A GRAPH ASSOCIATED WITH THE π-CHARACTER DEGREES OF A GROUP

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Abstract. Let G be a group and π be a set of primes. We consider the set $\mathrm{cd}^{\pi}(G)$ of character degrees of G that are divisible only by primes in π. In particular, we define $\Gamma^{\pi}(G)$ to be the graph whose vertex set is the set of primes dividing degrees in $\mathrm{cd}^{\pi}(G)$. There is an edge between p and q if pq divides a degree $a \in \mathrm{cd}^{\pi}(G)$. We show that if G is π-solvable, then $\Gamma^{\pi}(G)$ has at most two connected components.
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1. Introduction.

Throughout this note G will be a finite group. We focus on $\mathrm{Irr}(G)$, the irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) | \chi \in \mathrm{Irr}(G)\}$, the character degrees of G. Several recent papers have studied the influence of $\mathrm{cd}(G)$ on the structure of G (see **[4]**, **[10]**, **[12]**, and **[14]** for a few examples). The basic results on the relationship between $\mathrm{cd}(G)$ and the structure of G can be found in **[2]**, **[3]**, **[5]**, and **[9]**. In this note, we are particularly interested in the question of which sets of values for $\mathrm{cd}(G)$ force G to be nonsolvable. For example, we know that $\mathrm{cd}(A_5) = \{1, 3, 4, 5\}$ has a character degree set that cannot occur for a solvable group. We know this by looking at the graph $\Gamma(G)$ whose vertex set is $\rho(G)$, the set of primes that divide degrees in $\mathrm{cd}(G)$. There is an edge between p and q if pq divides some degree $a \in \mathrm{cd}(G)$. It is well-known that if G is solvable then $\Gamma(G)$ has at most two connected components. (One proof of this is in **[9]**.) When we look at $\Gamma(A_5)$, we see that it has three connected components, so its character degree set cannot occur for a solvable group.

It is easy to find a group P with $\mathrm{cd}(P) = \{1, 7\}$. We see that $\mathrm{cd}(P \times A_5) = \{1, 3, 4, 5, 7, 21, 28, 35\}$. One can ask: does there exist a solvable group with this character degree set? We see that $\Gamma(P \times A_5)$ is connected, so our earlier result about graphs does not apply. On the other hand, Pálfy has shown in **[11]** that if G is solvable and π is a set of primes that divide degrees in $\mathrm{cd}(G)$ with $|\pi| \geq 3$, then there exist distinct primes $p, q \in \pi$ so that pq divides some degree $a \in \mathrm{cd}(G)$. It is easy to see that $\mathrm{cd}(P \times A_5)$ violates this condition when $\pi = \{2, 3, 5\}$. Suppose that we have a group Q with $\mathrm{cd}(Q) = \{1, 14\}$. (For example, let Q be the semidirect product of a cyclic group of order 29 acted on by an automorphism of order 14.) In this case, we obtain $\mathrm{cd}(Q \times A_5) = \{1, 3, 4, 5, 14, 42, 56, 70\}$, and it is natural to ask whether this can be the character degree set of a solvable group. Notice that neither the graph result nor Pálfy's result applies in this case, but we will show in this paper that this also cannot be the character degree set of a solvable group.

The idea is to fix a set of primes π and to look at the set $\mathrm{cd}^\pi(G)$ of character degrees that are divisible only by primes in π. A priori, it seems unlikely that $\mathrm{cd}^\pi(G)$ should share any property with $\mathrm{cd}(G)$. In fact, we will show that $\mathrm{cd}^\pi(G)$ has at least one property in common with $\mathrm{cd}(G)$. To see this, we look at the graph $\Gamma^\pi(G)$ whose vertex set is $\rho^\pi(G)$, the set of primes dividing degrees in $\mathrm{cd}^\pi(G)$. There is an edge between p and q if pq divides a degree $a \in \mathrm{cd}^\pi(G)$. We will show when G is π-solvable that $\Gamma^\pi(G)$ has at most two connected components.

