EQUIVALENCES INVOLVING (p,q)-MULTI-NORMS

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Abstract. We consider (p,q)-multi-norms and standard t-multi-norms based on Banach spaces of the form $L^r(\Omega)$, and resolve some question about the mutual equivalence of two such multi-norms. We introduce a new multi-norm, called the $[p,q]$-concave multi-norm, and relate it to the standard t-multi-norm.

1. Introduction

1.1. **Definitions.** A theory of multi-norms based on a normed space E was first introduced by Dales and Polyakov in [8]. We recall the basic definitions of the theory.

We write $\mathbb{N}$ for the set of natural numbers, and set $\mathbb{N}_n = \{1,\dots,n\}$ for $n \in \mathbb{N}$; the collection of permutations of the set $\mathbb{N}_n$ is denoted by $\mathfrak{S}_n$.

Definition 1.1. Let $(E, \|\cdot\|)$ be a complex normed space. A *multi-norm* on the family $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following Axioms (A1)–(A4) are satisfied for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$:

(A1) $\left\|(x_{\sigma(1)},\dots,x_{\sigma(n)})\right\|_n = \|\boldsymbol{x}\|_n \quad (\sigma \in \mathfrak{S}_n)$;

(A2) $\|(\alpha_1 x_1,\dots,\alpha_n x_n)\|_n \leq (\max_{i\in\mathbb{N}_n}|\alpha_i|)\,\|\boldsymbol{x}\|_n \quad (\alpha_1,\dots,\alpha_n \in \mathbb{C})$;

(A3) $\|(x_1,\dots,x_n,0)\|_{n+1} = \|\boldsymbol{x}\|_n$;

(A4) $\|(x_1,\dots,x_{n-1},x_n,x_n)\|_{n+1} = \|\boldsymbol{x}\|_n$.

In this case, $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-normed space.*

We shall sometimes say that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm *based on* E; we write $\mathcal{E}_E$ for the family of all multi-norms based on E.

In the case where $(E, \|\cdot\|)$ is a Banach space, each space $(E^n, \|\cdot\|_n)$ is a Banach space, and $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is termed a *multi-Banach space.*

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In fact, Axiom (A3) is a consequence of Axioms (A1), (A2), and (A4) [8, Proposition 2.7]; to establish (A4), it suffices to show that

$$\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} \leq \|\boldsymbol{x}\|_n$$

for each element $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

Many properties of multi-norms were described in [8]; these properties included some strong connections with the theory of absolutely summing operators and with the theory of tensor norms. A study of multi-norms was continued in [9] and [10].

In [9], we explained how multi-norms correspond to certain tensor norms. We recall this briefly; details are given in [9, §3]. We write δ_i for the sequence $(\delta_{i,j} : j \in \mathbb{N})$ for $i \in \mathbb{N}$; c_0 is the Banach space of all complex-valued null sequences.

Definition 1.2. Let E be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a c_0-*norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$, with norm at most $\|T\|$, for each compact operator T on E.

We note that a c_0-norm on $c_0 \otimes E$ is a 'reasonable cross-norm' in the sense of [21, §6.1]; see [9, Lemma 3.3].

Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$, and set

$$\|(x_1, \dots, x_n)\|_n = \sum_{i=1}^{n} \delta_i \otimes x_i \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,.$$

Then $(\|\cdot\|_n :\ n \in \mathbb{N})$ is a multi-norm based on E.

A more general and detailed version of the following theorem is given as [9, Theorem 3.4].

Theorem 1.3. *Let E be a normed space. Then the above construction defines a bijection from the family of c_0-norms on $c_0 \otimes E$ onto $\mathcal{E}_E$.* □

The notion of the equivalence of two multi-norms was given in [8, §2.2.4], as follows.

Definition 1.4. Let $(E, \|\cdot\|)$ be a normed space. Suppose that the two multi-norms $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ belong to $\mathcal{E}_E$. Then

$$(\|\cdot\|_n^1) \leq (\|\cdot\|_n^2) \quad \text{if} \quad \|\boldsymbol{x}\|_n^1 \leq \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})\,,$$

and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ *dominates* $(\|\cdot\|_n^1 : n \in \mathbb{N})$, written $(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2)$, if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leq C\, \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})\,; \tag{1.1}$$

the two multi-norms are *equivalent*, written

$$(\|\cdot\|_n^1 : n\in\mathbb{N}) \cong (\|\cdot\|_n^2 : n\in\mathbb{N}) \quad \text{or} \quad (\|\cdot\|_n^1) \cong (\|\cdot\|_n^2)\,,$$

if each dominates the other.

A main theme of [10] was to determine when two multi-norms based on the same normed space are mutually equivalent. In particular, we discussed in [10] the '(p,q)-multi-norms based on a normed space E', and tried to determine when these multi-norms are mutually equivalent, especially on the Banach spaces of the form $L^r(\Omega)$. The question was resolved for most, but not all, cases. Here we resolves some of the remaining cases, and give simpler proofs of some results already established in [10]. We also consider the question whether a 'standard multi-norm' is ever equivalent to a (p,q)-multi-norm on a space $L^r(\Omega)$. For this, we introduce a new '$[p,q]$-concave multi-norm', and use some theorems of Maurey to show that 'usually' a standard t-multi-norm is not equivalent to any (p,q)-multi-norm on $L^r(\Omega)$. However there are special combinations of p, q, and r when this equivalence does hold, thereby refuting a conjecture of [10].

1.2. **Notation.** Let E be a normed space. The closed unit ball of E is denoted by $E_{[1]}$, and the dual space of E is E'; the action of $\lambda\in E'$ on $x\in E$ with respect to the duality gives the complex number denoted by $\langle x,\,\lambda\rangle$. Let E and F be Banach spaces. Then $\mathcal{B}(E,F)$ denotes the Banach space of all bounded linear operators from E to F, with the operator norm.

The standard Banach spaces of all complex-valued sequences on $\mathbb{N}$ that are bounded and $r-$summable (for $r\geq 1$) are denoted by ℓ^∞ and ℓ^r, respectively; the norms on ℓ^∞ and ℓ^r are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_r$, respectively, so that c_0 is a closed subspace of ℓ^∞. For $n\in\mathbb{N}$ and $r\in[1,\infty]$, the space $\mathbb{C}^n$ with the ℓ^r-norm is denoted by ℓ^r_n; it is regarded as a subspace of c_0 and ℓ^r by identifying $(x_1,\dots,x_n)\in\mathbb{C}^n$ with $(x_1,\dots,x_n,0,\dots)\in\mathbb{C}^{\mathbb{N}}$. The Banach space of all complex-valued, continuous functions on a compact space K, taken with the uniform norm, is denoted by $C(K)$.

Let Ω be a measure space, and take $r\geq 1$. Then we denote by $L^r(\Omega)$ or $L^r(\Omega,\mu)$ the usual Banach space of complex-valued, r-integrable functions with respect to a positive measure μ on Ω; here

$$\|f\|_r = \left(\int_\Omega |f(t)|^r \,\mathrm{d}\mu(t)\right)^{1/r} \quad (f\in L^r(\Omega))\,,$$

and we identify functions which are equal almost everywhere. For each r with $1\leq r<\infty$, the conjugate index to r is denoted by r', so that we have $1/r + 1/r' = 1$, and we regard 1 and ∞ as conjugates; throughout we interpret

$$\left(\sum_{i=1}^{n} |\zeta_i|^{r'}\right)^{1/r'} \quad \text{as} \quad \max\{|\zeta_1|, \dots, |\zeta_n|\}$$

when $r = 1$. The dual space of $L^r(\Omega)$ is identified with $L^{r'}(\Omega)$ in the usual manner.

It is standard [1, Proposition 6.4.1] that, in the case where $L^r(\Omega)$ is an infinite-dimensional space, we can regard ℓ^r as a closed, 1-complemented subspace of $L^r(\Omega)$.

Finally in this section, we recall that the generalized Hölder inequality implies the following. Take $q, s, u > 1$ such that $s < q$ and $1/u = 1/s - 1/q$. Then

(1.2)

$$\|(\beta_1, \dots, \beta_n)\|_q = \sup\left\{\|(\zeta_1\beta_1, \dots, \zeta_n\beta_n)\|_s : \zeta_1, \dots, \zeta_n \in \mathbb{C}, \sum_{j=1}^{n} |\zeta_j|^u \le 1\right\}$$

whenever $n \in \mathbb{N}$ and $\beta_1, \dots, \beta_n \in \mathbb{C}$. Indeed, $1/(u/s) + 1/(q/s) = 1$, and so

$$\begin{aligned}
\|(\beta_1, \dots, \beta_n)\|_q &= \|(|\beta_1|^s, \dots, |\beta_n|^s)\|_{q/s}^{1/s} \\
&= \sup\left\{\left|\sum_{j=1}^{n} \eta_j |\beta_j|^s\right|^{1/s} : \sum_{j=1}^{n} |\eta_j|^{u/s} \le 1\right\} \\
&= \sup\left\{\left(\sum_{j=1}^{n} |\zeta_j|^s |\beta_j|^s\right)^{1/s} : \sum_{j=1}^{n} |\zeta_j|^u \le 1\right\}, \\
&= \sup\left\{\|(\zeta_1\beta_1, \dots, \zeta_n\beta_n)\|_s : \sum_{j=1}^{\infty} |\zeta_j|^u \le 1\right\},
\end{aligned}$$

giving (1.2).

1.3. **The weak p−summing norm.** We recall the definition of the weak p–summing norms on a normed space; the following standard definition was given in [8, Definition 4.1.1] and [10, §2.3]. For further discussion, see [11, 13, 14].

Let E be a normed space, and take $p \ge 1$ and $n \in \mathbb{N}$. Following the notation of [8, 9, 14], we define $\mu_{p,n}(\boldsymbol{x})$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ by

$$\begin{aligned}
\mu_{p,n}(\boldsymbol{x}) &= \sup\left\{\left(\sum_{i=1}^{n} |\langle x_i, \lambda\rangle|^p\right)^{1/p} : \lambda \in E'_{[1]}\right\} \\
&= \sup\left\{\|(\langle x_1, \lambda\rangle, \dots, \langle x_n, \lambda\rangle)\|_p : \lambda \in E'_{[1]}\right\}.
\end{aligned}$$

Then $\mu_{p,n}$ is the *weak p–summing norm* (at dimension n).

Note that, for all $p \ge 1$, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\mu_{p,n}(\boldsymbol{x}) = \sup\left\{ \left\| \sum_{j=1}^n \zeta_j x_j \right\| : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \le 1 \right\}. \tag{1.3}$$

Let E be a normed space. Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, and define

$$T_{\boldsymbol{x}} : (\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^n \zeta_j x_j\,, \quad \mathbb{C}^n \to E\,.$$

It follows from (1.3) that

$$\mu_{p,n}(\boldsymbol{x}) = \left\| T_{\boldsymbol{x}} : \ell_n^{p'} \to E \right\| \tag{1.4}$$

for $p \ge 1$; the map $\boldsymbol{x} \mapsto T_{\boldsymbol{x}}$, $(E^n, \mu_{p,n}) \to \mathcal{B}(\ell_n^{p'}, E)$, is an isometric linear isomorphism.

1.4. $(q,p)-$**summing operators.** Let E and F be Banach spaces, and suppose that $1 \le p \le q < \infty$. We recall that an operator $T \in \mathcal{B}(E,F)$ is $(q,p)-$*summing* if there exists a constant C such that

$$\left(\sum_{i=1}^n \|Tx_i\|^q \right)^{1/q} \le C\, \mu_{p,n}(x_1, \dots, x_n) \quad (x_1, \dots, x_n \in E,\ n \in \mathbb{N})\,.$$

The smallest such constant C is denoted by $\pi_{q,p}(T)$. The set of these (q,p)–summing operators is denoted by $\Pi_{q,p}(E,F)$; it is a linear subspace of $\mathcal{B}(E,F)$, and $(\Pi_{q,p}(E,F), \pi_{q,p})$ is a Banach space; we write $(\Pi_p(E,F), \pi_p)$ for $(\Pi_{p,p}(E,F), \pi_{p,p})$. The latter space of all *p–summing operators* has been studied by many authors; see [11, 13, 14, 16, 21], for example.

