ON THE NUMBER OF CONJUGACY CLASS SIZES AND CHARACTER DEGREES IN FINITE p-GROUPS

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Abstract. In this note we prove that for any two integers $r, s > 1$ there exist finite p-groups G of class 2 such that $|\operatorname{cd}(G)| = r$ and $|\operatorname{cs}(G)| = s$.

1. Introduction

In the last decades a number of results have been proved concerning the sets $\operatorname{cs}(G)$ of conjugacy class sizes and $\operatorname{cd}(G)$ of complex irreducible character degrees of a finite group G. Many of the results about class sizes are dual to results about character degrees, though the reason for the existence of this duality is not understood yet. For instance, Isaacs [?] proved that given any set A of powers of a prime number p containing 1, there exists a p-group G of class ≤ 2 such that $\operatorname{cd}(G) = A$. The dual result for class sizes has been recently obtained in [?] by Cossey and Hawkes, who show that, for any set A as above, there always exists a p-group of class ≤ 2 such that $\operatorname{cs}(G) = A$. We raise more generally the following question: for which sets A and B of powers of p containing 1 is it possible to find a p-group G such that $\operatorname{cd}(G) = A$ and $\operatorname{cs}(G) = B$? Of course, if any of the two sets reduces to $\{1\}$ so does the other, hence we will assume that $|A|, |B| \geq 2$. Even in this case it is easy to see that there must be some kind of relation between A and B. As already pointed out by Burnside [?, page 126], if $\operatorname{cs}(G) = \{1, p\}$ then $|G'| = p$ and Theorem 7.5 in [?] yields that $\operatorname{cd}(G) = \{1, |G : Z(G)|^{1/2}\}$. On the other hand, according to Theorem 12.11 of [?], if $\operatorname{cd}(G) = \{1, p\}$ then either G has an abelian maximal subgroup or $|G : Z(G)| = p^3$ and consequently $|\operatorname{cs}(G)| \leq 3$. In this note we prove that, surprisingly enough, there is not any relation between the number of character degrees and the number of class sizes.

Theorem. *Given any two integers r and s greater than 1 there exists a p-group G of class 2 such that $|\operatorname{cd}(G)| = r$ and $|\operatorname{cs}(G)| = s$.*

The proof of this theorem is elementary, but still we think that the result is noteworthy and, together with the examples above, indicates that it may be very difficult to find a complete answer to the question raised.

Most of the work of the second author was done while he was visiting the University of Wisconsin-Madison. He would like to thank the Mathematics Department for its hospitality and I.M. Isaacs for encouraging us to think about this question. The authors also want to express their thanks to A. Jaikin-Zapirain for helpful discussions.

1991 *Mathematics Subject Classification.* Primary 20C15, 20D15.

Research of the second author supported by a grant of the Basque Government and by the University of the Basque Country grant UPV 127.310-EB160/98.

2. Proof of the Theorem

We first prove that, for any $n > 1$, there exist p-groups of class 2 with n character degrees and 2 class sizes and also groups with n class sizes and 2 character degrees. The theorem will then be a straightforward consequence of these results.

Let us denote by $\mathfrak{D}_p$ the variety of p-groups of class ≤ 2 and exponent p when p is odd, and the variety generated by the dihedral group of order 8 when $p = 2$. In our next lemma we see that the free groups in this variety provide examples for our first case.

Lemma 2.1. *Let F_n be the free group of rank $n \geq 2$ in the variety $\mathfrak{D}_p$. Then* $\mathrm{cd}(F_n) = \{1, p, \ldots, p^{[n/2]}\}$ *and* $\mathrm{cs}(F_n) = \{1, p^{n-1}\}$.

Proof. As observed in [**?**], any two elements of F_n that are independent modulo $\Phi(F_n)$ do not commute and, on the other hand, $\Phi(F_n) = Z(F_n)$ has index p^n in F_n. We derive from these facts that $\mathrm{cs}(F_n) = \{1, p^{n-1}\}$.

Let us examine now the character degrees of F_n. Since $|F_n : Z(F_n)| = p^n$, we deduce from Corollary 2.30 of [**?**] that $\chi(1) \leq p^{[n/2]}$ for any complex irreducible character χ of F_n. Conversely, choose any positive integer $i \leq [n/2]$. Since F_{2i} is a quotient of F_n, it suffices to show that $p^i \in \mathrm{cd}(F_{2i})$. If we consider an appropriate maximal subgroup N of F'_{2i} then F_{2i}/N is an extraspecial group of order p^{2i+1} and, according to [**?**, Example 7.6], $p^i \in \mathrm{cd}(F_{2i}/N) \subseteq \mathrm{cd}(F_{2i})$. The result follows. □

For the second case, we need an explicit construction. Recall from [**?**, Lemma 5.7] that if K and L are finite groups such that $\mathrm{cd}(K) = \{1, m\}$, $\mathrm{cd}(L) = \{1, n\}$ and $K' \cong L'$, then any product G of K and L with K' and L' amalgamated satisfies that $\mathrm{cd}(G) = \{1, mn\}$.

Lemma 2.2. *Let $K_n = \langle b \rangle [\langle a_1 \rangle \times \cdots \times \langle a_n \rangle]$ be a semidirect product, where b has order p, each a_i has order p^2 and the action of b is given by $a_i^b = a_i^{1+p}$. If $G_{l,n}$ denotes the canonical central product of $l \leq n$ copies of K_n then* $\mathrm{cd}(G_{l,n}) = \{1, p^l\}$ *and* $\mathrm{cs}(G_{l,n}) = \{1, p, \ldots, p^l, p^n\}$.