Main Theorem: *Let π be a set of primes, and let G be a π-solvable group, then $\Gamma^\pi(G)$ has at most 2 connected components.*

We note that $\rho^\pi(G) \subseteq \rho(G) \cap \pi$. It is possible that some prime $p \in \rho(G) \cap \pi$ divides no character degree in $\mathrm{cd}^\pi(G)$, so it may happen that $\rho^\pi(G) < \rho(G) \cap \pi$. Notice in our earlier example that $\mathrm{cd}^{7'}(Q \times A_5) = \{1, 3, 4, 5\}$, and so, $\Gamma^{7'}(Q \times A_5)$ has three connected components. We conclude that if $\mathrm{cd}(G) = \mathrm{cd}(Q \times A_5)$, then G is not solvable.

Our result extends a theorem of Beltrán. Suppose that a group A acts coprimely on a group G via automorphisms. We define $\mathrm{Irr}_A(G)$ to be the set of irreducible characters of G that are fixed by A, and $\mathrm{cd}_A(G) = \{\chi(1) | \chi \in \mathrm{Irr}_A(G)\}$. Let $\rho_A(G)$ be the primes that divide degrees in $\mathrm{cd}_A(G)$. We define the graph $\Gamma_A(G)$ to be the graph with vertices $\rho_A(G)$.

There is an edge between p and q if pq divides some degree $a \in \mathrm{cd}_A(G)$. In Theorem A of **[1]**, Beltrán proved when G is solvable that $\Gamma_A(G)$ has at most two connected components. Let $\pi = \rho(G)$, and let H be the semi-direct product of A acting on G. It is not difficult to show that $\mathrm{cd}^\pi(H) = \mathrm{cd}_A(G)$, and thus, $\Gamma^\pi(H) = \Gamma_A(G)$. Note that H is π-solvable, so our result implies that $\Gamma_A(G)$ has at most two connected components, which was Theorem A of **[1]**.

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2. Results.

We begin with several preliminary lemmas. This first lemma allows us to take a given character and replace it by a character with more desirable properties. In particular, we gain control over the kernel of the new character.

Lemma 1: *Let N be a normal subgroup of a π-separable group G. Suppose that the character $\chi \in \mathrm{Irr}(G)$ is nonlinear and has π-degree. Also, assume that the irreducible constituents of χ_N are linear. Then there exists a character $\psi \in \mathrm{Irr}(G)$ that is also nonlinear of π-degree, and there exists a normal subgroup K in G with $K \subseteq \ker(\psi)$, where $N' \subseteq K \subseteq N$ and N/K is either a π'-group or a q-group for some prime $q \in \pi$.*

Proof: It suffices to find a nonlinear character $\psi \in \mathrm{Irr}(G)$ with π-degree such that the irreducible constituents of ψ_N are linear and either have π'-order or else have q-power order where $q \in \pi$. Let ν be an irreducible constituent of χ_N, and note that the stabilizer of ν in G has π-index. Suppose first that the there is a prime $q \in \pi$ such that the q-part μ of ν is not extendible to G. The stabilizer of μ in G contains the stabilizer of ν, and thus, the stabilizer of μ has π-index in G. We now use Lemma 2.4 of **[6]** to see that some character $\psi \in \mathrm{Irr}(G|\mu)$ has π-degree. Since we are assuming that μ does not extend to G, the character ψ is not linear, and the result holds in this case.

We may now assume that the full π-part of the linear character ν extends to a linear character $\lambda \in \mathrm{Irr}(G)$. Then the character $\psi = \overline{\lambda}\chi$ is irreducible. Also, ψ is nonlinear and has π-degree. Finally, the irreducible constituents of ψ_N are linear with π'-order. ∎

This next lemma deals with a special case of the Main Theorem. We will later give an estimate on the upper bound for the diameter when our graph is connected, and the diameters of each connected component when the graph is disconnected. In the special case addressed by this lemma, we obtain better bounds for these diameters.