1.5. **The maximum and minimum multi-norm.** As in [8] and [9], there are a *maximum multi-norm* and *minimum multi-norm* based on a normed space E; they are denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, respectively, and they are defined by the property that

$$\|\boldsymbol{x}\|_n^{\min} \le \|\boldsymbol{x}\|_n \le \|\boldsymbol{x}\|_n^{\max} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for every multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on E. The formula for $\|\cdot\|_n^{\min}$ is

$$\|\boldsymbol{x}\|_n^{\min} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in E^n,\ n \in \mathbb{N})\,.$$

The dual of $\|\cdot\|_n^{\max}$ is the weak $1-$summing norm $\mu_{1,n}$ [8, Theorem 3.33], and hence

$$\|\boldsymbol{x}\|_n^{\max} = \sup\left\{ \left| \sum_{j=1}^n \langle x_j,\, \lambda_j \rangle \right| : \mu_{1,n}(\boldsymbol{\lambda}) \le 1 \right\}$$

for each $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ and $n \in \mathbb{N}$, where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$.

1.6. **The (p, q)-multi-norm.** The following definition was first given in [8, §4.1].

Definition 1.5. Let E be a normed space, and suppose that $1 \leq p \leq q < \infty$. For each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, define

$$\begin{aligned} \|\boldsymbol{x}\|_n^{(p,q)} &= \sup\left\{\left(\sum_{j=1}^n |\langle x_j,\, \lambda_j\rangle|^q\right)^{1/q} : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\right\} \\ &= \sup\left\{\|(\langle x_1,\, \lambda_1\rangle, \dots, \langle x_n,\, \lambda_n\rangle)\|_q : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\right\}, \end{aligned}$$

where the supremum is take over all $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$.

As noted in [8, Theorem 4.1], $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E; it is called the *(p, q)-multi-norm*.

Clearly, we have $(\|\cdot\|_n^{(p,q_1)}) \leq (\|\cdot\|_n^{(p,q_2)})$ whenever $1 \leq p \leq q_2 \leq q_1$ and $(\|\cdot\|_n^{(p_1,q)}) \leq (\|\cdot\|_n^{(p_2,q)})$ whenever $1 \leq p_1 \leq p_2 \leq q$.

Lemma 1.6. *Let E be a normed space, and take p, q_1, q_2 such that*

$$1 \leq p \leq q_1 < q_2 < \infty\,.$$

Then

$$\|\boldsymbol{x}\|_n^{(p,q_2)} = \sup\left\{\|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p,q_1)} : \sum_{j=1}^n |\zeta_j|^u \leq 1\right\}$$

for all $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ and $n \in \mathbb{N}$, where u is defined by the equation $1/u = 1/q_1 - 1/q_2$.

Proof. The result follows by applying the generalized Hölder's inequality (1.2) with $q = q_2$ and $s = q_1$ and with β_i taken to be the value $\langle x_i,\, \lambda_i\rangle$ for $i \in \mathbb{N}_n$ from the definition of the multi-norms. □

A key result from [10, Theorem 2.6] relates (p, q)-multi-norms to the known theory of absolutely summing operators.

Theorem 1.7. *Let E be a normed space, and suppose that $1 \leq p \leq q < \infty$. Then the (p, q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$.* □

Indeed, for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E' \to c_0)\,. \tag{1.5}$$

Further, it is shown in [10, Corollary 2.9] that, for $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$, we have $(\|\cdot\|_n^{(p_1,q_1)}) \cong (\|\cdot\|_n^{(p_2,q_2)})$ if and only if $\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0)$ as subsets of $\mathcal{B}(E', c_0)$.

Let F be a 1-complemented subspace of a Banach space E, and suppose that $1 \le p \le q < \infty$ and that $n \in \mathbb{N}$. Then it follows from [8, Proposition 4.3] that the restriction of the norm $\|\cdot\|_n^{(p,q)}$ on E^n to F^n is exactly $\|\cdot\|_n^{(p,q)}$ defined on F^n. In particular, to show that two (p,q)-multi-norms based on an infinite-dimensional space $L^r(\Omega)$ are not equivalent, it suffices to prove this for the corresponding (p,q)-multi-norms based on ℓ^r.

1.7. **The standard t-multi-norm.** Let (Ω, μ) be a measure space, take $r \ge 1$, and suppose that $r \le t < \infty$. In [8, §4.2] and [9, §6], there is a definition and discussion of the standard t-multi-norm on the Banach space $L^r(\Omega)$. We recall the definition.

Take $n \in \mathbb{N}$. For each ordered partition $\mathbf{X} = (X_1, \dots, X_n)$ of Ω into measurable subsets and each $f_1, \dots, f_n \in L^r(\Omega)$, we define

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \left(\sum_{i=1}^n \|P_{X_i} f_i\|^t\right)^{1/t}.$$

Here $P_{X_i} : f \mapsto f \mid X_i$ is the projection of $L^r(\Omega)$ onto $L^r(X_i)$, and $\|\cdot\|$ is the L^r-norm. Then we define

$$\|(f_1, \dots, f_n)\|_n^{[t]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \dots, f_n))\,,$$

where the supremum is taken over all such measurable ordered partitions $\mathbf{X}$. As in [8, §4.2.1], we see that $(\|\cdot\|_n^{[t]} :\ n \in \mathbb{N})$ is a multi-norm based on $L^r(\Omega)$; it is the *standard t-multi-norm* on $L^r(\Omega)$.

Clearly the norms $\|\cdot\|_n^{[t]}$ decrease as a function of $t \in [r, \infty)$, and so the maximum among these norms is $\|\cdot\|_n^{[r]}$.

For example, by [8, (4.9)], we have

$$\|(f_1, \dots, f_n)\|_n^{[t]} = \left(\|f_1\|^t + \cdots + \|f_n\|^t\right)^{1/t} \quad (n \in \mathbb{N})$$

whenever $f_1, \dots, f_n$ in $L^r(\Omega)$ have pairwise-disjoint supports, and, in particular,

$$\|(\delta_1, \dots, \delta_n)\|_n^{[t]} = n^{1/t} \quad (n \in \mathbb{N})\,, \tag{1.6}$$

where we regard δ_i as an element of ℓ^r. Further,

$$\|(f_1, \dots, f_n)\|_n^{[r]} = \|\,|f_1| \vee \cdots \vee |f_n|\,\| \quad (f_1, \dots, f_n \in L^r(\Omega),\ n \in \mathbb{N})\,; \tag{1.7}$$

this is equation (4.13) in [8]. Thus $(\|\cdot\|_n^{[r]})$ is the lattice multi-norm on $L^r(\Omega)$; see [8, §4.3].

Let Ω be a measure space, and take $t \geq 1$. By [8, Theorem 4.26], we have $\|\cdot\|_n^{[t]} = \|\cdot\|_n^{(1,t)}$ on $L^1(\Omega)$.

Lemma 1.8. *Let Ω be a measure space, and take r, t_1, t_2 such that*

$$1 \leq r \leq t_1 < t_2 < \infty\,.$$

Then

$$\|(f_1, \dots, f_n)\|_n^{[t_2]} = \sup\left\{\|(\zeta_1 f_1, \dots, \zeta_n f_n)\|_n^{[t_1]} : \sum_{j=1}^{n} |\zeta_j|^v \leq 1\right\}$$

for each $f_1, \dots, f_n \in L^r(\Omega)$ and $n \in \mathbb{N}$, where v satisfies $1/v = 1/t_1 - 1/t_2$.

Proof. Let $\mathbf{X} = (X_1, \dots, X_n)$ be an ordered partition of Ω into measurable subsets. Now the generalized Hölder's inequality (1.2) with $q = t_2$ and $s = t_1$ and with β_i taken to be the value $\|P_{X_i} f_i\|$ for $i \in \mathbb{N}_n$ shows that

$$r_{\mathbf{X}}((f_1, \dots, f_n)) = \sup\left\{r_{\mathbf{X}}((\zeta_1 f_1, \dots, \zeta_1 f_n)) : \sum_{j=1}^{n} |\zeta_j|^v \leq 1\right\}$$

for each $f_1, \dots, f_n \in L^r(\Omega)$ and $n \in \mathbb{N}$. Taking the supremum over all such ordered partitions $\mathbf{X}$ gives the result. □

It was conjectured in [10, §3.8] that, whenever $t \geq r > 1$, the standard t-multi-norm on an infinite-dimensional space $L^r(\Omega)$ is never equivalent to a (p, q)-multi-norm based on the same space. In §4, we shall extend the cases for which this is true, but, in §4.3, we shall give a counter-example to this conjecture.

1.8. **Earlier results.** The basic questions that we are concerned with in this paper are to determine, for a given normed space, when two (p, q)-multi-norms based on that space are mutually equivalent and when a (p, q)-multi-norm is equivalent to a standard t-multi-norm on the space.

Some elementary relations were given in [8]. For example, the following is [8, Theorem 4.6].

Theorem 1.9. *Let E be a normed space. Then $\|\mathbf{x}\|_n^{(1,1)} = \|\mathbf{x}\|_n^{\max}$ for each $\mathbf{x} \in E^n$ and $n \in \mathbb{N}$, and so $(\|\cdot\|_n^{(1,1)} : n \in \mathbb{N})$ is the maximum multi-norm based on E.* □

The mutual equivalence of different (p, q)-multi-norms is discussed more seriously in [10, §3]. The first general result is [10, Theorem 2.11]; it follows immediately from [13, Theorem 10.4] by using the connection between (p,q)-multi-norms and absolutely summing operators given in Theorem 1.7.

Theorem 1.10. *Let E be a normed space, and suppose that*

$$1 \le p_1 \le q_1 < \infty \quad \text{and} \quad 1 \le p_2 \le q_2 < \infty\,.$$

Then $(\|\cdot\|_n^{(p_2,q_2)}) \le (\|\cdot\|_n^{(p_1,q_1)})$ *on E when both* $1/p_1 - 1/q_1 \le 1/p_2 - 1/q_2$ *and* $q_1 \le q_2$. □

Given a $(\bar{p},\bar{q})$-multi-norm, the following figure illustrates the regions where the (p,q)-multi-norms are definitely smaller and larger than this particular $(\bar{p},\bar{q})$-multi-norm on each space $L^r(\Omega)$. We have not at this stage excluded the possibility that the shaded regions are larger; indeed, we shall show in §4 that the upper area can be larger for certain values of r.

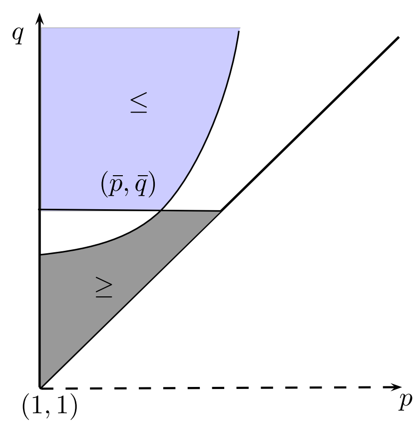

FIGURE 1. Regions where the (p,q)-multi-norms are smaller and are larger than a particular $(\bar{p},\bar{q})$-multi-norm.

To explain the main classification result obtained in [10], we refer to some curves $\mathcal{C}_c$ contained in the 'triangle'

$$\mathcal{T} = \{(p,q) : 1 \le p \le q < \infty\}\,.$$

For $c \in [0,1)$, the curve $\mathcal{C}_c$ is

$$\mathcal{C}_c = \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} = c\right\}\,,$$

so that $\mathcal{T}$ is the union of these curves. Note that, for $r > 1$, the curve $\mathcal{C}_{1/r}$ meets the line $p = 1$ at the point $(1, r')$.

