| \in \mathrm{cd}(GV)$. Since G is acting transitively on V, we have by the Fundamental Counting Principal $|G : I_G(\lambda)| = q^n - 1$, and so, $q^n - 1 \in \mathrm{cd}(GV)$.

A theorem of Huppert (Theorem 6.8 of [19]) classifies those solvable groups that act transitively on the nonidentity elements of a finite vector space. From that theorem, we have that either G is a subgroup of the semi-linear group $\Gamma(q^n)$ or $q^n = 3^2, 5^2, 7^2, 11^2, 23^2, \mathrm{or} 3^4$. Furthermore, the structure of G is explicitly given in these exceptional cases. In all of the exceptional cases, $\mathrm{cd}(G)$ contains a degree $t > 1$ with t a proper divisor of $q^n - 1$. Thus, conclusion (ii) is satisfied in the exceptional cases. We may assume that G is a subgroup of the semi-linear group $\Gamma(q^n)$ and G is not nilpotent. Note that $F < G$. Applying Corollary 6.6 and Lemmas 6.4 and 6.5 of [19], F and G/F are cyclic of orders dividing $q^n - 1$ and n, respectively.

If $\lambda \in \mathrm{Irr}(V)$ is not principal, then from the first paragraph there is a character $\chi \in \mathrm{Irr}(GV|\lambda)$ with $\chi(1) = |G : I_G(\lambda)| = q^n - 1$. Now, $I_F(\lambda) = 1$, so $I_G(\lambda)$ is isomorphic to a subgroup of G/F. Thus, $I_G(\lambda)$ is cyclic. By Gallagher's theorem (Corollary 6.17 of [8]), every character in $\mathrm{cd}(GV|\lambda)$ has degree $q^n - 1$, and hence, $\mathrm{cd}(GV|V) = \{q^n - 1\}$.

For the rest of this proof, we assume that $\mathrm{cd}(G) = \{1, |G : F|\}$, and notice that this implies $\mathrm{cd}(GV) = \{1, |G : F|, q^n - 1\}$. Certainly, F is maximal abelian normal subgroups of G. Also, V_F is homogeneous since G acts transitively on the nonprincipal characters in $\mathrm{Irr}(V)$. Since F is abelian and V_F is homogeneous, V_F is irreducible (see Lemma 2.2 of [19]). Because $|\mathrm{cd}(GV)| = 3$ and V_F is irreducible, the results of [13] apply. By Theorem B of [13], either (a) $\mathrm{cd}(GV) = \{1, |G : F|, |G|\}$ or (b) G/F is an r-group for some prime r and $\mathrm{cd}(GV) = \{1, |G : F|, |G|/r\}$. We know $|G|$ cannot lie in $\mathrm{cd}(G)$, so it must be that $|G| \in \mathrm{cd}(GV|V)$, and equating this with the value in the previous paragraph, $|G| = q^n - 1$ or $|G| = (q^n - 1)r$. Since $|G : F|$ divides n, it is well-known that $|G : F| \leq n < q^n - 1$. When $|G| = q^n - 1$, it follows that $|G : F|$ is a proper divisor of $q^n - 1$.

We now assume that G/F is an r-group for some prime r and $|G| = (q^n - 1)r$. Since $|G : F|$ divides n, we conclude that r divides n. By Theorem 5 of [13], either $|G : F|$ divides $|G|/r$ or $(|G : F|, |G|/r) = 1$. If $|G : F|$ divides $|G|/r = q^n - 1$, then $|G : F|$ will be a proper divisor of $q^n - 1$. We assume that $(|G : F|, |G|/r) = 1$. This implies that $|G : F| = r$, $|F| = |G|/r = q^n - 1$, and r does not divide $q^n - 1$. $\square$

We continue in the situation outlined at the beginning of this section. When looking at GV, we break the proof up into two cases depending on whether V is quasi-primitive or not. In the next lemma, we consider the case when V is quasi-primitive.

Theorem 5.2. *Suppose V is a faithful, irreducible, and quasi-primitive G-module for some group G and GV satisfies NDAD. If G is solvable with Fitting height 2, then $|\mathrm{cd}(GV)| = 3$.*

Proof. Notice that GV has Fitting height 3, so $|\mathrm{cd}(GV)| \geq 3$, and it suffices to prove $|\mathrm{cd}(GV)| \leq 3$. Set $F = \mathbf{F}(G)$. Now, V_F is homogeneous. If F is also abelian, then G acts semi-linearly on V (see [19]) and Corollary 2.3 implies that $|\mathrm{cd}(GV)| \leq 3$. Thus, we may assume that F is not abelian, and we will find a contradiction. Since G must satisfy NDAD, we may apply Lemma 4.5 to see that $F = P \times Z$ where P is a nonabelian Sylow p-subgroup of F and $Z \subseteq \mathbf{Z}(G)$.

We may apply Corollary 4.3 to conclude that G/F cyclic and $|G : F| \in \mathrm{cd}(G)$. Now, $F/\Phi(G)$ is a completely reducible and faithful module for the cyclic group G/F. Since $Z \subseteq \mathbf{Z}(G)$, we see that $P/(\Phi(G) \cap P)$ is a faithful G/F module. We deduce that p does not divide $|G : F|$.

We claim that $\mathrm{cd}(G)$ has no degree of the form p^a with $a \geq 1$. Suppose $\nu \in \mathrm{Irr}(G)$ with $\nu(1) = p^a$. Since $(\nu(1), |G : F|) = 1$, we know that ν_P is irreducible. Therefore, $p^a < |P|^{1/2}$. On the other hand, $\mathrm{Irr}(V)$ has a P-orbit $\mathcal{O}$ with $|\mathcal{O}| \geq |P|^{1/2}$ (see Theorem 4.7 of [19]). If $\tau \in \mathcal{O}$ and $\sigma \in \mathrm{Irr}(GV|\tau)$, then $|P : I_P(\tau)|$ divides $\sigma(1)$ using the fact that P is normal in G. It follows that $\nu(1)$ properly divides $\tau(1)$. In view of NDAD, it must be that $\nu(1) = 1$ and $a = 0$. Thus, the claim is proved.