Lemma 2: *Let π be a set of primes. Suppose that $N \subseteq M$ are normal subgroups of a group G so that G/M is a π'-group and M/N is a chief factor that is a p-group for some prime $p \in \pi$. Assume that $\mathrm{cd}^\pi(G/N') \setminus \{1\}$ is nonempty. If there exists a prime in $\rho^\pi(G)$ that is not adjacent to p in $\Gamma^\pi(G)$, then there exists a prime $q \in \rho^\pi(G)$ so that every prime not adjacent to p in $\Gamma^\pi(G)$ is adjacent to q. In particular, $\Gamma^\pi(G)$ has at most two connected components. If $\Gamma^\pi(G)$ is disconnected, then each connected component has diameter at most 2. If $\Gamma^\pi(G)$ is connected, then its diameter is at most 4.*

Proof: By assumption, there exists a nonlinear character $\chi \in \mathrm{Irr}(G)$ with π-degree such that $N' \subseteq \ker(\chi)$. Using Lemma 1, we may replace χ by another character, if necessary, and we may assume that there is a normal subgroup K in G so that $N' \subseteq K \subseteq N$ so that $K \subseteq \ker(\chi)$ and either N/K is a π'-group or N/K is a q-group for some prime $q \in \pi$. By Itô's theorem, all degrees in $\mathrm{cd}(G/N')$ divide $|G\colon N|$, so $\chi(1)$ divides $|G\colon N|_\pi = |M\colon N|$, and it follows that $\chi(1)$ is a power of p.

Suppose that r is a vertex in $\Gamma^\pi(G)$ that is not adjacent to p, and let $\psi \in \mathrm{Irr}(G)$ have π-degree divisible by r. If ψ_K is irreducible, then we may use Gallagher's theorem to see that $\psi\chi$ is irreducible, but $\psi\chi$ has π-degree divisible by pr. Since p and r are not adjacent in $\Gamma^\pi(G)$, it follows that ψ_K is not irreducible. Thus, $|G\colon K|$ is not coprime to $\psi(1)$, and hence, $|N\colon K|$ has common prime divisor with $\psi(1)$. This can only occur if N/K is a q-group for some prime $q \in \pi$, and q is adjacent to r, as desired.

Consider a character $\gamma \in \mathrm{Irr}(G)$ with $\gamma(1) \in \mathrm{cd}^\pi(G)$. If $(\gamma(1), pq) = 1$, then we use Corollary 11.29 of **[5]** to see that γ_K is irreducible. By Gallagher's theorem, we obtain $\gamma\chi \in \mathrm{Irr}(G)$. We have $\gamma(1)\chi(1) \in \mathrm{cd}^\pi(G)$. If there exists a prime $r \in \rho^\pi(G)$ that is not adjacent to p, then there is a character $\xi \in \mathrm{Irr}(G)$ where $\xi(1) \in \mathrm{cd}^\pi(G)$, r divides $\xi(1)$, p does not divide $\xi(1)$, and $\xi(1)\chi(1) \notin \mathrm{cd}^\pi(G)$. This forces q to divide $\xi(1)$, and we conclude that $q \in \rho^\pi(G)$, and every prime in $\rho^\pi(G)$ that is not adjacent to p is adjacent to q. It follows that $\Gamma^\pi(G)$ has at most two connected components, and if $\Gamma^\pi(G)$ is disconnected then each component has diameter at most 2.

We now assume that $\Gamma^\pi(G)$ is connected. When all the primes in $\rho^\pi(G)$ are adjacent to p, the diameter of $\Gamma^\pi(G)$ is at most 2. Suppose that there is some prime that is not adjacent to p, so that $q \in \rho^\pi(G)$. If p and q are adjacent, then the diameter of $\Gamma^\pi(G)$ is at most 3. Suppose that p and q are not adjacent. If p and q have a common neighbor, then the diameter of $\Gamma^\pi(G)$ is at most 4. Suppose that p and q do not have a common neighbor. Since $\Gamma^\pi(G)$ is connected we can find an edge between primes r and s where r is a neighbor of p and s is a neighbor of q. Thus, there is a character $\gamma \in \mathrm{Irr}(G)$ with $\gamma(1) \in \mathrm{cd}^\pi(G)$ and rs dividing $\gamma(1)$. Since s is not a neighbor of p and r is not a neighbor of q, we have $(\gamma(1), pq) = 1$. By the previous paragraph, we obtain $\gamma(1)\chi(1) \in \mathrm{cd}^\pi(G)$ which makes p and s neighbors, and thus is a contradiction. This proves the lemma. ■

The next lemma deals with fully ramified characters. A definition for fully ramified characters can be found in Problem 6.3 of **[5]**.