Following [10, §3.2], we say that two points $P_1 = (p_1,q_1)$ and $P_2 = (p_2,q_2)$ in $\mathcal{T}$ are *equivalent for a normed space E* if the corresponding multi-norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on E are equivalent.

The results in [10] on the equivalence of two such points in $\mathcal{T}$ for the Banach space $L^r(\Omega)$ are given in the following cases; here Ω is a measure space, $r \geq 1$, and we suppose that $L^r(\Omega)$ is infinite dimensional.

(I) The case where $r = 1$ is fully resolved in [10, Theorem 3.3].

Indeed, suppose that $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ are in $\mathcal{T}$. In the case where $q_1 \leq q_2$, we have $(\|\cdot\|_n^{(p_2,q_2)}) \preccurlyeq (\|\cdot\|_n^{(p_1,q_1)})$. Thus a necessary condition for the equivalence of P_1 and P_2 on $L^1(\Omega)$ is that $q_1 = q_2$; in this latter case, the points $P_1 = (p_1, q)$ and $P_2 = (p_2, q)$ are equivalent whenever $1 \leq p_1 \leq p_2 < q$, but (p, q) is not equivalent to (q, q) when $1 \leq p < q$.

(II) The case where $r \in (1, 2)$ is considered in [10, Theorem 3.16].

(III) The case where $r \geq 2$ is considered in [10, Theorem 3.18].

The above two cases will be fully described below.

Now take $r > 1$, and set $\overline{r} = \min\{r, 2\}$. We define the set

$$A_r := \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{\overline{r}}\right\} = \bigcup \{\mathcal{C}_c : c \in [1/\overline{r}, 1)\}\,.$$

Note that it follows from Theorem 1.10 that $(\|\cdot\|_n^{(p,q)}) \leq (\|\cdot\|_n^{(1,\overline{r}')})$ for each $(p, q) \in A_r$.

The following is [10, Theorem 3.9]. The proof uses Orlicz's theorem and some strong results on tensor norms; we shall give a direct proof of a somewhat more general result in Theorem 2.1, below.

Theorem 1.11. *Let Ω be a measure space, and take $r > 1$ and $(p, q) \in A_r$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|^{\min})$ on $L^r(\Omega)$.* □

Next, the theorems in [10] show that the two points P_1 and P_2 in $\mathcal{T}$ are not equivalent for $L^r(\Omega)$ (when $L^r(\Omega)$ is an infinite-dimensional space) when at least one point lies outside the region A_r, except perhaps in the following three cases, (A), (B), and (C).

(A) : *Both of the points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ lie on the same curve $\mathcal{C}_c$, where $c \in [0, 1/\overline{r})$ and, further, $p_1, p_2 \in [1, r)$ when $r < 2$ and $p_1, p_2 \in [1, 2]$ when $r \geq 2$.*

The question whether two such points P_1 and P_2 are indeed equivalent was already resolved in [10, Theorem 3.8] in the special case where $c = 0$: here, $P_1 = (p_1, p_1)$ and $P_2 = (p_2, p_2)$ are equivalent, and the corresponding multi-norms were shown to be equivalent to the maximum multi-norm whenever $p_1, p_2 \in [1, \overline{r})$. Further, in the case where $1 < r < 2$, so that $\overline{r} = r$, the point (r, r) is not equivalent to any point $P = (p, p)$ when $p \in [1, r)$ (this is a result of Kwapień [15, Theorem 7]; see also [3]), and, in the case where

$r \geq 2$, so that $\overline{r} = 2$, the point $(2,2)$ is equivalent to each point $P = (p,p)$ for $p \in [1,2)$, and hence is equivalent to the maximum multi-norm for $L^r(\Omega)$.

We shall prove in Theorem 2.5 that the above two points P_1 and P_2 specified in case (A) are indeed equivalent whenever $r > 1$. (The case (A) does not arise when $r = 1$.)

The second and third cases that were left open in [10] arise only when $r < 2$ (so that $\overline{r} = r$). Suppose that $c \in [1/2, 1/r)$ and the curve $\mathcal{C}_c$ meets the vertical line $\{(p,q) : p = r\}$ at the point (r, u_c), so that $u_c = r/(1-cr)$, and consider the horizontal line $\{(p,q) : q = u_c\}$. This line meets the curve $\mathcal{C}_{1/2}$ at the point (x_c, u_c), say, where $x_c = 2u_c/(2+u_c) = 2r/(2(1-cr)+r)$, as in [10, §3.5]. Let us denote by L_c the horizontal line segment

$$L_c = \{(p, u_c) : r \leq p \leq x_c\}\,.$$

(See Figure 3.) Then the following case was also left open in [10].

(B) : *Both of the points* $P_1 = (p_1, u_c)$ *and* $P_2 = (p_2, u_c)$ *lie on the line segment* L_c.

Further, the following case was left open.

(C) : $P_1 = (p_1, q_1)$ *lies on a curve* $\mathcal{C}_c$, *where* $c \in (0, 1/r)$ *and* $1 \leq p_1 < r$ *and* P_2 *is the point* $(r, r/(1-cr))$.

We regret that we have not been able to resolve whether P_1 and P_2 are equivalent in the case (B); we shall show that we do have equivalence in case (C) whenever $c \in (1/2, 1/r)$, but leave open the case where $0 < c \leq 1/2$.

Two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are mutually equivalent for a Banach space E if and only if $\Pi_{q_1,p_1}(E', F) = \Pi_{q_2,p_2}(E', F)$ for every Banach space F [10, Theorem 2.8]. Thus one method of showing that two such points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ are not equivalent for ℓ^r is to show that there is no constant $C > 0$ such that

$$\pi_{q_1,p_1}(I_n : \ell_n^{r'} \to \ell_n^r) \leq C\pi_{q_2,p_2}(I_n : \ell_n^{r'} \to \ell_n^r) \quad (n \in \mathbb{N})\,,$$

where I_n is the identity operator on $\mathbb{C}^n$. For example, it is shown in [3] that

$$\pi_{p,p}(I_n : \ell_n^{r'} \to \ell_n^r) \sim (n \log n)^{1/r} \quad \text{as} \quad n \to \infty$$

for $1 \leq p < r < 2$, whereas $\pi_{r,r}(I_n : \ell_n^{r'} \to \ell_n^r) \sim n^{1/r}$ as $n \to \infty$, and so (p,p) is not equivalent to (r,r) whenever $1 \leq p < r < 2$. There are several calculations related to these constants $\pi_{q,p}(I_n : \ell_n^{r'} \to \ell_n^r)$ in [5, 12, 19], but it appears that none of them resolve the points that we have left open.

The strongest earlier result about the equivalence of the standard t-multi-norm and a (p,q)-multi-norm on an infinite-dimensional space $L^r(\Omega)$

is given in [10, Theorem 3.22]. It shows that it is only possible for a multi-norm $(\|\cdot\|_n^{(p,q)})$ to be equivalent to $(\|\cdot\|_n^{[t]})$ on an infinite-dimensional space $L^r(\Omega)$ when $1 < r < 2$. Further, if $1 < r < 2$ and $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ on $L^r(\Omega)$, then necessarily $t \geq 2r/(2-r)$, $1/p - 1/q \geq 1/2$, and (p,q) lies on the same curve $\mathcal{D}_c$ (as defined in [10, §3.5]) as (r,t) with $p \leq 2t/(2+t)$. Stronger results will be given in §4.

2. Equivalences of (p,q)-multi-norms

2.1. **Rademacher functions and Khintchine's inequality.** We denote the Rademacher functions defined on $[0,1]$ by r_k for $k \in \mathbb{N}$; see [1, 6.2.1] or [13, p. 10], for example. Then $|r_k(t)| = 1$ $(t \in [0,1],\, k \in \mathbb{N})$ and

$$\int_0^1 r_i(t)r_j(t)\,\mathrm{d}t = 0 \quad (i,j \in \mathbb{N},\, i \neq j)\,.$$

We shall also use a form of Khintchine's inequality (see [1, Theorem 6.2.3] or [22, §I.B.8]): for each $u > 0$, there exist constants A_u and B_u such that

(2.1)
$$A_u\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} \leq \left(\int_0^1 \left|\sum_{j=1}^n \alpha_j r_j(t)\right|^u \mathrm{d}t\right)^{1/u} \leq B_u\left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

A normed space E has *type* u for $1 \leq u \leq 2$ if there is a constant $K \geq 0$ such that

$$\left(\int_0^1 \left\|\sum_{j=1}^n r_j(t)x_j\right\|^2 \mathrm{d}t\right)^{1/2} \leq K\left(\sum_{j=1}^n \|x_j\|^u\right)^{1/u} \tag{2.2}$$

for each $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$.

Theorem 2.1. *Let E be a Banach space with type $u \in [1,2]$, and take $s \in [1,u]$. Then there is a constant $K > 0$ such that*

$$\|\boldsymbol{x}\|_n^{(1,s')} \leq K\,\|\boldsymbol{x}\|_n^{\min} \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N})\,.$$

Proof. The constant K is defined by equation (2.2).

Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, and suppose that $\mu_{1,n}(\boldsymbol{\lambda}) \leq 1$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$. Then the following estimates hold: throughout the suprema are taken over all $\zeta_1, \dots, \zeta_n \in \mathbb{C}$ such that $\sum_{j=1}^n |\zeta_j|^s \leq 1$.

Indeed, we have

$$\begin{aligned}\left(\sum_{j=1}^{n}|\langle x_j,\,\lambda_j\rangle|^{s'}\right)^{1/s'} &= \sup\left\{\left|\sum_{j=1}^{n}\langle\zeta_j x_j,\,\lambda_j\rangle\right|\right\}\\ &= \sup\left\{\left|\int_0^1\left\langle\sum_{i=1}^{n}\zeta_i r_i(t)x_i,\sum_{j=1}^{n}r_j(t)\lambda_j\right\rangle\,\mathrm{d}t\right|\right\}\\ &\leq \sup\left\{\int_0^1\left\|\sum_{j=1}^{n}\zeta_j r_j(t)x_j\right\|\,\mathrm{d}t\right\}\end{aligned}$$

because $\left\|\sum_{j=1}^{n} r_j(t)\lambda_j\right\| \leq \mu_{1,n}(\boldsymbol{\lambda})$ by (1.3) (in the case where $p=1$), and so

$$\begin{aligned}\left(\sum_{j=1}^{n}|\langle x_j,\,\lambda_j\rangle|^{s'}\right)^{1/s'} &\leq \sup\left\{\left(\int_0^1\left\|\sum_{j=1}^{n}\zeta_j r_j(t)x_j\right\|^2\,\mathrm{d}t\right)^{1/2}\right\}\\ &\leq K\sup\left\{\left(\sum_{j=1}^{n}\|\zeta_j x_j\|^u\right)^{1/u}\right\}\quad\text{by}\quad(2.2)\\ &\leq K\max_{j\in\mathbb{N}_n}\|x_j\|\sup\left\{\left(\sum_{j=1}^{n}|\zeta_j|^u\right)^{1/u}\right\}\\ &= K\max_{j\in\mathbb{N}_n}\|x_j\|\end{aligned}$$

because $s\leq u$.

The result follows. □

2.2. **Calculations for the spaces $L^r(\Omega)$.**

We now make some calculations that are specific to the Banach space $L^r(\Omega)$. Again, we set $\overline{r}=\min\{r,2\}$ for $r\geq 1$.

The first result is a reprise of Theorem 1.11 with a more elementary proof; it follows immediately from Theorem 2.1 because a space $L^r(\Omega)$, for $r\geq 1$, has type $\min\{r,2\}$ [13, Corollary 11.7(a)].