Since V is quasi-primitive, we may use Corollary 1.10 of [19] to see that $P = ET$ for normal subgroups E and T of G with E extra-special and $E \cap T = \mathbf{Z}(E)$. Also, T has a subgroup U of index 1 or 2 that is cyclic and normal in G. In addition, $U = \mathbf{C}_T(U)$, $[E, T] = 1$, and $\mathbf{Z}(E) \subseteq \mathbf{Z}(F)$.

Suppose that $\mathbf{Z}(E) \subseteq \mathbf{Z}(G)$. Let $\lambda \in \mathrm{Irr}(\mathbf{Z}(E))$ be nonprincipal, and we know that λ is G-invariant. We know that $\mathbf{Z}(E)$ is the unique subgroup of order p in the cyclic group $\mathbf{Z}(P)$. Since $P' > 1$, we know that $P' \cap \mathbf{Z}(P) > 1$, and so, $\mathbf{Z}(E) \subseteq P'$. Let H be a p-complement for G. We know that $G = PH$ and $H \cap F = Z$. Now, H acts coprimely on P, and λ is invariant under the action of H. By Theorem 13.28 of [8], some irreducible constituent μ of λ^P is H-invariant. Notice that μ cannot be linear. Since $G = PH$, we see that μ is G-invariant, and hence, $\mu \times 1_Z \in \mathrm{Irr}(P \times Z) = \mathrm{Irr}(F)$ is G-invariant. Since G/F is cyclic, $\mu \times 1_Z$ extends to $\nu \in \mathrm{Irr}(G)$. Now, $\nu(1) = \mu(1) = p^a$ for some $a \geq 1$. This contradicts the earlier claim that $\mathrm{cd}(G)$ contains no nontrivial p-powers. Thus, $\mathbf{Z}(E)$ is not central in G, and since $|\mathbf{Z}(E)| = p$, we deduce that $p > 2$.

Since p is odd, T is cyclic and $T \subseteq \mathbf{Z}(F)$. But $\mathbf{Z}(E)$, which is the unique subgroup of T of order p is not central in G. It follows that $\mathbf{C}_T(H) = 1$.

We claim that $P = E$ and $T = \mathbf{Z}(E)$. We assume that the claim is not true, and we work to find a contradiction. Let S be the unique subgroup of index p in T, and observe that $\mathbf{Z}(E) \subseteq S$ and $P/S = ES/S \times T/S$. Now, ES/S is a direct product of G/F-modules, each of even dimension (see Theorem 1.9 of [19]). Since $\mathbf{C}_T(H) = 1$, it follows that T/S is nontrivial as a module for G/F, and $P/S = ES/S \times T/S$ is a direct product of nontrivial G/F-modules. For a nonprincipal character $\delta \in \mathrm{Irr}(P/S)$, we obtain $1 \neq |G : I_G(\delta)| \in \mathrm{cd}(G)$. Now, $F \subseteq I_G(\delta)$ and $|G : F| \in \mathrm{Irr}(G)$. Applying NDAD, we obtain $F = I_G(\delta)$ for all nonprincipal characters $\delta \in \mathrm{Irr}(P/S)$. In particular, T/S is a faithful G/F-module. Since $|T/S| = p$ and G/F is cyclic, we determine that $|G : F|$ divides $p - 1$. Every irreducible G/F-module over Z_p must have dimension 1. Now, ES/S is a direct product of irreducible G/F-modules of even dimension. It follows that $ES = S$ and $P = T$ is abelian. This is a contradiction since P is nonabelian. We deduce that $T = \mathbf{Z}(E) = \mathbf{Z}(P)$ and $P = E$. Also, $\mathbf{Z}(F) = Z \times \mathbf{Z}(E)$.

Now, $P/\mathbf{Z}(P) \cong F/\mathbf{Z}(F)$ is a direct product of irreducible G/F-modules that are nontrivial and have even dimension. Working as in the previous paragraph, we can show $I_G(\delta) = F$ for every nonprincipal character $\delta \in \mathrm{Irr}(F/\mathbf{Z}(F))$. Thus, $G/\mathbf{Z}(F)$ is a Frobenius group, whose only abelian subgroups are p-groups and p'-groups.

We know that P is not abelian, and since $p > 2$, we see that P cannot act Frobeniusly on $\mathrm{Irr}(V)$. There exists a character $\mu \in \mathrm{Irr}(V)$ with $1 < I_P(\mu) < P$. By Lemma 4.1, $I_G(\mu)$ is abelian. Now, $I_{\mathbf{Z}(F)}(\mu) = 1$, and thus, $I_G(\mu) \cong \mathbf{Z}(F)I_G(\mu)/\mathbf{Z}(F)$ is abelian and has order divisible by p. By the last paragraph, $\mathbf{Z}(F)I_G(\mu)/\mathbf{Z}(F)$ is a P-group, and thus, $I_G(\mu) \subseteq P \subseteq F$. We now have $|G : F||F : I_F(\mu)| = |G : I_G(\mu)| \in \mathrm{cd}(GV)$. Since $|G : F| \in \mathrm{cd}(G)$, this is a contradiction to NDAD. With this contradiction, the lemma is proved. □

We now turn to the case where V is not quasi-primitive. We will make use of the following lemma which is essentially Gluck's theorem on orbits of power sets for solvable primitive permutation groups (see Theorem 5.6 of [19]).

Lemma 5.3. *Suppose that S is a solvable primitive permutation group on Ω with $|\Omega| = n$.*

(i) *If S is abelian, then $|S| = |\Omega| = p$ for some prime p;*

(ii) *If S is nonabelian, then one of the following occurs:*

  (a) There exists $\Delta \subseteq \Omega$ such that $\mathbf{N}_S(\Delta) = 1$;
  (b) S has an irreducible character of prime degree;
  (c) $n = 5$ and S is a Frobenius group of order 20;
  (d) $n = 7$ and S is a Frobenius group of order 42.