Lemma 3: *Let $N \subseteq M$ be normal subgroups of G. Assume that M/N is a π-group and that G/M is a π'-group. Let H/N be a complement for M/N in G/N. Suppose that $\theta \in \mathrm{Irr}(M)$ and $\varphi \in \mathrm{Irr}(N)$ are G-invariant and fully ramified with respect to each other. Assume that θ and φ both have π-degree. Then θ extends to G if and only if φ extends to H.*

Proof: Let R/M be a Sylow r-subgroup of G/M for some prime r. By Corollary 11.31 of **[5]**, it suffices to prove that θ extends to R if and only if φ extends to $S = R \cap H$.

Suppose that $\hat{\theta}$ is an extension of θ to R. Since $\hat{\theta}(1)$ is not divisible by r, some irreducible constituent α of $(\hat{\theta})_S$ has degree not divisible by r. Thus, α_N is irreducible. But $(\hat{\theta})_N$ is a multiple of φ, and so $\alpha_N = \varphi$. We conclude that φ extends to S.

Conversely, assume that $\hat{\varphi}$ is an extension of φ to S. Then r does not divide $(\hat{\varphi})^R(1) = |R\colon S|\varphi(1)$. Therefore, $(\hat{\varphi})^R$ has an irreducible constituent β where r does not divide $\beta(1)$.

It follows that β_M is irreducible, and $((\hat{\varphi})^R)_M = \varphi^M$ is a multiple of θ. We conclude that $\beta_M = \theta$. ▪

To state the next lemma cleanly, it is convenient to make the following definition. Let $N \subseteq H \subseteq G$ and let p be a prime. Then the characters $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G)$ are *p-related* with respect to N if $\chi(1) = p^a \psi(1)$ for some nonnegative integer a and the restrictions χ_N and ψ_N have a common irreducible constituent.

Lemma 4: *Let π be a set of primes. Let M/N be a p-chief factor for G, where $p \in \pi$, and let G/M be a π'-group. Suppose that H/N is a complement for M/N in G/N. Then for each character $\psi \in \mathrm{Irr}(H)$ having π-degree, there exists a character $\chi \in \mathrm{Irr}(G)$ so that ψ and χ are p-related with respect to N. Conversely, for every character $\chi \in \mathrm{Irr}(G)$ having π-degree, there exists a character $\psi \in \mathrm{Irr}(H)$ so that χ and ψ are p-related with respect to N.*

Proof: Let $\psi \in \mathrm{Irr}(H)$ have π-degree, and let $\varphi = \psi_N$. Observe that φ is irreducible and H-invariant. Using Problem 6.12 of **[5]**, there are three possibilities, and we define a character $\chi \in \mathrm{Irr}(G)$ in each case. If φ^M is irreducible, then $\chi = \psi^G$ is irreducible. If φ extends to M, then φ has an extension $\chi \in \mathrm{Irr}(G)$ via Corollary 11.31 of **[5]**(since φ extends to H and $(|H\colon N|, |M\colon N|) = 1$. In the remaining case, φ is fully ramified with respect to $\theta \in \mathrm{Irr}(M)$, and we may apply Lemma 3 to obtain an extension of θ to G, written $\chi \in \mathrm{Irr}(G)$. In all three cases, it is not difficult to see that ψ is p-related to χ with respect to N.

