Dimension free estimates for the bilinear Riesz transform

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1 Introduction.

It is well known that the method of transference is a useful procedure for obtaining norm estimates independent of the dimension for classical operators acting on $L^p(\mathbb{R}^n, dx)$ (see for instance [1, 12, 13]) and even in the weighted situation (see for instance [7, 8, 9]). The aim of this note is to combine the techniques and methods at our disposal from the linear case (see [1, 6, 7, 17, 12]) and the "bilinear transference" method, introduced in [4] (and extended in [2, 3]), to show the boundedness of certain bilinear multipliers defined in $\mathbb{R}^n$ with the norm independent of the dimension n.

One particular case of interest in this note is the bilinear version of the classical *Riesz transforms* on $\mathbb{R}^n$, defined for $1 \leq k \leq n$ by

$$(R_k f)(x) = c_n \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < 1/\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}} dy, \quad k = 1, 2, ..., n \tag{1}$$

where $c_n = \Gamma(\frac{n+1}{2})\pi^{-\frac{n+1}{2}}$, or equivalently, and more useful, by

$$(R_k f)\hat{}(\xi) = \frac{-i\xi_k}{(\sum_{j=1}^n \xi_j^2)^{1/2}} \hat{f}(\xi), \quad k = 1, 2, ..., n \tag{2}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$. These operators are known to satisfy, for $1 < p < \infty$, the estimate

$$\|(\sum_{k=1}^n |R_k(f)|^2)^{1/2}\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)} \tag{3}$$

*Partially supported by the Spanish grant Proyecto MTM 2005-08350-C03-03

with a constant C independent of n.

The Riesz transforms are the basic examples of Calderón-Zygmund operators with kernels which are odd and homogeneous of degree 0.

Throughout the paper $K(x) = \frac{\Omega(x)}{|x|^n}$, where Ω is an odd function, homogeneous of degree 0 and integrable over Σ_{n-1}, i.e. $\Omega(-x) = -\Omega(x)$ and $\Omega(\lambda x) = \Omega(x)$ for $x \in \mathbb{R}^n$ and $\lambda > 0$, with $\Omega(u) \in L^1(\Sigma_{n-1})$. We define

$$T_\Omega(f) = c_n(\Omega) \lim_{\varepsilon \to 0} \int_{\varepsilon < |y| < 1/\varepsilon} f(x-y) \frac{\Omega(y)}{|y|^n} dy$$

where $c_n(\Omega)$ is chosen such that $\|T_\Omega\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = 1$, i.e.

$$c_n(\Omega)^{-1} = \|\hat{K}\|_{L^\infty(\mathbb{R}^n)}.$$

We use the notations $v_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ for the volume of the unit ball and write $d\sigma$ the normalized area measure of the sphere Σ_{n-1}. We shall see from our considerations that actually the following result holds true: The condition

$$n v_n c_n(\Omega) \|\Omega\|_{\Sigma_{n-1}} \leq C \tag{4}$$

implies

$$\|T_\Omega(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$ with a constant C independent of n.

In the last decade the *bilinear Hilbert transform*, given by

$$H(f,g)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| > \varepsilon} \frac{f(x-y)g(x+y)}{y} dy$$

for f, g belonging to the Schwarzt class $\mathcal{S}(\mathbb{R})$, was shown by M. Lacey and C. Thiele to be bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ solving an old question by A. Calderón. In the their fundamental work they discover that the parameter p_3 in the range space could go even below 1.

Theorem 1.1 *(see [10, 11]) Let* $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$ *and* $2/3 < p_3 < \infty$. *Then there exists a constant* $C > 0$ *such that*

$$\|H(f,g)\|_{L^{p_3}(\mathbb{R})} \leq C \|f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{p_2}(\mathbb{R})}. \tag{5}$$

In a similar way we shall define the bilinear version of the operator T_Ω and shall try to get its boundedness from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ under the same conditions on p_i. To analyze the independence of the dimension for the norm of the corresponding bilinear operator one needs to select the right normalization constant $b_n(\Omega)$. Let us introduce the natural choice in the following definition.