Proof. As a solvable primitive permutation group S has a unique minimal normal subgroup M that regularly and transitively permutes Ω, so that $n = |\Omega| = |M| = p^k$ for a prime power p^k. Also, M is a faithful irreducible module for S/M. If S is abelian, then $S = M$ has prime order. Thus, we may assume that S is nonabelian.

Gluck showed that S must satisfy (i) unless $n \leq 9$ (see Theorem 5.6 of [19]). The exceptions to Gluck's theorem are listed explicitly in Theorem 5.6 of [19]. In the exceptions when n is 8 or 9, S/M has an irreducible character of degree 2 or 3. The exceptions to Gluck's theorem when $n < 8$ are S_3, S_4, A_4, the Frobenius group of order 10, and those listed in (ii) (c) and (d) above. This proves the proposition as S_3, S_4, A_4, and the Frobenius group of order 10 each have an irreducible character of prime degree. $\square$

We now apply Lemma 5.3 to GV. Since we are working in the case where V is not quasi-primitive, there will be a normal subgroup N so that V_N is not homogeneous.

Lemma 5.4. *Suppose that V is a faithful, irreducible G-module for some group G where GV satisfies NDAD. If V_N is not homogeneous for some normal subgroup N of G, then either (1) $|\mathrm{cd}(GV)| \leq 3$ or (2) there exists a normal subgroup C with $N \subseteq C \subseteq G$ and $|G : C| = 2$ such that $V_C = V_1 \times V_2$ for homogeneous components V_i of V_C and C transitively permutes the nonprincipal characters in $\mathrm{Irr}(V_i)$ for each i.*

Proof. Choose C normal in G maximal with respect to V_C is not homogeneous and $N \subseteq C$. Then $V_C = V_1 \times \cdots \times V_n$ for homogeneous components V_i of V_C that are transitively and faithfully permuted by $S = G/C$. Furthermore, S primitively permutes $\Omega = \{V_1, \ldots, V_n\}$.

Let $\Delta \subseteq \Omega$. We claim that $|S : \mathbf{N}_S(\Delta)|$ divides $\chi(1)$ for some character $\chi \in \mathrm{Irr}(GV)$. To see this, we may assume that $\Delta = \{V_1, V_2, \ldots, V_j\}$ for some integer j. Let $\lambda_i \in \mathrm{Irr}(V_i)$ be a nonprincipal character for each i, and let $\beta = \lambda_1 \times \cdots \times \lambda_j \times 1 \times \cdots \times 1 \in \mathrm{Irr}(V)$. Then $I_G(\beta)$ fixes Δ, and so, $CI_G(\beta)/C \subseteq \mathbf{N}_S(\Delta)$. Thus, $|S : \mathbf{N}_S(\Delta)|$ divides $|G : I_G(\beta)|$ and divides $\chi(1)$ for every character $\chi \in \mathrm{Irr}(GV|\beta)$, proving the claim.

First, we assume that S is abelian, so that $|S| = |\Omega| = p$ for some prime p by Lemma 5.3. We also assume that $p > 2$. Since S acts primitively on Ω, we see that $\mathbf{N}_S(\Delta) = 1$ whenever Δ is not empty and proper in Ω. Let $\lambda_i \in \mathrm{Irr}(V_i)$ be nonprincipal for each i. Let $\alpha = \lambda_1 \times 1 \times \cdots \times 1 \in \mathrm{Irr}(V)$ and $\beta = 1 \times \lambda_2 \times 1 \times \cdots \times 1 \in \mathrm{Irr}(V)$. Since $\mathbf{N}_S(\Delta) = 1$ whenever Δ is a nonempty subset of $\{1, 2\}$, it follows that $I_G(\alpha)$, $I_G(\beta)$, and $I_G(\alpha\beta)$ all lie in C. Since C fixes each V_i,we see that $I_G(\alpha\beta) = I_C(\alpha\beta) = I_C(\alpha) \cap I_C(\beta) = I_G(\alpha) \cap I_G(\beta)$. By Lemma 4.1, we have $I = I_G(\alpha) = I_G(\beta)$. Since this is true for all nonprincipal characters $\lambda_1 \in \mathrm{Irr}(V_1)$, it follows that I centralizes V_1. By symmetry, I centralizes V_2. In a similar fashion, we can show that I centralizes $V_3, \ldots, V_n$. We determine that I centralizes V. Since V is a faithful G-module, this implies that $I = 1$. Therefore, $|G| \in \mathrm{cd}(GV)$. Now, V is an abelian normal subgroup of GV, so we may apply Itô's theorem to see that every degree in $\mathrm{cd}(GV)$ divides $|G|$. In view of NDAD, we conclude that $\mathrm{cd}(GV) = \{1, |G|\}$ and the result holds in this case.

The other possibility when S is abelian is that $p = 2 = |G : C|$. If C

acts transitively on the nonprincipal characters in $\mathrm{Irr}(V_i)$ for $i = 1, 2$, then (2) will hold, and we will be done. Since S transposes the V_i, it suffices to show that C acts transitively on the nonprincipal characters of one of them. If not, then there is some nonprincipal character $\delta_2 \in \mathrm{Irr}(V_2)$ that is not G-conjugate to $\lambda_1 \in \mathrm{Irr}(V_1)$. Then $\alpha = \lambda_1 \times 1$ and $\beta = 1 \times \delta_2 \in \mathrm{Irr}(V)$ are not G-conjugate. We have $I_G(\alpha\beta) = I_C(\alpha\beta) = I_C(\alpha) \cap I_C(\beta) = I_G(\alpha) \cap I_G(\beta)$. By Lemma 4.1, $I = I_G(\alpha) = I_G(\beta)$. Now, δ_2^c is not G-conjugate to λ_1 for all $c \in C$. Applying Lemma 4.1, we obtain $I = I_C(\beta_2^c) = I^c$ for all $c \in C$. Thus, I is normal in C and has a nontrivial centralizer in V_2. This implies that $I \subseteq \mathbf{C}_G(V_2)$ since C acts irreducibly on V_2. Likewise, $I \subseteq \mathbf{C}_G(V_1)$, and so, I centralizes V. Because G acts faithfully on V, we have $I = 1$. Again, $|G| \in \mathrm{cd}(GV)$, and as before, every degree in $\mathrm{cd}(GV)$ divides $|G|$, so by NDAD, $\mathrm{cd}(GV) = \{1, |G|\}$. This completes the lemma when S is abelian.