Conversely, consider a character $\chi \in \mathrm{Irr}(G)$ having π-degree, and let $\theta = \chi_M$. It is not difficult to see that θ is irreducible and G-invariant. By Theorem 6.18 of **[5]**, there are three cases to consider. If θ_N is irreducible, then χ_N is irreducible, so we take $\psi = \chi_H \in \mathrm{Irr}(H)$. If $\theta = \varphi^M$ for some character $\varphi \in \mathrm{Irr}(N)$, then the stabilizer of φ in G must be conjugate to H. Replacing φ by some conjugate, we may assume H is the stabilizer of φ in G. We take ψ to be the Clifford correspondent for χ with respect to φ. Finally, if θ is fully ramified with respect to $\varphi \in \mathrm{Irr}(N)$, then we may apply Lemma 3, and we find an extension $\psi \in \mathrm{Irr}(H)$ of φ. In all three cases, it is not difficult to see that ψ is p-related to χ with respect to N. ▪

With our preliminary lemmas proved, we now move to the Main theorem.

Proof of Main Theorem: We work by induction on $|G|$. Take $M = \mathbf{O}^{\pi'}(G)$, and note that we may assume $M > 1$. (If $M = 1$, then G is a π'-group, and the result is clear.) Choose N to be a normal subgroup of G contained in M so that M/N is a chief factor for G. We know that M/N is a p-group for some prime $p \in \pi$. Let H/N be a complement for M/N in G/N, and note that $H < G$. By the inductive hypothesis, we see that $\Gamma^\pi(H)$ has at most two connected components. Using Lemma 4, we see that the irreducible π-degrees of H divide the irreducible π-degrees of G. This implies that $\Gamma^\pi(H)$ is a subgraph of Γ^π. In fact, we see that $\rho^\pi(G) \subseteq \rho^\pi(H) \cup \{p\} \subseteq \rho^\pi(G) \cup \{p\}$.

When $\rho^\pi(G) = \rho^\pi(H)$, the result is immediate. In the remaining case, we have $\rho^\pi(H) = \rho^\pi(G) - \{p\}$. If p is joined to some other vertex in $\Gamma^\pi(G)$, then $\Gamma^\pi(G)$ cannot have more connected components than $\Gamma^\pi(H)$, and we are done in this case also. Therefore, we may assume that p is an isolated vertex in $\Gamma^\pi(G)$, and there is a character $\chi \in \mathrm{Irr}(G)$

so that $\chi(1)$ is a power of p. By Lemma 4, there exists a character $\psi \in \mathrm{Irr}(H)$ that is p-related to χ with respect to N. Since p does not occur in $\rho^\pi(H)$, we deduce that ψ is linear, and thus, the irreducible constituents of χ_N are linear. We now apply Lemma 2 to see that $\Gamma^\pi(G)$ has at most two connected components. ∎

We now find an upper estimate on the diameter of $\Gamma^\pi(G)$ when $\Gamma^\pi(G)$ is connected and G is π-solvable. We also obtain bounds on the diameters of both connected components when $\Gamma^\pi(G)$ is disconnected. We believe that our estimates are higher than the actual bounds. The largest diameter that we know actually occurs when the graph is connected is 3. To see this example look at [**8**]. That paper exhibits a group G where $\Gamma(G)$ has diameter 3. Taking $\pi = \rho(G)$, we have $\Gamma^\pi(G) = \Gamma(G)$. To make this example less trivial, we consider a solvable group H where $\rho(H) \cap \pi$ is empty. Then $\mathrm{cd}^\pi(G \times H) = \mathrm{cd}(G)$ and $\Gamma^\pi(G \times H) = \Gamma(G)$. When the graph is disconnected, we do not know of any examples where the connected components are not complete graphs. Lemma 2 also gives us an indication that our estimates of the diameters can be improved.

Lemma 5: *Let π be a set of primes, and let G be a π-solvable group. If $\Gamma^\pi(G)$ is connected, then its diameter is at most* 6. *If $\Gamma^\pi(G)$ is disconnected, then each connected component has diameter at most* 3.