Definition 1.2 *Given Ω as above we define*

$$B_\Omega(f,g)(x) = b_n(\Omega) \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}dy,$$

where $b_n(\Omega)$ is chosen in such a way that

$$\|B_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} = 1.$$

Let us also mention the formulation in terms of Fourier transforms which is left to the reader.

Remark 1.1 *Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then*

$$B_\Omega(f,g)(x) = b_n(\Omega) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{K}(\xi-\eta)e^{2\pi i\langle(\xi+\eta),x\rangle}d\xi d\eta. \qquad (6)$$

Let us estimate $b_n(\Omega)$ and calculate $c_n(\Omega)$ for particular cases.

Proposition 1.3 *Let Ω be defined as above. Then*

$$b_n(\Omega) \le c_n(\Omega).$$

Proof. Denote

$$\tilde{B}_\Omega(f,g)(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}dy$$

and

$$\tilde{T}_\Omega(f)(x) = \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)\frac{\Omega(y)}{|y|^n}dy.$$

We shall show that

$$\|\tilde{B}_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} \ge \|\tilde{T}_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}.$$

If $f, g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\|\tilde{B}_\Omega(f,g)\|_{L^1(\mathbb{R}^n)} &\geq |\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x-y)g(x+y)dx)\frac{\Omega(y)}{|y|^n}dy| \\
&= |\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x)g(x+2y)dx)\frac{\Omega(y)}{|y|^n}dy| \\
&= |\int_{\mathbb{R}^n} (\int_{\mathbb{R}^n} f(x)g(x-y)dx)\frac{\Omega(y)}{|y|^n}dy| \\
&= |\int_{\mathbb{R}^n} f(x)\tilde{T}_\Omega(g)(x)dx|.
\end{aligned}$$

Now taking the supremum over $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} = 1$ we obtain $\|\tilde{B}_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} \geq \|\tilde{T}_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$ and the result follows. ■

Proposition 1.4 *Let $a \in \mathbb{R}^n \setminus \{0\}$ and $\Omega_a(x) = \frac{\langle a,x\rangle}{|x|}$. Then, for $e_1 = (1,0,...,0)$,*

$$b_n(\Omega_a) = |a|^{-1} b_n(\Omega_{e_1}).$$

Proof. Let A be an orthogonal transformation of $\mathbb{R}^n$ such as $Ae_1 = \frac{a}{|a|}$ and write $f_A(x) = f(Ax)$. Then, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}
\tilde{B}_{\Omega_a}(f,g)(Ax) &= \int_{\mathbb{R}^n} f(Ax-y)g(Ax+y)\frac{\langle a,y\rangle}{|y|^{n+1}}dy \\
&= |a|\int_{\mathbb{R}^n} f_A(x-u)g_A(x+u)\frac{u_1}{|u|^{n+1}}du \\
&= |a|\tilde{B}_{\Omega_{e_1}}(f_A, g_A)(x).
\end{aligned}$$

This allows to conclude the result. ■

Proposition 1.5 $c_n(\Omega_a) = |a|^{-1}\pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2}) = \frac{c_n}{|a|}$.

Proof. It is elementary to show that if Ω is odd then

$$\hat{K}(\xi) = \frac{i\pi n v_n}{2}\int_{\Sigma_{n-1}} \Omega(u) sign\langle u, \xi\rangle d\sigma(u).$$

Hence $|\hat{K}(\xi)| \leq \frac{\pi n v_n}{2}\|\Omega\|_{L^1(\Sigma_{n-1})}$. In particular for $\Omega = \Omega_a$ one gets $\hat{K}(a) = \frac{i\pi n v_n}{2}\int_{\Sigma_{n-1}} |\Omega_a(u)| d\sigma(u)$.