Theorem 2.2. *Let Ω be a measure space, and take $r>1$ $(p,q)\in A_r$. Then $(\|\cdot\|_n^{(p,q)})\cong(\|\cdot\|^{\min})$ on $L^r(\Omega)$.* □

We shall use the following elementary calculation, given in [10, (2.5)], concerning (p,q)-multi-norms based on ℓ^r, where $r\geq 1$. Recall that, for each $k\in\mathbb{N}$, we write δ_k for the sequence $(\delta_{j,k} : j\in\mathbb{N})$. Indeed, for each $(p,q)\in\mathcal{T}$ and each $n\in\mathbb{N}$, we have

$$\Delta_n(p,q) = \begin{cases} n^{1/r+1/q-1/p} & \text{when } p < r \text{ and } 1/p - 1/q \leq 1/r, \\ 1 & \text{when } 1/p - 1/q > 1/r, \\ n^{1/q} & \text{when } p \geq r\,, \end{cases} \tag{2.3}$$

where $\Delta_n(p,q) = \|(\delta_1,\dots,\delta_n)\|_n^{(p,q)}$ for $(p,q) \in \mathcal{T}$.

The next result is a simple part of [10, Theorem 3.11]; it follows by inspecting the proof of that theorem.

Proposition 2.3. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite dimensional, where $r > 1$. Suppose that $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ lie on curves $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_2}$, respectively, where $c_2 < \min\{c_1, 1/\overline{r}\}$ and $p_1, p_2 \in [1, \overline{r}]$. Then it is not the case that $(\|\cdot\|_n^{(p_2,q_2)}) \preccurlyeq (\|\cdot\|_n^{(p_1,q_1)})$, and so P_1 and P_2 are not equivalent for $L^r(\Omega)$.* □

The next lemma is essentially the 'factorization theorem' given as [13, Lemma 2.23], combined with results related to Grothendieck's constant, K_G.

Lemma 2.4. *Let $F = L^s(\Omega)$, where Ω is a measure space and $s \geq 1$. Take $u > s$ and $u = 2$ in the cases where $s > 2$ and $s \in [1,2]$, respectively. Then there is a constant $K_u > 0$ such that, for each $n \in \mathbb{N}$ and each $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, there exist $\zeta_1,\dots,\zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} = (\nu_1,\dots,\nu_n) \in F^n$ such that:*

(i) $\lambda_j = \zeta_j \nu_j \ \ (j \in \mathbb{N}_n)$ *;*

(ii) $\sum_{j=1}^n |\zeta_j|^u \leq 1$ *;*

(iii) $\mu_{u',n}(\boldsymbol{\nu}) \leq K_u$.

In the case where $s \in [1,2]$, we can take $K_u = K_G$.

Proof. First, suppose that $s \in [1,2]$. By [13, Theorem 3.7], each operator $T \in \mathcal{B}(\ell^\infty, F)$ is 2-summing, with $\pi_2(T) \leq K_G \|T\| \ \ (T \in \mathcal{B}(\ell^\infty, F))$. Second, suppose that $s > 2$, and take $u > s$. By [13, Corollary 10.10], each operator $T \in \mathcal{B}(\ell^\infty, F)$ is u-summing, and so there is a constant K_u (depending on u) such that $\pi_u(T) \leq K_u \|T\| \ \ (T \in \mathcal{B}(\ell^\infty, F))$.

Now take $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, and define an operator $T_{\boldsymbol{\lambda}} \in \mathcal{B}(\ell^\infty, F)$ by requiring that $T_{\boldsymbol{\lambda}}(\delta_j) = \lambda_j \ \ (j \in \mathbb{N}_n)$ and $T_{\boldsymbol{\lambda}}(\delta_j) = 0 \ \ (j > n)$. We note that $\|T_{\boldsymbol{\lambda}}\| = \mu_{1,n}(\boldsymbol{\lambda}) = 1$ by (1.4), and so, in each case, T is u-summing, with $\pi_u(T_{\boldsymbol{\lambda}}) \leq K_u$.

We now use [13, Lemma 2.23] (taking $r = 1$ in that result) to see that there exist $\zeta_1,\dots,\zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} \in F^n$ with the required properties. □

2.3. **The open case (A).** The following result resolves the first open case, (A), specified on page 10.

Theorem 2.5. *Let Ω be a measure space, and take $r > 1$. Consider two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ lying on the same curve $\mathcal{C}_c$ with $0 \le c < 1$. Suppose, further, that $p_1, p_2 \in [1, r)$ in the case where $1 < r < 2$ and $p_1, p_2 \in [1, 2]$ in the case where $r \ge 2$. Then P_1 and P_2 are equivalent for $L^r(\Omega)$.*

Proof. We set $E = L^r(\Omega)$, $s = r'$, and $F = E' = L^s(\Omega)$.

Take $p < r$ in the case where $1 < r < 2$ and $p = 2$ when $r \ge 2$. We shall first show that there is a constant $K_p > 0$ such that

$$\|\boldsymbol{x}\|_n^{(1,1)} \le K_p \|\boldsymbol{x}\|_n^{(p,p)} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N}). \tag{2.4}$$

Indeed, take $u = p' > s$ when $1 < r < 2$ and $u = 2$ when $r \ge 2$. Let K_p be the constant K_u specified in Lemma 2.4, and take $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$; we adopt the notation of the factorization in Lemma 2.4. Take $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$. Then

$$\sum_{j=1}^n |\langle x_j,\, \lambda_j\rangle| = \sum_{j=1}^n |\langle x_j,\, \zeta_j\nu_j\rangle| = \sum_{j=1}^n |\zeta_j|\, |\langle x_j,\, \nu_j\rangle| \le \left(\sum_{j=1}^n |\langle x_j,\, \nu_j\rangle|^{u'}\right)^{1/u'}$$

by Hölder's inequality, noting that $\sum_{j=1}^n |\zeta_j|^u \le 1$, and so

$$\sum_{j=1}^n |\langle x_j,\, \lambda_j\rangle| \le \left(\sum_{j=1}^n |\langle x_j,\, \nu_j\rangle|^p\right)^{1/p} \le \|\boldsymbol{x}\|_n^{(p,p)} \mu_{p,n}(\boldsymbol{\nu}) \le K_p \|\boldsymbol{x}\|_n^{(p,p)},$$

giving (2.4). This covers the case where $c = 0$.

For the case where $c > 0$, consider a point $P = (p_0, q_0)$ which lies on a curve $\mathcal{C}_{1/v}$, where $v > 1$, and is such that $p_0 \in [1, r)$ in the case where $1 < r < 2$ and $p_0 \in [1, 2]$ in the case where $r \ge 2$; we recall that $(1, v')$ is a point of $\mathcal{C}_{1/v}$. It follows from Theorem 1.10 that it suffices to prove that $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{(p_0,q_0)})$. Again take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

By Lemma 1.6 with $p = s = 1$ and $q = v'$, we have

$$\|\boldsymbol{x}\|_n^{(1,v')} = \sup\left\{\|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(1,1)} : \sum_{j=1}^n |\zeta_j|^v \le 1\right\}.$$

By (2.4),

$$\|\boldsymbol{x}\|_n^{(1,v')} \le K_{p_0} \sup\left\{\|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p_0,p_0)} : \sum_{j=1}^n |\zeta_j|^v \le 1\right\}.$$

However, again by Lemma 1.6, now with $s = p_0$ and $q = q_0$, we have

$$\|\boldsymbol{x}\|_n^{(p_0,q_0)} = \sup\left\{\|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p_0,p_0)} : \sum_{j=1}^n |\zeta_j|^v \leq 1\right\}$$

because $1/v = 1/p_0 - 1/q_0$. Thus $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{(p_0,q_0)})$, as required. $\square$

It remains to be decided whether $P = (r, r/(1-cr)) = (r, u_c)$ is equivalent to $(1, 1/(1-c))$ when $1 < r < 2$; we shall discuss this further later.

We summarize the situation in the case where $r \geq 2$, where we have a full solution to the question concerning the equivalence of (p,q)-multi-norms.

Theorem 2.6. *Let Ω be a measure space such that $E := L^r(\Omega)$ is an infinite-dimensional space, where $r \geq 2$. Then the triangle $\mathcal{T}$ is decomposed into the following* (*mutually disjoint*) *equivalence classes:*

(i) *the region $\mathcal{T}_{\min} := A_r = \{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\}$;*

(ii) *the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c : 1 \leq p \leq 2\}$, for $c \in (0, 1/2)$;*

(iii) *the line segment $\mathcal{T}_{\max} := \{(p,p) : 1 \leq p \leq 2\}$;*

(iv) *the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with $p > 2$.*

Moreover:

(v) *there is a constant $K > 0$ such that*

$$\|\cdot\|_n^{\min} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,2)} \leq K\,\|\cdot\|_n^{\min} \quad (n \in \mathbb{N}),$$

and so the (p,q)-multi-norm is equivalent to the minimum multi-norm for E, for each $(p,q) \in \mathcal{T}_{\min}$;

(vi) *for each $c \in (0,1/2)$ and each $(p,q) \in \mathcal{T}_c$, we have*

$$\|\cdot\|_n^{(2,2/(1-2c))} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,1/(1-c))} \leq K_G\,\|\cdot\|_n^{(2,2/(1-2c))} \quad (n \in \mathbb{N});$$

(vii) *for each $(p,p) \in \mathcal{T}_{\max}$, the (p,p)-multi-norm is equivalent to the maximum multi-norm for E, and the $(1,1)$-multi-norm is equal to the maximum multi-norm.*

Proof. It follows from Theorem 2.2 that $\mathcal{T}_{\min}$ is an equivalence class and that clause (v) holds. By Theorems 1.9 and 2.5, $\mathcal{T}_c$ is an equivalence class for each $c \in [0, 1/2)$ and clause (vi) holds, noting that the constant in equation (2.4) can be taken to be K_G because $s = r' \in [1,2]$.

It remains to show that there are no other equivalences than those specified above. Again it is sufficient to prove the result for the space ℓ^r. This was established in [10, Theorem 3.18] with the help of Khintchine's inequalities and classical results about Schatten classes. $\square$

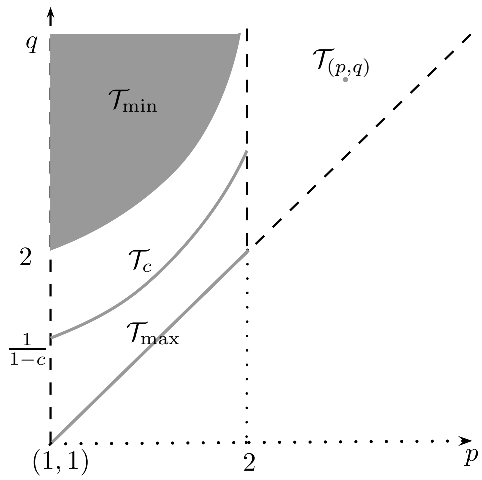

FIGURE 2. The various mutually disjoint equivalence classes of (p,q)-multi-norms on $L^r(\Omega)$ for $r \geq 2$.

We now summarize the situation in the case where $1 < r < 2$. Most of the result is contained in [10, Theorem 3.16]; this is combined with the new information given Theorem 2.5. Clause (vii) will be extended in Proposition 4.10.