Proof: Suppose that $\Gamma^\pi(G)$ is connected with diameter at least 7. Thus, we can find primes p_1 and p_8 whose shortest path connecting them in $\Gamma^\pi(G)$ is $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_8$. Thus, we can find characters $\chi_1, \chi_2, \chi_3 \in \mathrm{Irr}(G)$ where $\chi_i(1) \in \mathrm{cd}^\pi(G)$ for $i \in \{1, 2, 3\}$ and p_1p_2 divides $\chi_1(1)$, p_4p_5 divides $\chi_2(1)$, and p_7p_8 divides $\chi_3(1)$. Observe that the primes dividing $\chi_1(1)$ cannot divide a degree in $\mathrm{cd}^\pi(G)$ that is divisible by a prime dividing $\chi_2(1)$, or we would have a shorter path between p_1 and p_8. Similarly, the primes dividing $\chi_1(1)$ or $\chi_2(1)$ cannot divide a degree in $\mathrm{cd}^\pi(G)$ that is divisible by any prime dividing $\chi_3(1)$. If τ is the set of primes that divide $\chi_i(1)$ for $i \in \{1, 2, 3\}$, then it follows that $\Gamma^\tau(G)$ has three connected components. Since $\tau \subseteq \pi$, we see that G is τ-solvable, so the Main theorem applies, and we have a contradiction. A similar proof works when $\Gamma^\pi(G)$ is disconnected to show that each component has diameter at most 3. ∎

Let G be a group, and let p be a prime. In Section 20 of [**9**], Manz and Wolf define a graph $\Gamma_p(G)$ whose vertices are the primes dividing the degrees of the irreducible p-Brauer characters, and the edges occur when two primes divide the degree of the same irreducible p-Brauer character. They show when G is solvable that $\Gamma_p(G)$ has at most two connected components. We can obtain an analog of this result in our case. As always, let π be a set of primes, and suppose that G is a π-solvable group. We let $\rho_p^\pi(G)$ be the set of all primes that divide the degrees of irreducible p-Brauer characters of G of π-degree. We can define $\Gamma_p^\pi(G)$ to be the graph whose vertex set is $\rho_p^\pi(G)$ and edges occur when two primes divide the degree of some irreducible p-Brauer character of π-degree. Using a proof similar to the proof of the Main theorem one can show that $\Gamma_p^\pi(G)$ has at most two connected components. Since the concepts of this proof are identical to those of the main theorem, we do not feel the need to include the proof here. (A similar result occurs for the irreducible Isaacs δ-partial characters when δ is a set of primes and G is a δ-separable group. Define the graph $\Gamma_\delta^\pi(G)$ in the usual fashion. We can show that $\Gamma_\delta^\pi(G)$ has at most two connected components.)

3. Further Questions.

We would like to close with some questions. We have shown that $\mathrm{cd}^{\pi}(G)$ shares at least one property in common with $\mathrm{cd}(G)$. Accordingly, it seems likely that these two sets should share other properties, as well. For example, it may be possible to prove a generalization of Pálfy's. In particular, if G is solvable and δ is a set of primes dividing degrees in $\mathrm{cd}^{\pi}(G)$ with $|\delta| \geq 3$, then must there exist distinct primes $r, s \in \delta$ so that rs divides some degree in $\mathrm{cd}^{\pi}(G)$?

We know that if G is solvable and $\Gamma(G)$ is disconnected, then G has Fitting height of at most 4. Does a similar result hold when $\Gamma^{\pi}(G)$ is disconnected? Do we have a bound on the Fitting height of a Hall π-subgroup of G in this situation? The following example shows that there is no bound on the Fitting height of a Hall π-subgroup of G when G is solvable and Γ^{π} is disconnected.

Theorem 6: *Let n be any integer. There is a solvable group G and a set of primes π so that the Fitting height of a Hall π-complement of G is at least n and $\Gamma^{\pi}(G)$ is disconnected.*

Proof: Let H be a solvable group where $\Gamma(H)$ is disconnected. Take $\pi = \rho(H)$, and fix distinct primes $r, s \in \pi$. Choose a prime $p \notin \pi$, and find a p-group P of order p^n. By Theorem B of **[13]**, there is a solvable $\{r, s\}$-group Q of Fitting height n so that P acts coprimely on Q and $\mathbf{C}_Q(P) = 1$. Let K be the semi-direct product of P acting on Q. Observe that p divides every nontrivial degree in $\mathrm{cd}(K)$. Let $G = H \times K$. Since $\mathrm{cd}^{\pi}(K) = \{1\}$ and $\mathrm{cd}(H) = \mathrm{cd}^{\pi}(H)$, we deduce that $\mathrm{cd}^{\pi}(G) = \mathrm{cd}(H)$. Therefore, $\Gamma^{\pi}(G) = \Gamma(H)$ is disconnected. Now, HQ is the Hall π-subgroup of G, and its Fitting height is at least the Fitting height of Q which is n. This proves the theorem. ∎