We now assume that S is not abelian. If S has prime character degree, then so does G, and thus, $|\mathrm{cd}(GV)| \leq 3$ by Corollary 2.4. If there exists $\Delta \subseteq \Omega$ such that $\mathbf{N}_S(\Delta) = 1$; it follows from the second paragraph that $|S|$ divides some degree $a \in \mathrm{cd}(GV|V)$. Since S is not abelian, there is a degree $b \in \mathrm{cd}(S) \subseteq \mathrm{cd}(G)$ so that $1 < b < |S|$. Now, b must divide $|S|$, and so, b is a proper divisor of a in violation of NDAD. We may now apply Lemma 5.3 to see that either $n = 5$ and S is a Frobenius group of order 20 or $n = 7$ and S is a Frobenius group of order 42.

Let M be the Fitting subgroup of S. We know that $|M|$ is either 5 or 7, so $N_M(\Delta) = 1$ for all nonempty proper subsets $\Delta \subseteq \Omega$. Now, S/M is isomorphic to the point stabilizer S_α for any $\alpha \in \Omega$. Note that S_α is a cyclic group of order $p - 1$. The S_α-orbits in Ω are $\{\alpha\}$ and $\Omega - \{\alpha\}$. Thus, for any nonempty, proper subset Δ of Ω, $N_S(\Delta)$ is cyclic of order dividing $p-1$ and must be contained in some point stabilizer. If also, $1 < |\Delta| < p - 1$, then $N_S(\Delta)$ is not a point stabilizer, so it must have order 2 or 3.

Let t be an involution in S. Then t is in a unique point stabilizer, and so the cycle structure of t is without loss of generality, either (1)(23)(45) or (1)(23)(45)(67). Thus, $\langle t \rangle$ is the stabilizer in S and S_1 of the sets $\{2, 3\}$, $\{4, 5\}$, and when $p = 7$, $\{6, 7\}$. But the normalizer in S of each of these sets lies in a point stabilizer, and the point stabilizers are disjoint. Thus, S_1 contains the normalizers in S of $\{2, 3\}$, $\{4, 5\}$, and when $p = 7$, $\{6, 7\}$. Likewise, S_1 will also contain the normalizers in S of $\{1, 2, 3\}$ and $\{1, 4, 5\}$. Let J be the pre-image in G of S_1, so $J/C \cong S_1$. Observe that $J < G$.

For each i, let $\lambda_i \in \mathrm{Irr}(V_i)$ be nonprincipal. Set $\alpha = \lambda_1 \times 1 \times \cdots \times 1$, $\beta = 1 \times \lambda_2 \times \lambda_3 \times 1 \times 1$, (or $\beta = 1 \times \lambda_2 \times \lambda_3 \times 1 \times 1 \times 1 \times 1$ when $p = 7$) and $\delta = 1 \times 1 \times 1 \times \lambda_4 \times \lambda_5$ (or $\delta = 1 \times 1 \times 1 \times \lambda_4 \times \lambda_5 \times 1 \times 1$ when $p = 7$) in $\mathrm{Irr}(V)$. It follows from the last paragraph that the inertia groups of α, β, δ, $\alpha\beta$, and $\alpha\delta$ all lie in J. Since J stabilizes $\{1\}$, $\{2, 3\}$ $\{4, 5\}$, and $\{6, 7\}$ (when

$p = 7$), it follows that $I_G(\alpha\beta) = I_J(\alpha\beta) = I_J(\alpha) \cap I_J(\beta) = I_G(\alpha) \cap I_G(\beta)$. By Lemma 4.1, we obtain $I_G(\alpha) = I_G(\beta)$. Similarly, we obtain $I_G(\alpha) = I_G(\delta)$. Set $I = I_G(\alpha) = I_G(\beta) = I_G(\delta)$.

Now, $I = I_G(\alpha)$ for all nonprincipal character $\alpha \in \mathrm{Irr}(V_1)$. It follows that I centralizes V_1. Also, I fixes $\beta = 1 \times \lambda_2 \times \lambda_3 \times 1 \times \cdots \times 1$ for all nonprincipal characters $\lambda_2 \in \mathrm{Irr}(V_2)$ and $\lambda_3 \in \mathrm{Irr}(V_3)$. If $x \in I$ exchanges V_2 and V_3, then $\lambda_2^x = \lambda_3$ for all nonprincipal $\lambda_3 \in \mathrm{Irr}(V_3)$. This can only occur if $|V_i| = 2$. If $|V_i| = 2$, then C will centralize each V_i and hence, V. This is a contradiction since V_C is not homogeneous. It must follow that $I \subseteq C$. In addition, I fixes every nonprincipal character in $\mathrm{Irr}(V_2)$, so I centralizes V_2. Similarly, I will centralize all the V_i's, and thus, V. Since V is faithful, this implies $I = 1$, and $|G| \in \mathrm{cd}(GV)$. This is a contradiction to NDAD since S (and hence G) are not abelian, and thus, $\mathrm{cd}(GV)$ will have a nontrivial degree that is a proper divisor of $|G|$. With this contradiction the lemma is proved. □

Finally, we are able to prove that $|\mathrm{cd}(GV)| \leq 3$ when V is not quasi-primitive. Notice that we are not making any assumption on the Fitting height of G.