Theorem 2.7. *Let Ω be a measure space such that $E := L^r(\Omega)$ is an infinite-dimensional space, where $1 < r < 2$. Then the triangle $\mathcal{T}$ is decomposed into the following (mutually disjoint) sets. Further, two points in distinct sets are not equivalent, and each specified set is an equivalence class, except possibly as noted:*

(i) *the region $\mathcal{T}_{\min} := A_r = \{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/r\}$;*

(ii) *the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c : 1 \leq p \leq r\} \cup \{(p,u_c) : r \leq p \leq x_c\}$, where $1/r - 1/u_c = c$ and $1/x_c - 1/u_c = 1/2$ for some $c \in (1/2, 1/r)$;*

(iii) *the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c : 1 \leq p \leq r\}$, for some $c \in (0, 1/2]$;*

(iv) *the line segment $\mathcal{T}_{\max} := \{(p,p) : 1 \leq p < r\}$;*

(v) *the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with either $p = q = r$ or both $p > r$ and $1/p - 1/q < 1/2$.*

Moreover:

(vi) *there is a constant $K > 0$ such that*

$$\|\cdot\|_n^{\min} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,r')} \leq K \|\cdot\|_n^{\min} \quad (n \in \mathbb{N}),$$

and so the (p,q)-multi-norm is equivalent to the minimum multi-norm for E, for each $(p,q) \in \mathcal{T}_{\min}$;

(vii) *in* $\mathcal{T}_c$ *for* $c \in (0, 1/r)$*, the* (p,q)*-multi-norms with* $1 \leq p < r$ *are all equivalent to the* $(1, 1/(1-c))$*-multi-norm, but we cannot say whether any two* (p,q)*-multi-norms on the horizontal segment* L_c *(when* $c > 1/2$*) are mutually equivalent, or whether the* (r, u_c)*-multi-norm is equivalent to the* $(1, 1/(1-c))$*-multi-norm;*

(viii) *for each* $(p,p) \in \mathcal{T}_{\max}$*, the* (p,p)*-multi-norm is equivalent to the maximum multi-norm for* E*, and the* $(1,1)$*-multi-norm is equal to the maximum multi-norm.* □

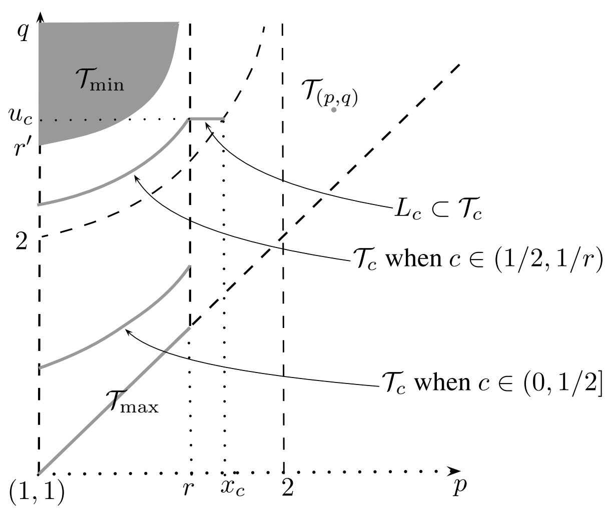

FIGURE 3. The various mutually inequivalent sets of (p,q)-multi-norms on $L^r(\Omega)$ for $1 < r < 2$.

3. The $[p,q]$-concave multi-norms on Banach lattices

In this section, we shall introduce a new class of multi-norms on general Banach lattices, and relate some of them to standard t-multi-norms: these multi-norms are of interest in their own right, and also will help us to settle at least one of the above questions about the equivalence of the (p,q)-multi-norms and to resolve the conjecture on the equivalence of (p,q)- and standard t-multi-norms on ℓ^r.

Let $(L, \|\cdot\|)$ be a (complex) Banach lattice. A summary of all necessary background in Banach lattice theory is given in [8, §1.3].

Throughout, L' denotes the dual Banach lattice to L. We write $|x|$ for the modulus of an element $x \in L$. Take $n \in \mathbb{N}$ and an n-tuple $(x_1, \ldots, x_n)$ in L^n. Recall that, for each $p \geq 1$, we can define the element $\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \in L$

by the Krivine calculus, and that

$$\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \sup\left\{\left|\sum_{j=1}^n \zeta_j x_j\right| : \zeta_1,\dots,\zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \le 1\right\},$$

where the supremum is taken in the Banach lattice sense; for more details, see [8] and [17, II.1.d], although only real Banach lattices were considered in the latter source. In fact, it can be seen that

$$\begin{aligned}\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} &= \sup\left\{\Re\left(\sum_{j=1}^n \zeta_j x_j\right) : \zeta_1,\dots,\zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \le 1\right\}\\ &= \sup\left\{\sum_{j=1}^n |\zeta_j x_j| : \zeta_1,\dots,\zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \le 1\right\}.\end{aligned}$$

It is also obvious that

$$\mu_{p,n}(x_1,\dots,x_n) \le \left\|\left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\right\|, \tag{3.1}$$

with equality whenever L is a $C(K)$-space.

Definition 3.1. Let $(L, \|\cdot\|)$ be a Banach lattice, and take $p, q \ge 1$ and $n \in \mathbb{N}$. For each $\boldsymbol{x} \in L^n$, define

$$\|\boldsymbol{x}\|_n^{[p,q]} = \sup\left\{\left(\sum_{j=1}^n |\langle x_j,\ \lambda_j\rangle|^q\right)^{1/q} : \left\|\left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}\right\| \le 1\right\},$$

where $\lambda_1,\dots,\lambda_n \in L'$. Then $\|\cdot\|_n^{[p,q]}$ is the n^{th} $[p,q]$-*concave norm* on L^n.

Clearly, we have $(\|\cdot\|_n^{[p,q_1]}) \le (\|\cdot\|_n^{[p,q_2]})$ when $1 \le p \le q_2 \le q_1$ and $(\|\cdot\|_n^{[p_1,q]}) \le (\|\cdot\|_n^{[p_2,q]})$ when $1 \le p_1 \le p_2 \le q$.

We shall prove that $(\|\cdot\|_n^{[p,q]} : n \in \mathbb{N})$ is a multi-norm on L whenever $1 \le p \le q < \infty$, and then we shall call the sequence $(\|\cdot\|_n^{[p,q]} : n \in \mathbb{N})$ *the* $[p,q]$-*concave multi-norm on* L. For the remainder of this section, we suppose that $L = (L, \|\cdot\|)$ is a Banach lattice.

Lemma 3.2. *Suppose that* $1 \le p \le q_1 < q_2 < \infty$. *Then*

$$\|\boldsymbol{x}\|_n^{[p,q_2]} = \sup\left\{\|(\zeta_1 x_1,\dots,\zeta_n x_n)\|_n^{[p,q_1]} : \sum_{j=1}^n |\zeta_j|^u \le 1\right\}$$

for each $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$ *and* $n \in \mathbb{N}$, *where* u *satisfies the equation* $1/u = 1/q_1 - 1/q_2$.

Proof. This is essentially the same as the proof of Lemma 1.6. □

Following the argument in [2, Proposition 3], we obtain the following basic result.

Proposition 3.3. *Suppose that $1 \le p \le q < \infty$, and let $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ be any map. Denote by $i_1, \dots, i_m$ the distinct elements of $\sigma(\mathbb{N}_n)$. Then*

$$\left\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\right\|_n^{[p,q]} \le \|(x_{i_1}, \dots, x_{i_m})\|_m^{[p,q]} \quad (x_1, \dots, x_n \in L)\,.$$

Proof. Let $\lambda_1, \dots, \lambda_n \in L'$ with $\left\|\left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}\right\| \le 1$. Then

$$\sum_{j=1}^n \left|\langle x_{\sigma(j)},\, \lambda_j\rangle\right|^q = \sum_{k=1}^m \sum_{\sigma(j)=i_k} \left|\langle x_{\sigma(j)},\, \lambda_j\rangle\right|^q \le \sum_{k=1}^m \left(\sum_{\sigma(j)=i_k} \left|\langle x_{\sigma(j)},\, \lambda_j\rangle\right|^p\right)^{q/p}$$
$$= \sum_{k=1}^m \left|\sum_{\sigma(j)=i_k} \langle x_{\sigma(j)},\, \lambda_j\rangle\zeta_j\right|^q$$

for some $\zeta_j \in \mathbb{C}$ with $\sum_{\sigma(j)=i_k} |\zeta_j|^{p'} \le 1$, and so

$$\sum_{j=1}^n \left|\langle x_{\sigma(j)},\, \lambda_j\rangle\right|^q = \sum_{k=1}^m |\langle x_{i_k},\, \mu_k\rangle|^q\ ,$$

where $\mu_k = \sum_{\sigma(j)=i_k} \zeta_j\lambda_j \in L'$.

We see that, for every $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ with $\sum_{k=1}^n |\alpha_k|^{p'} \le 1$, we have

$$\left|\sum_{k=1}^m \alpha_k\mu_k\right| = \left|\sum_{k=1}^m \sum_{\sigma(j)=i_k} \alpha_k\zeta_j\lambda_j\right| \le \left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}$$

because $\sum_{k=1}^m \sum_{\sigma(j)=i_k} |\alpha_k\zeta_j|^{p'} \le \sum_{k=1}^n |\alpha_k|^{p'} \le 1$. It follows that

$$\left(\sum_{k=1}^m |\mu_k|^p\right)^{1/p} \le \left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}\ ,$$

and so $\left\|(\sum_{k=1}^m |\mu_k|^p)^{1/p}\right\| \le 1$.

The result now follows. □

Theorem 3.4. *Let $(L, \|\cdot\|)$ be a Banach lattice. Then the sequence*

$$(\|\cdot\|_n^{[p,q]}\,:\ n \in \mathbb{N})$$

is a multi-norm based on L whenever $1 \le p \le q < \infty$.

Proof. The multi-norm axioms follows easily, using Proposition 3.3. □

Let E be a Banach space, and suppose that $1 \le p \le q < \infty$. Recall from [13, page 330] that a bounded linear operator $T : L \to E$ is (q,p)-*concave* if there is a constant $C > 0$ such that

$$\left(\sum_{j=1}^{n} \|Tx_j\|^q\right)^{1/q} \le C \left\| \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} \right\| \quad (x_1, \dots, x_n \in L,\, n \in \mathbb{N})\,;$$

the least such constant C is denoted by $K_{q,p}(T)$. We write $\mathcal{C}_{q,p}(L,E)$ for the space of (q,p)-concave operators; $\mathcal{C}_{q,p}(L,E)$ is a Banach space with respect to the norm $K_{q,p}(\,\cdot\,)$. The Banach lattice L is (q,p)-*concave* if the identity operator $I_L : L \to L$ is (q,p)-concave.

Proposition 3.5. *Let L be a Banach lattice, and and take p, q such that $1 \le p \le q < \infty$. Then L' is (q,p)-concave if and only if the $[p,q]$-concave multi-norm is equivalent to the minimum multi-norm on L.*

Proof. Suppose first that L' is (q,p)-concave, so that $C := K_{q,p}(I_L) < \infty$. Then, for each $n \in \mathbb{N}$, $x_1, \dots, x_n \in L$, and $\lambda_1, \dots, \lambda_n \in L'$, we have

$$\begin{aligned}\left(\sum_{j=1}^{n} |\langle x_j,\, \lambda_j\rangle|^q\right)^{1/q} &\le \max_{j \in \mathbb{N}_n} \|x_j\| \cdot \left(\sum_{j=1}^{n} \|\lambda_j\|^q\right)^{1/q} \\ &\le C \max_{j \in \mathbb{N}_n} \|x_j\| \cdot \left\| \left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p} \right\| .\end{aligned}$$

Hence $\|(x_1, \dots, x_n)\|_n^{[p,q]} \le C \max_{j \in \mathbb{N}_n} \|x_j\| = C\, \|(x_1, \dots, x_n)\|_n^{\min}$.