We look for other invariants that might be bounded when G is π-solvable and $\Gamma^{\pi}(G)$ is disconnected. The one that appears to most closely mimic the Fitting height is the π-length. Thus, we ask: is it possible to get a bound on the π-length of G when G is π-solvable and $\Gamma^{\pi}(G)$ is disconnected?

We now return to the question of comparing $\mathrm{cd}(G)$ and $\mathrm{cd}^{\pi}(G)$. One consequence of the Main theorem is that if the distinct degrees in $\mathrm{cd}^{\pi}(G)$ are relatively prime and G is π-solvable, then $|\mathrm{cd}^{\pi}(G)| \leq 3$. A similar condition is true for $|\mathrm{cd}(G)|$ when G is solvable. In **[7]**, the first author generalized this condition by studying $\mathrm{cd}(G)$ when G was a solvable group satisfying the one-prime hypothesis. (A group G satisfies the *one-prime* hypothesis if whenever a and b are distinct degrees in $\mathrm{cd}(G)$ their greatest common divisor is either 1 or a prime.) It was shown there that $|\mathrm{cd}(G)| \leq 14$. We can now ask: suppose that G is a π-solvable group and that $\mathrm{cd}^{\pi}(G)$ satisfies the one-prime hypothesis. Can we obtain a bound on the size of $|\mathrm{cd}^{\pi}(G)|$? In some sense all these questions would be answered if we could prove the following statement: if G is a π-solvable group for some set of primes π, then there exists a solvable group H so that $\mathrm{cd}^{\pi}(G) = \mathrm{cd}(H)$. This seems unlikely to be true, but we do not know of any examples where the statement is not true. It would be interesting to find a set of primes π and a π-solvable group G where there is no (solvable) group H with $\mathrm{cd}(H) = \mathrm{cd}^{\pi}(G)$.

Finally, it makes sense to ask what happens when one does not assume that G is π-solvable. We know that $\Gamma(G)$ has at most three connected components, and it may be true that $\Gamma^{\pi}(G)$ has at most three connected components.

References:

[1] A. Beltrán, The invariant degree graph of a solvable group, *Archiv. der Math.* **77** (2001), 369-372.

[2] B. Huppert, Research in representation theory at Mainz (1984-1990), *Progr. Math.* **95** (1991), 17-36.

[3] B. Huppert, *Character Theory of Finite Groups*, deGruyter Expositions in Mathematics 25, Berlin, 1998.

[4] B. Huppert, Some simple groups which are determined by the set of their character degrees. I, *Illinois J. Math.* **44** (2000), 828-842.

[5] I. M. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[6] I. M. Isaacs, Abelian normal subgroups of M-groups, *Math. Z.* **182** (1983), 205-221.

[7] M. L. Lewis, Solvable groups having almost relatively prime distinct irreducible character degrees, *J. Algebra* **174** (1995), 197-216.

[8] M. L. Lewis, A solvable group whose character degree graph has diameter 3, *Proc. AMS* **130** (2002), 625-630.

[9] O. Manz and T. R. Wolf, *Representations of Solvable Groups*, Cambridge University Press, Cambridge, 1993.

[10] T. Noritzsch, Groups having three complex irreducible character degrees, *J. Algebra* **175** (1995), 767-798.

[11] P. P. Pálfy, On the character degree graph of solvable groups, I: three primes, *Period. Math. Hungar.* **36** (1998), 61-65.

[12] J. M. Riedl, Fitting heights of solvable groups with few character degrees, *J. Algebra* **233** (2000), 287-308.

[13] A. Turull, Generic fixed point free action of arbitrary finite groups, *Math. Z.* **187** (1984), 491-503.

[14] J. Zhang, A note on character degrees of finite solvable groups, *Comm. Algebra* **28** (2000), 4249-4258.