Theorem 5.5. *Suppose V is a faithful irreducible G-modules for a solvable group G and GV satisfies NDAD. If V is not quasi-primitive, then $|\mathrm{cd}(GV)| \leq 3$.*

Proof. By Lemma 5.4, we may assume that there is a normal subgroup C in G so that $|G : C| = 2$, $V_C = V_1 \times V_2$ for homogeneous components V_i of V_C, and C transitively permutes the nonprincipal characters in $\mathrm{Irr}(V_i)$. Let F be the Fitting subgroup of G. By Lemma 5.4, we may assume that either V_F is homogeneous or $F \subseteq C$.

Observe that V is the Fitting subgroup of GV. If G is nilpotent, then GV has Fitting height 2. Since $V' = 1$, we have from Corollary 4.3 that $\mathrm{cd}(GV) = \{1, |G|\}$, and the theorem is proved. Thus, we may assume that $F < G$. Applying Corollary 4.4 to the NDAD group G, we see that G/F_{l-1} is cyclic and $|G : F_{l-1}| \in \mathrm{cd}(G)$ where l is the Fitting height of G and the F_i's are the terms of the Fitting series for G.

Suppose C is nilpotent. Since G is not nilpotent, and $|G : C| = 2$, we deduce that $C = F$, and it follows that $2 = |G : F| \in \mathrm{cd}(G) \subseteq \mathrm{cd}(GV)$. Applying Corollary 2.4, we have $|\mathrm{cd}(GV)| \leq 3$. This proves the result in this case, and so, we assume that C is not nilpotent.

Let $\lambda_1 \in \mathrm{Irr}(V_1)$ be nonprincipal. Since C transitively permutes the nonprincipal characters in $\mathrm{Irr}(V_i)$ and G transitively permutes $\{V_1, V_2\}$, the

G-orbit of $\lambda_1 \times 1$ is precisely

$$\mathcal{S} = \{\delta_1 \times \delta_2 | \delta_i \in \mathrm{Irr}(V_i) \text{ and exactly one } \delta_i \text{ is principal}\}.$$

Since $\lambda_1 \times 1$ extends to its inertia group in GV, it follows that $|\mathcal{S}| = |G : I_G(\lambda_1 \times 1)| \in \mathrm{cd}(GV)$. Now, $|V_i| = q^n$ for some prime q and some integer n. We have $2(q^n - 1) = |\mathcal{S}| \in \mathrm{cd}(GV)$.

Set $M_i = \mathbf{C}_C(V_i) = \mathbf{C}_C(V_i)$ so that $M_1 \cap M_2 = \mathbf{C}_G(V) = 1$, and observe that M_1 and M_2 are G-conjugate. If C/M_1 has an irreducible character of degree t, then so does C. It follows that either t or $2t$ must lie in $\mathrm{cd}(G)$. If $1 \neq t$ is a proper divisor of $q^n - 1$, then this will violate NDAD as $2(q^n - 1)$ is in $\mathrm{cd}(GV)$. Also, C/M_i is not nilpotent, since C is not nilpotent. We now apply Lemma 5.1 to C/M_i. We have seen that conclusions (i) and (ii) do not apply, so we must have conclusion (iii). Setting F_i/M_i to be the Fitting subgroup of C/M_i, we conclude that F_i/M_i is cyclic of order dividing $q^n - 1$ and that C/F_i is cyclic of order dividing n. Because G/M_i is not nilpotent, $F_i < C$. Since $M_1 \cap M_2 = 1$, routine arguments show that $E = \mathbf{F}(C) = F_1 \cap F_2$. Since C/F_i and F_i/M_i are cyclic, it follows that E and C/E are abelian. Now, C/E is a normal abelian subgroup of index 2 in G/E. If G/E is not abelian, we use Itô's theorem to see that $2 \in \mathrm{cd}(G)$, and the result follows from Corollary 2.4. Therefore, we may assume that G/E is abelian. Also, F_1/E and F_2/E are G-conjugate, so $F_1 = F_2 = E$. Since M_1 and M_2 lie in F_1 and F_2, we have $M_1, M_2 \subseteq E$.

Notice that $F \cap C = E$. If $F > E$, then $FC > C$, and $G = FC$ since $|G : C| = 2$. This implies that $|F : E| = 2$. By the choice of C, we know that V_F is homogeneous. If F is abelian, then F acts irreducibly on V by Corollary 2.2 of [19], and $|\mathrm{cd}(GV)| \leq 3$ by Corollary 2.3. Thus, we may assume that F is not abelian. Since E is abelian and $|F : E| = 2$, it follows that $F = T \times Z$ where T is a nonabelian 2-group and Z is an abelian of odd order. By Lemma 4.5, $Z \subseteq \mathbf{Z}(G)$. In particular, $E/M_1 = T_1/M_1 \times S_1/M_1$ where T_1/M_1 is the Sylow 2-subgroup of E/M_1 and $S_1/M_1 \subseteq \mathbf{Z}(C/M_1)$. Since E/M_1 is the Fitting subgroup of the solvable group C/M_1 and S_1/M_1 is central in C/M_1, it follows that C/F is isomorphic to a subgroup of $\mathrm{Aut}(T_1/M_1)$. Because T_1/M_1 is a cyclic 2-group, C/F will be a 2-group. This implies that C/S_1 is a 2-group, and as S_1/M_1 is central in C/M_1, we determine that C/M_1 is nilpotent. Hence, $C = F_1 = F$, and C is nilpotent, which is a contradiction since we earlier assumed that C is not nilpotent. Therefore, we must have $F = E$.