Conversely, suppose that the $[p,q]$-concave multi-norm is equivalent to the minimum multi-norm on L, so that there is a constant $C > 0$ such that

$$\|(x_1, \dots, x_n)\|_n^{[p,q]} \le C\, \|(x_1, \dots, x_n)\|_n^{\min} \quad (x_1, \dots, x_n \in L,\, n \in \mathbb{N})\,.$$

Let $\lambda_1, \dots, \lambda_n \in L'$. Take $\eta > 1$ and $j \in \mathbb{N}_n$, and choose $x_j \in L$ with $\|x_j\| = 1$ and such that $\|\lambda_j\| \le \eta\, |\langle x_j,\, \lambda_j\rangle|$. Then

$$\begin{aligned}\left(\sum_{j=1}^{n} \|\lambda_j\|^q\right)^{1/q} &\le \eta \left(\sum_{j=1}^{n} |\langle x_j,\, \lambda_j\rangle|^q\right)^{1/q} \\ &\le \eta\, \|(x_1, \dots, x_n)\|_n^{[p,q]} \cdot \left\| \left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p} \right\| \\ &\le C\eta \left\| \left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p} \right\| .\end{aligned}$$

Thus L' is (q,p)-concave, with $K_{q,p}(L) \le C$. □

Note that we simply say 'p-concave' for '(p,p)-concave'; in the case where $p=1$, '$(q,1)$-concave' is also called 'having a lower q-estimate' in [17, II.1.f].

Let E be a Banach space. By theorems of Maurey (see [18] and [13, Corollaries 16.6 and 16.7]), we have

$$\mathcal{C}_{q,p}(L,E)=\mathcal{C}_{q,1}(L,E)\subset\mathcal{C}_{r,r}(L,E)$$

whenever $1\le p<q<r<\infty$, and

$$\mathcal{C}_{q,1}(L,E)=\Pi_{q,1}(L,E)\quad\text{whenever } q>2\,.$$

The proof of [13, Corollary 16.7] also gives the inclusion

$$\mathcal{C}_{2,2}(L,E)\subset\Pi_{2,1}(L,E)\,.$$

We also have the following more elementary inclusion, which follows immediately from the definitions and inequality (3.1):

$$\Pi_{q,p}(L,E)\subset\mathcal{C}_{q,p}(L,E)\quad\text{with}\quad K_{q,p}(T)\le\pi_{q,p}(T)\quad(T\in\Pi_{q,p}(L,E))$$

whenever $1\le p<q<\infty$; moreover, $\Pi_{q,p}(C(K),E)=\mathcal{C}_{q,p}(C(K),E)$ with $K_{q,p}(T)=\pi_{q,p}(T)$ $(T\in\Pi_{q,p}(C(K),E))$ for a compact space K.

We remark also that, by [13, Theorems 10.4 and 16.5], the inclusion

$$\mathcal{C}_{q_1,p_1}(L,E)\subset\mathcal{C}_{q_2,p_2}(L,E)$$

holds, with $K_{p_2,q_2}(T)\le K_{p_1,q_1}(T)$ $(T\in\mathcal{C}_{q_1,p_1}(L,E))$ whenever we have $1\le p_1\le q_1<\infty$, $1\le p_2\le q_2<\infty$, and both $1/p_1-1/q_1\le 1/p_2-1/q_2$ and $q_1\le q_2$.

The following result is similar to equation (1.5).

Theorem 3.6. *Let L be a Banach lattice, and suppose that $1\le p\le q<\infty$. Then*

$$\|\boldsymbol{x}\|_n^{[p,q]}=K_{q,p}(T'_{\boldsymbol{x}}:L'\to\ell_n^\infty)\quad(\boldsymbol{x}\in L^n,\,n\in\mathbb{N})\,.$$

Proof. Set $\boldsymbol{x}=(x_1,\dots,x_n)$ and $K_{q,p}=K_{q,p}(T'_{\boldsymbol{x}}:L'\to\ell_n^\infty)$.

We see that

$$\begin{aligned}K_{q,p}&=\sup\left\{\left(\sum_{j=1}^n\|T'_{\boldsymbol{x}}\lambda_j\|_{\ell_n^\infty}^q\right)^{1/q}:\left\|\left(\sum_{j=1}^n|\lambda_j|^p\right)^{1/p}\right\|\le 1\right\}\\&=\sup\left\{\left(\sum_{j=1}^n\sup_{k\in\mathbb{N}_n}|\langle x_k,\lambda_j\rangle|^q\right)^{1/q}:\left\|\left(\sum_{j=1}^n|\lambda_j|^p\right)^{1/p}\right\|\le 1\right\}\\&\ge\sup\left\{\left(\sum_{j=1}^n|\langle x_j,\lambda_j\rangle|^q\right)^{1/q}:\left\|\left(\sum_{j=1}^n|\lambda_j|^p\right)^{1/p}\right\|\le 1\right\}\\&=\|(x_1,\dots,x_n)\|_n^{[p,q]}\,,\end{aligned}$$

where $\lambda_1,\dots,\lambda_n \in L'$. In particular, this gives $\|\boldsymbol{x}\|_n^{[p,q]} \le K_{q,p}$.

On the other hand, take $\lambda_1,\dots,\lambda_n \in L'$ with $\left\|\left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}\right\| \le 1$. For each $j \in \mathbb{N}_n$, let $k_j \in \mathbb{N}_n$ be such that $\sup_{k\in\mathbb{N}_n} |\langle x_k, \lambda_j\rangle| = \left|\langle x_{k_j}, \lambda_j\rangle\right|$, and set $\sigma(j) = k_j$. Then we see that

$$\left(\sum_{j=1}^n \sup_{k\in\mathbb{N}_n} |\langle x_k, \lambda_j\rangle|^q\right)^{1/q} \le \left\|(x_{\sigma(1)},\dots,x_{\sigma(n)})\right\|_n^{[p,q]} \le \|\boldsymbol{x}\|_n^{[p,q]}\,.$$

Hence $K_{q,p} \le \|\boldsymbol{x}\|_n^{[p,q]}$. □

Consequently, we have the following conclusions.

Corollary 3.7. *Let L be a Banach lattice, and consider multi-norms based on L. Then:*

(i) $(\|\cdot\|_n^{[p_2,q_2]}) \le (\|\cdot\|_n^{[p_1,q_1]})$ *whenever we have* $1 \le p_1 \le q_1 < \infty$ *and* $1 \le p_2 \le q_2 < \infty$ *and both* $1/p_1 - 1/q_1 \le 1/p_2 - 1/q_2$ *and* $q_1 \le q_2$;

(ii) $(\|\cdot\|_n^{[p,q]}) \le (\|\cdot\|_n^{(p,q)})$ *whenever* $1 \le p \le q < \infty$;

(iii) $(\|\cdot\|_n^{[p,q]}) \cong (\|\cdot\|_n^{[1,q]}) \not\succcurlyeq (\|\cdot\|_n^{[r,r]})$ *whenever* $1 \le p < q < r < \infty$;

(iv) $(\|\cdot\|_n^{[1,q]}) \cong (\|\cdot\|_n^{(1,q)})$ *in the case where* $q > 2$;

(v) $(\|\cdot\|_n^{(1,2)}) \preccurlyeq (\|\cdot\|_n^{[2,2]})$. □

Proposition 3.8. *Let E be a Banach space, and take $r \ge 1$. Then the map*

$$T \mapsto (T(\delta_j))\,, \quad \mathcal{C}_{1,1}(\ell^{r'}, E) \to \ell^{\,r}(E)\,,$$

is an isometric isomorphism.

Proof. Take $T \in \mathcal{C}_{1,1}(\ell^{r'}, E)$. Then, for each $n \in \mathbb{N}$, there are $\alpha_1,\dots,\alpha_n \in \mathbb{C}$ with

$$\sum_{j=1}^n |\alpha_j|^{r'} \le 1 \quad \text{and} \quad \left(\sum_{j=1}^n \|T(\delta_j)\|^r\right)^{1/r} = \sum_{j=1}^n \|T(\alpha_j\delta_j)\|\,.$$

Therefore

$$\left(\sum_{j=1}^n \|T(\delta_j)\|^r\right)^{1/r} \le K_{1,1}(T)\left\|\sum_{j=1}^n |\alpha_j\delta_j|\right\|_{\ell^{r'}} = K_{1,1}(T)\,.$$

Conversely, take $\boldsymbol{x} = (x_j) \in \ell^{\,r}(E)$, and set $T(\delta_j) = x_j\ \ (j \in \mathbb{N})$; extend T to be a linear map from c_{00} into E. Then, for each $n \in \mathbb{N}$ and each

$f_1, \dots, f_n \in c_{00}$, we see that

$$\begin{aligned}\sum_{k=1}^n \|T(f_k)\| &\le \sum_{k=1}^n \sum_{j=1}^\infty |f_k(j)|\, \|T(\delta_j)\| = \sum_{j=1}^\infty \sum_{k=1}^n |f_k(j)|\, \|x_j\| \\ &\le \left(\sum_{j=1}^\infty \left(\sum_{k=1}^n |f_k(j)|\right)^{r'}\right)^{1/r'} \left(\sum_{j=1}^\infty \|x_j\|^r\right)^{1/r} \\ &= \left\|\sum_{k=1}^n |f_k|\right\|_{\ell^{r'}} \|\boldsymbol{x}\|_{\ell^r(E)} .\end{aligned}$$

Thus T extends uniquely to an operator in $\mathcal{C}_{1,1}(\ell^{r'}, E)$ with the 1-concave norm at most $\|\boldsymbol{x}\|_{\ell^r(E)}$. □

We can now give a key relationship between a standard t-multi-norm and certain concave multi-norms.

Theorem 3.9. *Suppose that $1 \le r \le t < \infty$, and set $1/v = 1/r - 1/t$. Then the standard t-multi-norm is equal to the $[1, v']$-concave multi-norm on ℓ^r.*

Proof. By Lemmas 1.8 and 3.2, it is sufficient to consider only the case where $r = t$, so that $v' = 1$. Thus we need to show that

$$\|\boldsymbol{x}\|_n^{[1,1]} = \|\boldsymbol{x}\|_n^{[r]} \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell^r)^n,\, n \in \mathbb{N}) .$$

However, we have seen that

$$\begin{aligned}\|\boldsymbol{x}\|_n^{[1,1]} &= K_{1,1}(T'_{\boldsymbol{x}} : \ell^{r'} \to \ell_n^\infty) = \left(\sum_{j=1}^n \|T'_{\boldsymbol{x}}(\delta_j)\|^r\right)^{1/r} \\ &= \||x_1| \vee \cdots \vee |x_n|\|_{\ell^r} ,\end{aligned}$$

and this gives the result. □

4. Equivalence of the standard t-multi-norm and a (p, q)-multi-norm

4.1. **Notation.** We now consider when a standard t-multi-norm is equivalent to a (p,q)-multi-norm on an infinite-dimensional space $L^r(\Omega)$. In fact, this problem clearly divides into two separate questions: determine when $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$ and when $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]})$.

We define two new subsets of the triangle $\mathcal{T}$: for $1 \le r \le t$, we set

$$B_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \le 1/r - 1/t,\ q \le t\}$$

and

$$C_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \ge 1/r - 1/t\} \cup \{(p,q) \in \mathcal{T} : q \ge t\} ,$$

so that $B_{r,t}$ and $C_{r,t}$ intersect in the curve

$$L_{r,t} := \{(p,q) \in \mathcal{T} : 1/p - 1/q = 1/r - 1/t,\ p \leq r\} \cup \{(p,t) \in \mathcal{T} : r \leq p \leq t\}\,.$$

Further, we set $B_r = B_{r,r} = \{(p,p) : 1 \leq p \leq r\}$ and $C_r = C_{r,r} = \mathcal{T}$. Note that

$$B_{1,t} = \{(p,q) \in \mathcal{T} : q \leq t\} \quad \text{and} \quad C_{1,t} = \{(p,q) \in \mathcal{T} : q \geq t\}\,.$$

The answer to the first question is easy.

Theorem 4.1. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite dimensional, where $r \geq 1$. Then $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$ for $L^r(\Omega)$ if and only if $(p,q) \in B_{r,t}$.*

Proof. Let S be the set of points $(p,q) \in \mathcal{T}$ with $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$.