Proof. The result about $\mathrm{cd}(G_{l,n})$ follows from the remark before the lemma, since $\mathrm{cd}(K_n) = \{1, p\}$ by Ito's Theorem [**?**, Theorem 6.15]. We prove the claim about $\mathrm{cs}(G_{l,n})$ by induction on l. For $l = 1$, $G_{1,n} = K_n$ and the result is clear. We assume now that $l > 1$ and $\mathrm{cs}(G_{l-1,n}) = \{1, p, \ldots, p^{l-1}, p^n\}$.

We have that $G_{l,n}$ is the quotient of the direct product $T = G_{l-1,n} \times K_n$ by the normal subgroup $N = \{(x, x^{-1}) \mid x \in Z(K_n)\}$, after identifying the centre of $G_{l-1,n}$ with the centre of K_n. Let us use the bar notation in $G_{l,n}$. Since the class of an element does not increase its size when passing to a quotient, for $x \in G_{l-1,n}$, $y \in K_n$ we have that

$$|\,\mathrm{Cl}_{G_{l,n}}(\overline{(x,y)})| \leq |\,\mathrm{Cl}_T((x,y))| = |\,\mathrm{Cl}_{G_{l-1,n}}(x)||\,\mathrm{Cl}_{K_n}(y)|.$$

On the other hand,

$$\begin{aligned} |\,\mathrm{Cl}_{G_{l,n}}(\overline{(x,y)})| &\geq |\{\overline{(x,y)}^{(g,1)} \mid g \in G_{l-1,n}\}| \\ &= |\{\overline{(x^g,1)} \mid g \in G_{l-1,n}\}| = |\,\mathrm{Cl}_{G_{l-1,n}}(x)| \end{aligned}$$

and similarly $|\,\mathrm{Cl}_{G_{l,n}}(\overline{(x,y)})| \geq |\,\mathrm{Cl}_{K_n}(y)|$. Therefore

$$\max\{|\,\mathrm{Cl}_{G_{l-1,n}}(x)|, |\,\mathrm{Cl}_{K_n}(y)|\} \leq |\,\mathrm{Cl}_{G_{l,n}}(\overline{(x,y)})| \leq |\,\mathrm{Cl}_{G_{l-1,n}}(x)||\,\mathrm{Cl}_{K_n}(y)|. \quad (1)$$

If we use (**??**) with $y = 1$, we deduce that the size of the class of $\overline{(x, 1)}$ in $G_{l,n}$ is the same as that of the class of x in $G_{l-1,n}$. Hence $\{1, p, \dots, p^{l-1}, p^n\} \subseteq \mathrm{cs}(G_{l,n})$. On the other hand, if we take into acount that $\mathrm{cs}(K_n) = \{1, p, p^n\}$, the induction hypothesis and that the size of a class in $G_{l,n}$ is at most $|G'_{l,n}| = p^n$, it also follows from (**??**) that $\mathrm{cs}(G_{l,n}) \subseteq \{1, p, \dots, p^l, p^n\}$.

So the theorem will be proved once we show that $p^l \in \mathrm{cs}(G_{l,n})$. Let x be an element whose class in $G_{l-1,n}$ has size p^{l-1}. Since $Z(K_n) = K'_n$ is an elementary abelian group of order p^n generated by the commutators $[a_i, b]$ and $l - 1 < n$, some $[a_i, b]$ is not contained in $[x, G_{l-1,n}]$. Then

$$|\operatorname{Cl}_{G_{l,n}}(\overline{(x, a_i)})| = |\overline{[x, G_{l-1,n}] \times [a_i, K_n]}| = p^l,$$

as we wanted to prove. □

Finally, we proceed to prove our theorem. In the proof, E_n will stand for an extraspecial group of order p^{2n+1} and U_n will denote a Sylow p-subgroup of $SL(3, p^n)$. Then $|U_n| = p^{3n}$ and, according to [**?**, Lemma 4], U_n is a semiextraspecial group (that is, the quotient by any maximal subgroup of the centre is extraspecial) with centre of order p^n. It follows from Theorem A in [**?**] that $\mathrm{cd}(U_n) = \mathrm{cs}(U_n) = \{1, p^n\}$.

Proof of the Theorem. If $r \le s$ it suffices to consider the group $G = G_{n,n} \times E_n \times \overset{r-2}{\cdots} \times E_n$, where $n = s - r + 1$.

Assume now that $s \le r$. If $s = 2l$ is even then for any $n \ge l$ the group $G = F_{2n} \times E_1 \times \overset{l-1}{\cdots} \times E_1$ satisfies that $|\,\mathrm{cs}(G)| = 2l$ and $|\,\mathrm{cd}(G)| = n + l$, so we are done by choosing $n = r - l$. If $s = 2l + 1 > 3$ is odd then for $n \ge l$ the group $G_{l-1,2n} \times F_{2n+1}$ has $2l + 1$ class sizes and $n + l$ character degrees, and again it suffices to take $n = r - l$. Lastly, for $s = 3$ consider the groups $F_{2n+1} \times F_{2n+1}$ and $U_{2n} \times F_{2n+1}$, where $n \ge 1$. The number of class sizes is 3 for any of these groups and, on the other hand, the number of character degrees is respectively $2n + 1$ and $2n + 2$, hence it may equal any $r \ge 3$. □

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