Now G has Fitting height 2 and satisfies NDAD and F is abelian, so we can use Corollary 4.3 to see that G/F is cyclic and $\mathrm{cd}(G) = \{1, |G : F|\}$. Also, $|G : F| = 2|C : F| \neq 2$. Since G/F is cyclic, any irreducible character of F will either extend to G or induce irreducibly to G. It follows that $\mathrm{cd}(C/M_i) = \{1, |C : F|\}$. We may apply Lemma 5.1 to the semi-direct product $(C/M_i)V_i$ to conclude that either $|C : F|$ is a proper divisor of $q^n - 1$ or that $|C : F| = r$ and $|F : M_i| = q^n - 1$ for a prime r that does not divide $q^n - 1$. In the first case, $|G : F| = 2|C : F|$ will be a proper divisor of $2(q^n - 1)$, contradicting NDAD as $|G : F|, 2(q^n - 1) \in \mathrm{cd}(GV)$. Thus, $|C : F| = r$ and $|F : M_i| = q^n - 1$ for a prime r that does not divide $q^n - 1$.

Let $\lambda_1 \in \mathrm{Irr}(V_1)$ be nonprincipal, and let $I = I_G(\lambda_1 \times 1)$. Since C acts transitively on the nonprincipal characters in $\mathrm{Irr}(V_1)$, it follows that I/M_1 has order r. By Lemma 4.1, I is abelian. But, F is also abelian, and so M_1 is central in $FI = C$. Likewise, M_2 and $M_1M_2 = M_1 \times M_2$ are central in C. All the Sylow subgroups of C/M_1 are cyclic as r does not divide $q^n - 1$. Hence, every character in $\mathrm{Irr}(M_1M_2)$ extends to C. If G has a character of degree 2, then the result would hold by Corollary 2.4, and so, we assume that every character $\delta \in \mathrm{Irr}(M_1M_2)$ extends to G. It follows that $\ker(\delta)$ is normal in G. Since this is true for every character $\delta \in \mathrm{Irr}(M_1 \times M_2)$, we conclude that M_1 is normal in G. Since M_1 and M_2 are G-conjugate and $M_1 \cap M_2 = 1$, we obtain $M_1 = M_2 = 1$.

Now, F is cyclic of order $q^n - 1$ and acts Frobeniusly on $\mathrm{Irr}(V)$. It follows that $q^n - 1$ divides every degree in $\mathrm{cd}(GV|V)$. Since $|G : F| = 2r$, it follows that $\mathrm{cd}(GV|V) \subseteq \{q^n - 1, 2(q^n - 1), r(q^n - 1), 2r(q^n - 1)\}$. We already know that $2(q^n - 1) \in \mathrm{cd}(GV)$, so by NDAD, we determine that $(q^n - 1)$ and $2r(q^n - 1)$ do not lie in $\mathrm{cd}(GV|V)$. Since $\mathrm{cd}(G) = \{1, |G : F|\}$, the result will hold if $\mathrm{cd}(GV|V) = \{2(q^n - 1)\}$. If $r = 2$, then this is true. Therefore, we assume that $r > 2$. By way of finding a contradiction, we assume that $r(q^n - 1) \in \mathrm{cd}(GV|V)$. Because $r \leq n$, we have $2 < q^n - 1$. Since $2r = |G : F|$ also lies in $\mathrm{cd}(GV)$, we apply NDAD to see that $2r$ cannot be a divisor of $r(q^n - 1)$, and so, 2 does not divide $q^n - 1$. Now, $|G : F| = 2r$ and $|F| = q^n - 1$ are coprime, and G has a cyclic subgroup X of order $2r$.

Let R be the Sylow r-subgroup of X, and t be the involution in X. There is a nonprincipal character $\lambda_1 \in \mathrm{Irr}(V_1)$ so that the stabilizer of $\lambda_1 \times 1$ in G is R. Notice that $\lambda^t \in \mathrm{Irr}(V_2)$ and t stabilizes $\lambda_1 \times \lambda_1^t$. But $R = R^t \subseteq I_G(\lambda_1^t)$, and thus, R stabilizes $\lambda_1 \times \lambda_1^t$. This implies that $X \subseteq I_G(\lambda_1 \times \lambda_1^t)$, but $I_F(\lambda_1 \times \lambda_1^t) = 1$. We obtain $X = I_G(\lambda_1 \times \lambda_1^t)$, and hence, $q^n - 1 = |G : X| \in \mathrm{Irr}(GV)$. This contradicts NDAD as $2(q^n - 1) \in \mathrm{cd}(GV)$. Therefore, $\mathrm{cd}(GV) = \{1, 2r, 2(q^n - 1)\}$, and the theorem is proved. □

We are ready to prove Theorem B for groups of Fitting height bigger than 2. Together with Theorem 4.9 this proves Theorem B in general.

Theorem 5.6. *If G is a solvable group of Fitting height larger than 2 that satisfies NDAD, then $|\mathrm{cd}(G)| = 3$.*

Proof. Since G has Fitting height at least 3, we know via Garrison's theorem

(Corollary 12.21 of [8]) that $|\mathrm{cd}(G)| \geq 3$. Notice that the groups lying in $\mathcal{C}$ have Fitting height 2, so we know that $G \notin \mathcal{C}$. Hence, it suffices to prove that $|\mathrm{cd}(G)| \leq 3$. That is, we have to prove that if G has the condition NDAD and the Fitting height of G is at least 3, then $|\mathrm{cd}(G)| \leq 3$. We work by induction on $|G|$.

Suppose first that G has a nontrivial normal subgroup N so that G/N has Fitting height at least 3. By the inductive hypothesis, $\mathrm{cd}(G/N) = \{1, a, b\}$. If a and b are not relatively prime, then Theorem 5.6 of [21] would imply that G/N has Fitting height 2, which is a contradiction. Thus, a and b are relatively prime. Applying Lemma 4.1 (a) of [10], one of a or b must be a prime number. This implies that $\mathrm{cd}(G)$ contains a prime. By Corollary 2.4, $|\mathrm{cd}(G)| \leq 3$ and this contradicts the choice of G. Thus, we may assume that G/N has Fitting height 2 for every nontrivial normal subgroup N.