By [8, Theorem 4.22], $(\|\cdot\|_n^{[t]}) \leq (\|\cdot\|_n^{(r,t)})$, and so $(r,t) \in S$. By Theorem 1.10, we increase $(\|\cdot\|_n^{(p,q)})$ when we move from (r,t) to any point $(p,q) \in \mathcal{T}$ with $1/p - 1/q \leq 1/r - 1/t$ and $q \leq t$, and so $B_{r,t} \subset S$.

Conversely, let $(p,q) \in S$. In the case where $p \geq r$, we have seen that $\Delta_n(p,q) = n^{1/q}\ \ (n \in \mathbb{N})$, and so, by (1.6), we also have $q \leq t$ In the case where $p \in [1,r)$, by (2.3) and (1.6) again, we must have $1/p - 1/q \leq 1/r - 1/t$, which implies also that $q \leq t$. Thus in both case $(p,q) \in B_{r,t}$, and so $S \subset B_{r,t}$. □

We now consider the second question.

Definition 4.2. Let Ω be a measure space, set $E = L^r(\Omega)$, where $r \geq 1$, and take $t \geq r$. Then

$$D_{r,t} = \{(p,q) \in \mathcal{T} : (\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]}) \quad \text{on} \quad E\}\,,$$

with $D_r = D_{r,r}$.

Note that $D_{r,t_2} \subset D_{r,t_1}$ whenever $r \leq t_1 \leq t_2$, and hence, in particular, $D_{r,t} \subset D_r$ whenever $t \geq r$. It is clear that $A_r \subset D_{r,t}$ for $t \geq r \geq 1$ because $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{\min})$ when $(p,q) \in A_r$ by Theorem 2.2. By comparing the values of $\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)}$ and $\|(\delta_1, \dots, \delta_n)\|_n^{[t]}$ given in equations (2.3) and (1.6), we see that $D_{r,t} \subset C_{r,t}$ for $t \geq r$.

We now work on the spaces $\ell^{\,r}$, where $r \geq 1$.

4.2. **The case where $r = 1$.** We first give a full solution to our questions in the case where $r = 1$. Recall that we have $(\|\cdot\|_n^{[1]}) = (\|\cdot\|_n^{(1,1)}) = (\|\cdot\|_n^{\max})$ on $\ell^{\,1}$, and so $D_{1,1} = \mathcal{T}$.

Proposition 4.3. *Take $t > 1$. Then*

$$D_{1,t} = \{(p,q) : q \geq \max\{t,p\}\} \setminus \{(t,t)\} = C_{1,t} \setminus \{(t,t)\}\,.$$

Proof. We know that

$$D_{1,t} \subset C_{1,t} = \{(p,q) : q \geq \max\{t,p\}\}\,.$$

Also, it is proved in [8, Theorem 4.26] that $(\|\cdot\|_n^{[q]}) = (\|\cdot\|_n^{(1,q)})$ on ℓ^1 for each $q \geq 1$, and so $(1,t) \in D_{1,t}$. By [9, Theorem 5.6] (which depends on [20, Corollary 2.5], *cf.* [13, Theorem 10.9]), we have $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{(1,q)})$ for $1 \leq p < q$, and so $(p,t) \in D_{1,t}$ for $1 \leq p < t$.

Take $(p,q) \in \mathcal{T}$. It follows from the previous paragraph and Theorem 1.10 that $(p,q) \in D_{1,t}$ whenever $q \geq t$ and $q > p$. It remains to consider the case where $q = p$. If $q = p > t$, then, by [9, Theorem 5.6] again, we have

$$(\|\cdot\|_n^{(p,p)}) \preccurlyeq (\|\cdot\|_n^{(1,t)}) = (\|\cdot\|_n^{[t]})\,,$$

and so $(p,p) \in D_{1,t}$. On the other hand, in the case where $p = q = t$, we certainly have $(\|\cdot\|_n^{(1,t)}) \leq (\|\cdot\|_n^{(t,t)})$. However, by [10, Theorem 3.2], $(\|\cdot\|_n^{(1,t)}) \ncong (\|\cdot\|_n^{(t,t)})$, and so it follows that $(\|\cdot\|_n^{(t,t)}) \not\preccurlyeq (\|\cdot\|_n^{(1,t)}) = (\|\cdot\|_n^{[t]})$. Thus $(t,t) \notin D_{1,t}$. □

Theorem 4.4. *Suppose that $t \geq 1$ and $1 \leq p \leq q < \infty$. Then*

$$(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$$

on the space ℓ^1 if and only if $p = q = t = 1$ or $p < q = t$.

Proof. This follows from Theorem 4.1 and Proposition 4.3. □

4.3. **The case where $r > 1$.** We now turn to the case where $r > 1$.

Lemma 4.5. *Take $t \geq r > 1$ and $1 \leq p \leq q < \infty$, and consider the space ℓ^r. Then*

$$A_r \subset D_{r,t} \subset \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}\right\} \subsetneq C_{r,t}\,.$$

Proof. Let $n \in \mathbb{N}$. As shown in the proof of [10, Theorem 3.22], there exists an element $\boldsymbol{g} = (g_1, \dots, g_n) \in (\ell^r)^n$ such that $\|\boldsymbol{g}\|_n^{[t]} \leq 1$ and

$$\|\boldsymbol{g}\|_n^{(p,q)} \sim \|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \quad \text{as} \quad n \to \infty\,,$$

where we are now regarding $\delta_1, \dots, \delta_n$ as elements of ℓ^2. Now suppose that $1/p - 1/q < 1/2$. Then it follows from (2.3) that $\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \geq n^\alpha$, where $\alpha = \min\{1/2 + 1/q - 1/p, 1/q\} > 0$. Hence $(p,q) \notin D_{r,t}$. □

The following theorem, which is essentially [10, Theorem 3.22], determines fully the relation between the multi-norms $(\|\cdot\|_n^{(p,q)})$ and $(\|\cdot\|_n^{[t]})$ on the space ℓ^r in the case where $r \geq 2$.

Theorem 4.6. *Suppose that $t \geq r \geq 2$ and $1 \leq p \leq q < \infty$, and consider the space ℓ^r. Then $(\|\cdot\|^{(p,q)}) \preccurlyeq (\|\cdot\|^{[t]})$ if and only if $1/p - 1/q \geq 1/2$, and $(\|\cdot\|^{[t]}) \preccurlyeq (\|\cdot\|^{(p,q)})$ if and only if $(p,q) \in B_{r,t}$. In particular, $(\|\cdot\|_n^{(p,q)})$ and $(\|\cdot\|_n^{[t]})$ are not equivalent on ℓ^r for any $(p,q) \in \mathcal{T}$ and any $t \geq r$.*

Proof. Since $r \geq 2$, the set A_r is equal to $\{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\}$, giving the first clause. The second clause is Theorem 4.1. □

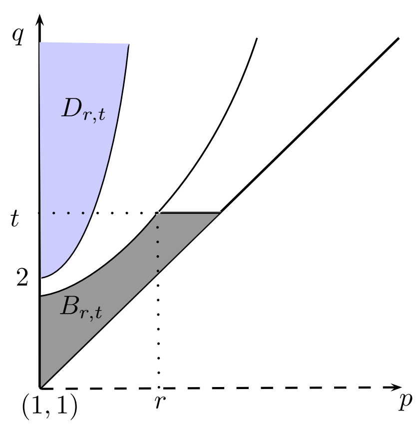

FIGURE 4. The sets $B_{r,t}$ and $D_{r,t}$ for $r \geq 2$

It remains to consider the case where $1 < r < 2$, and again it is this case that is the more difficult. Throughout we fix $t \geq r$ and define v by

$$\frac{1}{v} = \frac{1}{r} - \frac{1}{t},$$

taking $v = \infty$ when $t = r$.

Proposition 4.7. *Suppose that $r \in (1,2)$, $t \geq r$, and $1 \leq p \leq q < \infty$. Then:*

(i) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q \geq 1/v$ and $v < 2$;*

(ii) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q > 1/2$ and $2 \leq v < \infty$;*

(iii) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q \geq 1/2$ and $v = \infty$.*

Proof. (i) By Theorem 1.10, it suffices to show that $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{[t]})$. By Theorem 3.9, $(\|\cdot\|_n^{[t]}) = (\|\cdot\|_n^{[1,v']})$. Also it follows from Corollary 3.7(iv) that $(\|\cdot\|_n^{(1,v')}) \cong (\|\cdot\|_n^{[1,v']})$, where we note that $v' > 2$.

(ii) By Theorem 1.10, it suffices to show that $(\|\cdot\|_n^{(1,u)}) \preccurlyeq (\|\cdot\|_n^{[t]})$ whenever $u > 2$. But now

$$(\|\cdot\|_n^{[t]}) = (\|\cdot\|_n^{[1,v']}) \geq (\|\cdot\|_n^{[1,u]}) \cong (\|\cdot\|_n^{(1,u)}) \quad \text{on } \ell^r,$$

as required.

(iii) By Corollary 3.7(v), we have $(\|\cdot\|_n^{(1,2)}) \preccurlyeq (\|\cdot\|_n^{[2,2]})$; by Corollary 3.7(i), we have $(\|\cdot\|_n^{[2,2]}) \leq (\|\cdot\|_n^{[1,1]})$; by Theorem 3.9, $(\|\cdot\|_n^{[1,1]}) = (\|\cdot\|_n^{[t]})$. This gives the stated result. □

We interpret the above proposition in Figures 5 and 6, below.

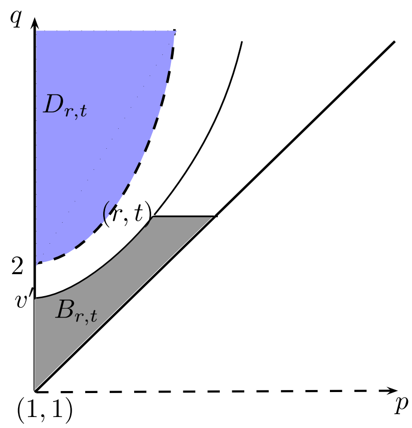

FIGURE 5. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t \leq 1/2$. When $r \geq 2$, the set $D_{r,t}$ contains the dotted line.

It follows from Figure 5 that, in the case where $1 \leq r \leq t$ and $v > 2$, the multi-norms $(\|\cdot\|_n^{(p,q)})$ are never equivalent to the multi-norm $(\|\cdot\|_n^{[t]})$, as remarked on page 12.

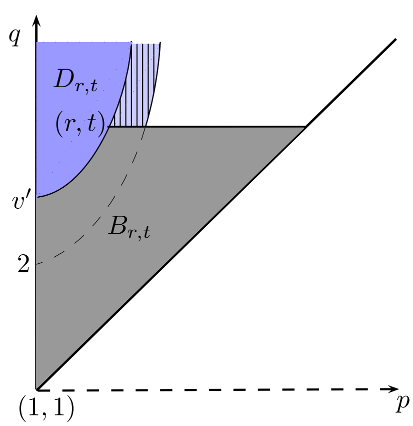

FIGURE 6. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t > 1/2$

Corollary 4.8. *Suppose that $r > 1$ and that $1 \leq p \leq q < \infty$. Then $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r if and only if $1/p - 1/q \geq 1/2$.*

Proof. Suppose that $(p,q) \in D_r$. Then $1/p - 1/q \geq 1/2$ by Lemma 4.5.

Suppose that $1/p - 1/q \geq 1/2$. Then $(p,q) \in D_r$ on ℓ^r: this follows from Theorem 4.6 when $r \geq 2$ and from Proposition 4.7(iii) when $r \in (1,2)$. □

Thus $A_r \subset D_{r,t} \subset D_r = A_2$ and $D_{r,t} \subset C_{r,t}$.

We now have the following counter to the conjecture in [10, §3.8] on the equivalence of (p,q)-multi-norms and standard t-multi-norms.

