On the number of constituents of products of characters

by

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Let φ and ψ be faithful characters of a finite p-group P. What can be said about the number of different irreducible constituents of the product $\varphi\psi$? At first sight, it does not seem reasonable to expect strong restrictions for the possible values of this number. However, in [1] it was proved that if the number of constituents of this product is bigger than one, then it is at least $(p+1)/2$. (The proof of this result depends on some of the ideas of [2] and could be simplified following the argument of [2] more closely.)

It seems reasonable to ask what further restrictions can be found. In p. 237 of [1] it was conjectured that if the number of constituents of the product of two faithful characters of a finite p-group, for $p \geq 5$, is bigger than $(p+1)/2$, then it is at least p. The goal of this note is to give a counterexample to this conjecture.

Theorem. *Let $P = C_p \wr C_p$ for $p \geq 5$. There exist $\varphi, \psi \in \mathrm{Irr}(P)$ faithful such that $\varphi\psi$ has exactly $p-1$ distinct irreducible constituents.*

Proof. Write $P = CA$, where A is the base group, which is elementary abelian of order p^p. It is clear that the non-linear characters of P are induced from characters of A. In particular, they have degree p. Fix any non-principal character $\lambda \in \mathrm{Irr}(C_p)$. Then any character of A can be written in the form $\nu = \lambda^{i_1} \times \cdots \times \lambda^{i_p}$ for some integers $i_j = 0, \ldots, p-1$. Thus, we can identify the character ν with the p-tuple $(i_1, \ldots, i_p)$. It is clear that $\nu^G \in \mathrm{Irr}(G)$ if and only if not all the i_j's are equal.

We have that $Z(P) = \{(x, \ldots, x) \mid x \in C_p\}$ is the unique minimal normal subgroup of P. Also, if $\nu^G \in \mathrm{Irr}(G)$, it follows from Lemma 5.11 of [3] that ν^G is faithful if and only if $(x, \ldots, x) \notin \mathrm{Ker}\, \nu^G$. Notice that if $\lambda(x) = \varepsilon$ for a primitive pth root of unity, then

$$\nu^G(x, \ldots, x) = p\varepsilon^{i_1 + \cdots + i_p}.$$

Thus ν^G is faithful if and only if $\sum_{j=1}^{p} i_j \not\equiv 0 \pmod{p}$.

Consider the characters of A associated to the p-tuples $(1, 0, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$. They induce faithful irreducible characters of P, φ and ψ respectively. We claim that $\varphi\psi$ has $p-1$ distinct irreducible constituents.

The character φ_A decomposes as the sum of the characters associated to $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$. We can argue similarly with ψ_A. The product of two characters of A corresponds to the componentwise sum of the associated p-tuples and two characters of A are P-conjugate if and only we can go from the p-tuple associated to one of the characters to the other by a cyclic permutation of the components. Now, it is easy to see that the number of constituents of the character $\varphi\psi$ is the number of characters of P lying over the characters of A corresponding to the p-tuples $(2, 1, 0, 0, \ldots, 0), (1, 2, 0, 0, \cdots, 0), (1, 1, 1, 0, \ldots, 0), (1, 1, 0, 1, \ldots, 0), \ldots,$

$(1, 1, 0, 0, \ldots, 1)$. Here we have p different p-tuples. The third of these p-tuples and the last one correspond to P-conjugate characters of A, so they induce the same character of P. It is easy to see that no other pair of p-tuples are conjugate. The claim follows. □

We have been unable to find any example where the number of constituents of the product of two faithful characters of a p-group has more than $(p+1)/2$ distinct irreducible constituents but less than $p-1$. So the following modification of the conjecture could still be true.

Question. *Let φ and ψ be faithful irreducible characters of a finite p-group P. Assume that $\varphi\psi$ has more than $(p+1)/2$ distinct irreducible constituents. Does it necessarily have at least $p-1$ irreducible constituents?*

References

[1] E. Adan-Bante, Products of characters and finite p-groups, *J. Algebra* **277** (2004), 236–255.

[2] E. Adan-Bante, M. Loukaki, A. Moretó, Homogeneous products of characters, *J. Algebra* **274** (2004), 587–593.

[3] M. Isaacs, *Character Theory of Finite Groups*, Dover, New York, 1994.