Recall that the Fitting height of $G/\Phi(G)$ is the same as G where $\Phi(G)$ is the Frattini subgroup of G. If $\Phi(G) > 1$, then by the previous paragraph $G/\Phi(G)$ and thus, G will have Fitting height 2, a contradiction to the assumption G has Fitting height greater than 2. Therefore, $\Phi(G) = 1$. Let F be the Fitting subgroup of G. By a theorem of Gaschütz, Satz III.4.5 of [4], F is a direct product of minimal normal subgroups of G. In particular, F is abelian, so we may apply Hilfsatz III.4.4 of [4] to see that there is a subgroup H so that $G = HF$ and $H \cap F = 1$. Thus, $H \cong G/F$. By the last paragraph, we know that H has Fitting height at most 2. Since G has Fitting height at least 3, we conclude that H has Fitting height 2.

If F is not minimal normal in G, then we can find normal subgroups M and N in G so that $F = M \times N$. Since $M \cap N = 1$, the Fitting height of G will be the maximum of the Fitting heights of G/M and G/N. Since G has Fitting height at least 3, one of G/N or G/M must have Fitting height at least 3. We can now apply the second paragraph to obtain the result. Therefore, F is minimal normal in G. This is equivalent to saying that F is irreducible under the action of H. When F is not quasi-primitive as a module for H, the result holds by Theorem 5.5. If F is quasi-primitive as a module for H, then since H has Fitting height 2, we may apply Theorem 5.2 to obtain the result. In either case, the theorem is proved. □

6 Examples

In this section, we show that the set $\mathcal{C}$ is nonempty.

We begin by choosing a prime p that is congruent to 1 mod 3 and is not a Mersenne prime. Since p is not a Mersenne prime, we can find an odd prime q that divides $p + 1$. As p is odd, there is some integer a with $1 < a < p$ so that a is not a square mod p. Because p is congruent to 1 mod 3, there is some integer b with $1 < b < p$ so that b^3 is congruent to 1 mod p. We take F to be the Galois field of order p and E to be the splitting field of $x^2 - a$ over F. Since a is not a square mod p, it follows that the polynomial $x^2 - a$ is irreducible over F. We know that $x^2 - a$ has a root $\lambda \in E$, and in fact, $E = F[\lambda]$. Since q divides $p + 1$, we can find $\alpha \in E$ so that α has order q. We write $\alpha = \alpha_1 + \alpha_2\lambda$ where α_1 and α_2 are integers between 0 and $p - 1$.

We will construct a solvable group G of order $p^7 3q$ for each choice of p, q, a, b, and α. We start by constructing a Camina p-group P of nilpotence class 3 using the construction in [14]. The group P will have exponent p. The group P will have the generators: $\{a_1, a_2, a_3, a_4\}$. Also, P' will be generated by $\{b_1, b_2, c\}$ and $P^3 = \mathbf{Z}(P)$ is generated by $\{c\}$. We have the relations: $[a_1, a_2] = 1$, $[a_1, a_3] = b_1$, $[a_1, a_4] = b_2^a c^{x_2}$, $[a_2, a_3] = b_2$, $[a_2, a_4] = b_1 c^{x_4}$, $[a_3, a_4] = c$, $[b_1, a_1] = 1$, $[b_2, a_1] = c$, $[b_1, a_2] = c$, and $[b_2, a_2] = 1$. In addition, b_1 and b_2 will commute with a_3 and a_4. The parameters x_2 and x_4 are defined to be the integers between 0 and $p-1$ that satisfy the congruences mod p: $2x_2 \equiv a$ and $2x_4 \equiv 1$. Notice that $x_2 \equiv x_4 a$. Also, the polynomial used in [14] to determine this group is $x^2 - a$.

We now define commuting automorphisms σ and τ on P as follows. Recall that b is element of multiplicative order 3 in F, and $\alpha = \alpha_1 + \alpha_2\lambda$ has order q in E. First, $\tau^3 = 1$ and $\sigma^q = 1$. We define τ by $a_1^\tau = a_1^b$, $a_2^\tau = a_2^b$, $a_3^\tau = a_3^{b^2}$, $a_4^\tau = a_4^{b^2}$, $b_1^\tau = b_1$, $b_2^\tau = b_2$, and $c^\tau = c^b$. We define σ as by $a_1^\sigma = a_1^{\alpha_1} a_2^{-\alpha_2 a}$, $a_2^\sigma = a_1^{-\alpha_2} a_2^{\alpha_1}$, $b_1^\sigma = b_1^{\alpha_1} b_2^{\alpha_2 a}$, $b_2^\sigma = b_1^{\alpha_2} b_2^{\alpha_1}$, and $c^\sigma = c$. In addition, we define $a_3^\sigma = a_3^{\gamma_1} a_4^{\gamma_2} c^{y_3}$ and $a_4^\sigma = a_3^{\gamma_2 a} a_4^{\gamma_1} c^{y_4}$ where the following equivalences mod p define the parameters: $\gamma_1 \equiv \alpha_1^2 + \alpha_2^2 a$, $\gamma_2 \equiv 2\alpha_1\alpha_2$, $y_3 \equiv -\gamma_1\gamma_2/2$, and $y_4 \equiv y_3 a$. The group G will be the group generated by P, σ, and τ. We claim that $\mathrm{cd}(G) = \{1, 3q, p^2q, p^3 3\}$. The fact that the group P exists is proved in [14]. To see that σ and τ are automorphisms of P, it suffices to check that σ and τ preserve the relations of P. We leave these computations to the reader. Also, the reader should check that $x^{\sigma\tau} = x^{\tau\sigma}$ for every element $x \in P$.