Theorem 4.9. *Suppose that $1 < r < 2$, that $t \geq r$, and that $1 \leq p \leq q < \infty$, and consider the space ℓ^r. Suppose further that $1/r - 1/t > 1/2$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ whenever*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{t} \quad \text{and} \quad 1 \leq p \leq r\,.$$

Proof. Take v as above, so that $v < 2$, and suppose that $1/p - 1/q = 1/v$. By Proposition 4.7(i), $(p,q) \in D_{r,t}$, and, by Theorem 4.1, $(p,q) \in B_{r,t}$ whenever $1 \leq p \leq r$. □

In fact, in the case specified in the above theorem, we know that

$$\left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{r} - \frac{1}{t}\right\} \subset D_{r,t} \subset \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}\right\},$$

but this is all that we know; if we could resolve Case (B), above, positively, we would know that

$$D_{r,t} = \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}\right\}.$$

The above theory does allow us to improve clause (vii) of Theorem 2.7. We recall that $u_c = r/(1-cr)$.

Proposition 4.10. *Suppose that $1 < r < 2$, and consider the space ℓ^r. Suppose further that $1/2 < c < 1/r$. Then the points $(1, 1/(1-c))$ and (r, u_c) are equivalent, and there is a constant K such that*

$$\|\cdot\|_n^{(r,u_c)} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,1/(1-c))} \leq K\,\|\cdot\|_n^{(r,u_c)} \quad (n \in \mathbb{N})$$

whenever $(p,q) \in \mathcal{C}_c$ and $1 \leq p \leq r$.

Proof. The new information is that $(\|\cdot\|_n^{(r,u_c)}) \cong (\|\cdot\|_n^{[u_c]}) \cong (\|\cdot\|_n^{(1,1/(1-c))})$ by Theorem 4.9. □

5. Regular operators

The above results actually have the following interesting consequence concerning the regularity of operators from ℓ^r into ℓ^q.

For a sequence $\alpha = (\alpha_j) \in \mathbb{C}^{\mathbb{N}}$, we set $|\alpha|$ to be the sequence $(|\alpha_j|)$; we say that $\alpha \geq 0$ whenever $\alpha_j \geq 0$ $(j \in \mathbb{N})$. Take $r, q \geq 1$ and $T \in \mathcal{B}(\ell^r, \ell^q)$. Then T specifies an infinite matrix $(T_{i,j} : i, j \in \mathbb{N})$, where $T_{i,j} = (T\delta_j)_i$ $(i, j \in \mathbb{N})$. The matrix $(|T_{i,j}|)$ then specifies a linear map $|T|$ from ℓ^r to $\mathbb{C}^{\mathbb{N}}$. Another way to define $|T|$ is as follows. A map $T \in \mathcal{B}(\ell^r, \ell^q)$ is *positive* if $T\alpha \geq 0$ in ℓ^q whenever $\alpha \geq 0$ in ℓ^r, and T is *regular* if it is a linear combination of positive operators; the collection of regular operators from ℓ^r to ℓ^q is denoted by $\mathcal{B}_r(\ell^r, \ell^q)$. Thus $T \in \mathcal{B}_r(\ell^r, \ell^q)$ if and only if $|T| \in \mathcal{B}(\ell^r, \ell^q)$. In fact, T is regular if and only if it is order-bounded [8, Theorem 1.31]. For $T \in \mathcal{B}_r(\ell^r, \ell^q)$, we define $|T|$ by

$$|T|\,(u) = \sup\{|Tz| : |z| \leq u\} \quad (u \geq 0)\,,$$

and extend T linearly. For a summary of properties of the space $\mathcal{B}_r(\ell^r, \ell^q)$ and its connections with 'multi-bounded operators', see [8, §§1.3.4, 6.4.1].

It is well-known that $\mathcal{B}_r(\ell^r, \ell^q) \subsetneq \mathcal{B}(\ell^r, \ell^q)$ when $1 < r, q < \infty$ (*cf.* [6], where more general results are proved).

Theorem 5.1. *Take $r \geq 1$. Then the following conditions on $(p, q) \in \mathcal{T}$ are equivalent:*

(a) $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ *on* ℓ^r *;*

(b) *there exists a constant $C > 0$ such that*

$$\|\,|A| : \ell^r_m \to \ell^q_n\| \leq C\,\|A : \ell^r_m \to \ell^p_n\|$$

for every $m, n \in \mathbb{N}$ and every $n \times m$ matrix A ;

(c) $T \in \mathcal{B}_r(\ell^r, \ell^q)$ *whenever* $T \in \mathcal{B}(\ell^r, \ell^p)$.

Proof. We set $s = r'$.

(a) $\iff$ (b) From the definition, we see that $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r if and only if there is a constant $C > 0$ such that, for every $n \in \mathbb{N}$, every $f_1, \ldots, f_n \in \ell^r$, and every $\lambda_1, \ldots, \lambda_n \in \ell^s$, we have

$$\left(\sum_{j=1}^{n} |\langle f_j,\, \lambda_j\rangle|^q\right)^{1/q} \leq C\mu_{p,n}(\lambda_1, \ldots, \lambda_n)\,\|(f_1, \ldots, f_n)\|_n^{[r]}\,.$$

Set $f = |f_1| \vee \cdots \vee |f_n|$. Then $f \in (\ell^r)^+$ and $\|(f_1, \ldots, f_n)\|_n^{[r]} = \|f\|$. So the statement above is equivalent to the condition that there is a constant $C > 0$

such that, for every $n \in \mathbb{N}$, every $f \in (\ell^r)^+$, and every $\lambda_1, \dots, \lambda_n \in \ell^s$, we have

$$\sup\left\{ \left(\sum_{j=1}^n |\langle f_j,\, \lambda_j\rangle|^q\right)^{1/q} : f_1, \dots, f_n \in \ell^r \ \text{ with } |f_1| \vee \cdots \vee |f_n| = f \right\}$$
$$\leq C\mu_{p,n}(\lambda_1, \dots, \lambda_n)\,\|f\|\,.$$

Since the supremum above is attained when $|f_1| = \cdots = |f_n| = f$ and when each $f_j\lambda_j$ is a positive sequence, this inequality can be rewritten as

$$\left(\sum_{j=1}^n \langle f,\, |\lambda_j|\rangle^q\right)^{1/q} \leq C\mu_{p,n}(\lambda_1, \dots, \lambda_n)\,\|f\|$$

for every $n \in \mathbb{N}$, every $f \in (\ell^r)^+$, and every $\lambda_1, \dots, \lambda_n \in \ell^s$.

By a standard approximation argument, we can reduce the above further by requiring that the preceding inequality hold for every $m, n \in \mathbb{N}$, every $f \in (\ell^r_m)^+$, and every $\lambda_1, \dots, \lambda_n \in \ell^s_m$.

In the latter case, we set $\lambda_j = (\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{m,j})$ for $j \in \mathbb{N}_n$ and set $f = (\alpha_1, \alpha_2, \dots, \alpha_m)$. Then the preceding inequality becomes

$$\left(\sum_{j=1}^n \left(\sum_{i=1}^m \alpha_i\,|\lambda_{i,j}|\right)^q\right)^{1/q} \leq C\mu_{p,n}(\lambda_1, \dots, \lambda_n)\,\|(\alpha_i)\|_{\ell^r}$$

for every $m, n \in \mathbb{N}$, every $(\alpha_i) \in (\ell^r_m)^+$ and every $\lambda_1, \dots, \lambda_n \in \ell^s_m$.

As usual, $(\lambda_{i,j} : i \in \mathbb{N}_m,\, j \in \mathbb{N}_n)$ forms an $m \times n$ matrix, say Λ, whose columns are the vectors $\lambda_1, \dots, \lambda_n$. The above argument shows that $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r if and only if there is a constant $C > 0$ such that, for every $m \times n$ matrix Λ, we have

$$\|\,|\Lambda|^t : \ell^r_m \to \ell^q_n\,\| \leq C\left\|\Lambda : \ell^{p'}_n \to \ell^s_m\right\|\,,$$

where M^t is the transpose of a matrix M and we are using equation (1.4). In other words, the condition in (a) is equivalent to the existence of a constant $C > 0$ such that,

$$\|\,|A| : \ell^r_m \to \ell^q_n\| \leq C\,\|A : \ell^r_m \to \ell^p_n\|$$

for every $m, n \in \mathbb{N}$ and every $n \times m$ matrix A.

This establishes the equivalence of (a) and (b).

(b) $\Rightarrow$ (c) Clearly, (b) implies that $|A| \in \mathcal{B}(\ell^r, \ell^q)$ whenever $A \in \mathcal{B}(\ell^r, \ell^p)$, and hence that $A \in \mathcal{B}_r(\ell^r, \ell^q)$ whenever $A \in \mathcal{B}(\ell^r, \ell^p)$.

(c) $\Rightarrow$ (b) Assume that (b) does not hold. Then there exists a sequence (A_n) of finite-dimensional matrices such that $\|\,|A_n|\, : \ell^r_* \to \ell^q_*\| \geq n$ whereas

$\|A_n : \ell^r_* \to \ell^p_*\| \leq 1$, where $*$ represents suitable indices. Now set

$$A := A_1 \oplus A_2 \oplus \cdots ,$$

so that A is the block-diagonal matrix where the blocks are the finite-dimensional matrices A_n. Then $A \in \mathcal{B}(\ell^r, \ell^p)$, but $|A| \notin \mathcal{B}(\ell^r, \ell^q)$. Hence (c) fails, a contradiction. □

The discussion above leads to the following result, possibly new, about matrices.

Corollary 5.2. *Take $r > 1$ and $1 \leq p \leq q < \infty$. Then there exists a constant $C > 0$ such that*

$$\| \, |A| : \ell^r_m \to \ell^q_n \| \leq C \, \|A : \ell^r_m \to \ell^p_n\| \tag{5.1}$$

for every $m, n \in \mathbb{N}$ and every $n \times m$ matrix A if and only if $1/p - 1/q \geq 1/2$.

Proof. This follows from the equivalence of (a) and (b) in the above proposition and Corollary 4.8 . □

In terms of operators, we similarly have:

Corollary 5.3. *Take $r > 1$ and $1 \leq p \leq q < \infty$. Then $T \in \mathcal{B}_r(\ell^r, \ell^q)$ for every operator $T \in \mathcal{B}(\ell^r, \ell^p)$ if and only if $1/p - 1/q \geq 1/2$.* □

One implication of Corollary 5.2 was already known (in a stronger form) by a result of G. Bennett. Indeed, by [4, Proposition 3.2], there exist a constant K and, for each $m, n \in \mathbb{N}$, an $n \times m$ matrix A whose entries are all ± 1 such that

$$\|A : \ell^r_m \to \ell^p_n\| \leq K \max\{n^{1/p} m^{(1/2-1/r)^+}, m^{1/r'} n^{(1/p-1/2)^+}\} .$$

It is easy to see that

$$\||A| : \ell^r_m \to \ell^q_n\| = n^{1/q} m^{1/r'} ,$$

and so

$$\frac{\|A : \ell^r_m \to \ell^q_n\|}{\||A| : \ell^r_m \to \ell^p_n\|} \leq K \max\{n^{1/p-1/q} / m^{1/r' - (1/2-1/r)^+}, n^{(1/p-1/2)^+ - 1/q}\} .$$

Now suppose that $1/p - 1/q < 1/2$. Then $(1/p - 1/2)^+ - 1/q < 0$ and $1/r' - (1/2 - 1/r)^+ > 0$, and so the right-hand side of the above inequality is $K \max\{n^{1/p-1/q} m^{-\alpha}, n^{-\beta}\}$ for some $\alpha, \beta > 0$ which depend on only p, q, and r, and this expression can be made arbitrarily small by making a suitable choice of first $n \in \mathbb{N}$ and then $m \in \mathbb{N}$. Thus, for a matrix A of restricted form, there is no constant $C > 0$ such that equation (5.1) holds.

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