ON THE NUMBER OF DIFFERENT PRIME DIVISORS OF ELEMENT ORDERS

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ABSTRACT. We prove that the number of different prime divisors of the order of a finite group is bounded by a polynomial function of the maximum of the number of different prime divisors of the element orders. This improves a result of J. Zhang.

1. Introduction

Given a finite group G, let $\rho(G)$ be the number of different prime divisors of $|G|$ and let $\alpha(G)$ be the maximum number of different prime divisors of the orders of the elements of G. It was proved by J. Zhang in [5] that if G is solvable, then $\rho(G)$ is bounded by a quadratic function of $\alpha(G)$ and that for arbitrary G, $\rho(G)$ is bounded by a superexponential function of $\alpha(G)$. The result for solvable groups was improved by T. M. Keller in [2], where he proved that $\rho(G)$ is bounded by a linear function of $\alpha(G)$. The purpose of this short note is to provide a proof of a better bound in the case of arbitrary finite groups.

Theorem A. *There exist universal (explicitly computable) constants C_1 and C_2 such that for every finite group $G > 1$ the inequality*

$$\rho(G) \leq C_1 \alpha(G)^4 \log \alpha(G) + C_2$$

holds.

This result will be used in [3].

2. Proof

First, we prove that for simple groups there is an essentially cubic bound. We begin with the alternating groups.

Lemma 2.1. *There exists a constant C_1 such that $\rho(A_n) \leq C_1 \alpha(A_n)^2$ for every $n \geq 5$.*

Proof. Let p_j be the jth prime number. Let k be the maximum integer such that

$$4 + \sum_{j=2}^{k} p_j \leq n.$$

It is clear that the elements of A_n that can be written as the product of two 2-cycles, one p_2-cycle, one p_3-cycle,..., one p_{k-1}-cycle and one p_k-cycle, with all these cycles

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pairwise disjoint are divisible by $\alpha(A_n) = k$ different primes. It follows from p. 190 of [4], for instance, that $p_j \leq 10j \log j$. Therefore

$$\alpha(A_n) \geq \max\{l \mid 4 + 10\sum_{j=2}^{l} j \log j \leq n\} \geq \max\{l \mid 4 + 10l^2 \log l \leq n\} = t.$$

In particular, we have that $n < 4 + 10(t+1)^2 \log(t+1)$. By p. 160 of [4], for instance, we have that $\rho(A_n)$ is bounded by a quadratic function of t. The result follows. □

All the inequalities that appear in this proof have reversed inequalities of the same order of magnitude. This implies that there exists for constant K_1 such that $\rho(A_n) \geq K_1\alpha(A_n)$ for every $n \geq 5$. This means that it is not possible to improve our cubic bound in Theorem A to anything better than a quadratic bound.

Next, we consider the simple groups of Lie type.

Lemma 2.2. *There exists a constant C_2 such that $\rho(G) \leq C_2\alpha(G)^3 \log \alpha(G)$ whenever G is a simple group of Lie type.*

Proof. It suffices to argue as in the proof of Lemma 5 of [5] using the proof of Lemma 2.1 instead of the proof of Lemma 4 of [5]. □

Now, we are ready to prove Theorem A.

Proof of Theorem A. We know by [2] that there exists $n_0 > 1$ such that if H is solvable and $\alpha(H) \geq n_0$ then $\rho(H) < 5\alpha(H)$. We consider groups G with $\alpha(G) = k \geq n_0$ and we want to prove that $\rho(G) \leq Ck^4 \log k$, where $C = 10\max\{C_1, C_2, C_3, 5\}$ and C_3 is defined in such a way that $\rho(G) \leq C_3k^3$ whenever $\alpha(G) = k < n_0$ or G is sporadic.

Let G be a minimal (nonsolvable) counterexample. We define the series $1 = S_0 \leq R_1 < S_1 < R_2 < S_2 < \cdots < R_m < S_m \leq R_{m+1} = G$ as follows: R_1 is the largest normal solvable subgroup of G and for any $i \geq 1$, S_i/R_i is the socle of G/R_i and R_{i+1}/S_i is the largest normal solvable subgroup of G/S_i. Notice that for $i \geq 1$ S_i/R_i is a direct product of non-abelian simple groups.

We claim that $m \leq 5k$. In order to see this, we are going to prove first that there exists a prime divisor q_i of $|S_i/R_i|$ that is coprime to $|G/S_i||R_i|$ for $i = 1, \ldots, m$. This argument is due to Zhang [5]. Let P be a Sylow 2-subgroup of S_i. By the Frattini argument, $G = S_iN_G(P)$. Put $T = R_iN_G(P)$. Then T is a proper subgroup of G. If every prime divisor of $|S_i/R_i|$ divides $|G/S_i||R_i|$ then we would have $\rho(T) = \rho(G)$. Since the theorem holds for T, it also holds for G. This contradiction implies that such q_i exists.

Now, let Q_m be a q_m-Sylow subgroup of G. We have that Q_m acts coprimely on R_m and using Glauberman's Lemma (Lemma 13.8 of [1]), we deduce that there exists $Q_{m-1} \in \mathrm{Syl}_{q_{m-1}}(R_m)$ that is Q_m- invariant. Now, we consider the action of $Q_{m-1}Q_m$ on R_{m-1} and conclude that there exists a $Q_{m-1}Q_m$-invariant Sylow q_{m-2}-subgroup of G. In this way, we build a solvable subgroup $H = Q_mQ_{m-1}\ldots Q_1$. By [2], we have that $m \leq 5\alpha(H) \leq 5\alpha(G)$, as claimed.

Using Lemmas 2.1 and 2.2 together with [2], we have that

$$\rho(S_i/S_{i-1}) \leq (C/5)k^3 \log k.$$

Finally we deduce that

$$\rho(G) \leq m \cdot \max_i \rho(S_i/S_{i-1}) \leq Ck^4 \log k.$$

This contradiction completes the proof. □

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