There are several other ways to construct Camina p-groups of nilpotence class 3. (See [15], [16], and [2].) We would not be surprised if examples could be constructed using automorphisms of those Camina p-groups. We would expect that other examples can be constructed using the groups from [14] where an irreducible polynomial of the form $x^2 + a_1 x + a_0$ is used or where E is the Galois field of order p^e where e is an even integer larger than 2. The examples constructed in these situations are surely going to be more complicated than the example we presented here.

7 Further comments

As already mentioned in the Introduction, Huppert [5] has conjectured that simple groups are determined by their sets of character degrees (up to abelian direct factors). As we see in our next result, sometimes it is possible to determine simple groups just in terms of certain properties of their character degrees.

Corollary 7.1. *Let G be a finite group with square-free character degrees and the NDAD property. If $|\mathrm{cd}(G)| \geq 4$, then $G \cong A_7 \times B$, where B is abelian.*

Proof. By Theorem B, if G were solvable, then there is some prime p so that $\mathrm{cd}(G)$ has two degrees with different nontrivial p-parts. Since these two p-parts cannot both be square-free, we deduce that G is not solvable. It was proved in [6] that if a non-solvable group has square-free character degrees then $G \cong A_7 \times B$, where B is solvable. Now, the NDAD property forces B to be abelian, as desired. □

Let G be a group. The set $\mathrm{cd}(G) - \{1\}$ can be partially ordered by divisibility. If G satisfies NDAD, then the chains in this partial ordering all have length 1. We can generalize NDAD by supposing that all the chains have length at most k for some integer k. It seems reasonable to ask whether there is a function f on the natural numbers so that $|\mathrm{cd}(G)| \leq f(k)$. Taking direct products of groups in $\mathcal{C}$, it is not difficult to see that if $f(k)$ exists, then $f(k) \geq 4^k$.

When G is a solvable group, Garrison has shown (Corollary 12.19 of [8]) that the Fitting height of G is bounded by $|\mathrm{cd}(G)|$. Since we have just conjectured that $|\mathrm{cd}(G)|$ is bounded by a function in k where k is the length of the longest chain of divisibility in $\mathrm{cd}(G) - \{1\}$, it follows that the Fitting height of G should be bounded by a function of k. We will now show that this is indeed the case. Note that our conjecture might suggest that this function is exponential in k, but we show that it is in fact linear in k. Our proof is an immediate consequence of a result by the second author and Wolf, [20].

Lemma 7.2. *Let G be a solvable group, and let k be the length of the longest chain of divisibility in $\mathrm{cd}(G) - \{1\}$. Then the Fitting height of G is at most $10k + 1$.*

Proof. We will prove the contrapositive. We will show that if G has Fitting height greater than or equal to $10k + 2$, then $\mathrm{cd}(G) - \{1\}$ has a chain of divisibility of length at least $k+1$. Let $F_i = F_i(G)$ for every $i \geq 0$. Using Theorem C of [20], we can find for every integer i with $1 \leq i \leq k$ a character degree $a_i \in \mathrm{cd}(G/F_{10(i-1)})$ so that $|G\colon F_i|$ divides a_i. Since G/F_{10k} is not nilpotent, we can find a degree $1 \neq a_{k+1} \in \mathrm{cd}(G/F_{10k})$. For each $i = 1, \ldots, k+1$, it is not difficult to see that a_i is a proper divisor of $|G\colon F_{10(i-1)}|$, and for $i = 2, \ldots, k+1$, we have $|G\colon F_{10(i-1)}|$ is a divisor of a_{i-1}. It follows that $a_{k+1}, a_k, \ldots, a_1$ is a chain of divisibility in $\mathrm{cd}(G) - \{1\}$ of length $k+1$. This proves the lemma. □

There is another way to view Conjecture A. For any group G, we can define $\mathrm{nd}(G)$ to be the minimum integer n such that there exist $b_1, \ldots, b_n \in \mathrm{cd}(G)$ such that for all $a \in \mathrm{cd}(G)$, a divides b_i for some $i = 1, \ldots, n$. With this terminology, Corollary C yields that if G is solvable and $|\mathrm{cd}(G)| \geq 5$, then $\mathrm{nd}(G) \leq |\mathrm{cd}(G)| - 2$. We believe that the "right" order of magnitude should be sublinear.

Conjecture 7.3. *There exist real numbers $\alpha < 1$ and $C > 0$ such that $\mathrm{nd}(G) < |\mathrm{cd}(G)|^{\alpha} + C$ for every finite group G.*

Taking direct products of n groups in $\mathcal{C}$, we can find groups G with $|\mathrm{nd}(G)| = 3^n$ and $|\mathrm{cd}(G)| = 4^n$. In particular, the value of α in Conjecture 7.3 cannot be smaller than $\log_4(3)$.

Finally, we consider class sizes and prove the analog of Conjecture A in this setting. In this case, the problem is much easier than the corresponding one for character degrees.

Lemma 7.4. *Let G be a solvable group and assume that if $1 < a < b$ are sizes of conjugacy classes of G then a does not divide b. Then G has at most 3 different conjugacy class sizes.*

Proof. Our hypothesis implies that if $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(y)$, then $\mathbf{C}_G(x) = \mathbf{C}_G(y)$. Groups with this property where studied in [22]. Using the main result of that paper, we have that solvable groups with this property are divided into five families. It is not difficult to see in all the cases that the groups cannot have more than 3 conjugacy classes if we assume that there is not any divisibility relation among the different class sizes of non-central elements. □

The same interpretations that we have presented in this section for the problem on character degrees can be done for the corresponding one for class sizes.

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M. L. Lewis, Department of Mathematical Sciences, Kent State University, Kent OH 44242 USA

E-mail: lewis@math.kent.edu

A. Moretó, Departament d'Àlgebra, Universitat de València, 46100 Burjassot. València SPAIN

E-mail: Alexander.Moreto@uv.es

T. R. Wolf, Department of Mathematics, Ohio University, Athens OH 45701 USA

E-mail: wolf@ohiou.edu