Hence

$$\pi n v_n c_n(\Omega_a)\|\Omega_a\|_{L^1(\Sigma_{n-1})} = 2. \tag{7}$$

On the one hand, using polar coordinates, one has

$$\int_{|x|\leq 1} |\langle a, x\rangle| dx = \frac{n}{n+1} v_n \|\Omega_a\|_{L^1(\Sigma_{n-1})},$$

and, on the other hand, using Fubini's theorem, one also has

$$\int_{|x|\leq 1} |\langle a, x\rangle| dx = |a| \int_{|x|\leq 1} |x_1| dx = |a| \frac{2v_{n-1}}{n+1}.$$

Hence $n v_n \|\Omega_a\|_{L^1(\Sigma_{n-1})} = 2|a| v_{n-1}$ which gives

$$c_n(\Omega_a) = \frac{1}{|a|\pi v_{n-1}} = |a|^{-1}\pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2}).$$

■

Definition 1.6 *For $a = e_k$, $\Omega(x) = \frac{x_k}{|x|}$, $k = 1, 2, ..., n$, the bilinear Riesz transform is given by*

$$\begin{aligned}(R_k(f,g))(x) &= b_n \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} f(x-y)g(x+y) \frac{y_k}{|y|^{n+1}} dy \quad (8)\\ &= -i\frac{b_n}{c_n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta) \frac{\xi_k - \eta_k}{|\xi-\eta|} e^{2\pi i\langle(\xi+\eta),x\rangle} d\xi d\eta, \quad (9)\end{aligned}$$

where $b_n^{-1} = \|\tilde{B}_{\Omega_{e_1}}\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)}$.

Hence $B_{\Omega_a} = |a|^{-1}\sum_{k=1}^n a_k R_k$, $a \in \mathbb{R}^n \setminus \{0\}$.

Our aim is to show that the transforms R_k (and more generally B_Ω for certain Ω) define bounded bilinear maps from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ for $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$ and certain values of p_3 with norm independent of the dimension. As in the linear case we shall make use of the method of rotations and a transference result.

We now define the directional bilinear Hilbert transform $\mathbb{R}^n$ as follows: Given $u \in \Sigma_{n-1}$ we denote

$$H^u(f,g)(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon<|t|<1/\varepsilon} f(x-tu)g(x+tu)\frac{dt}{t}.$$

We also use the notation

$$H(f,g)(x,y) = H^{\frac{y}{|y|}}(f,g)(x), x \in \mathbb{R}^n, y \in \mathbb{R}^n, y \neq 0.$$

Here is our version of the method of rotations in the bilinear case.

Theorem 1.7 *Let $\Omega \in L^1(\Sigma_{n-1})$ be odd and homogeneous of degree 0 and let $\psi_n \in L^1(\mathbb{R}^+, \frac{dr}{r})$. Define $d\mu_n(x) = \psi_n(|x|)dx$ and $\langle f, g\rangle_{\mu_n} = \int_{\mathbb{R}^n} f(x)g(x)\psi_n(|x|)dx$. Then*

$$B_\Omega(f,g)(x) = \frac{\pi}{2} n v_n b_n(\Omega) \int_{\Sigma_{n-1}} H(f,g)(x,u)\Omega(u)d\sigma(u), x \in \mathbb{R}^n \tag{10}$$

$$B_\Omega(f,g)(x) = \frac{\pi b_n(\Omega)}{2\|\psi_n\|_{L^1(\mathbb{R}^+, \frac{dr}{r})}} \langle H(f,g)(x,.), K\rangle_{\mu_n}, x \in \mathbb{R}^n \tag{11}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Use the spherical coordinates to obtain (10).

$$\begin{aligned} B_\Omega(f,g)(x) &= n v_n b_n(\Omega) \lim_{\varepsilon\to 0} \int_{\Sigma_{n-1}} \int_{\varepsilon<t<1/\varepsilon} f(x-tu)g(x+tu)\frac{\Omega(u)}{t} d\sigma(u)dt \\ &= \frac{n v_n}{2} b_n(\Omega) \lim_{\varepsilon\to 0} \int_{\Sigma_{n-1}} \int_{\varepsilon<|t|<1/\varepsilon} f(x-tu)g(x+tu)\frac{\Omega(u)}{t} d\sigma(u)dt \\ &= \frac{\pi}{2} n v_n b_n(\Omega) \int_{\Sigma_{n-1}} \Omega(u) H(f,g)(x,u)d\sigma(u). \end{aligned}$$

Now

$$\begin{aligned} \langle H(f,g)(x,\cdot), K\rangle_{\mu_n} &= \int_{\mathbb{R}^n} H(f,g)(x,y)\frac{\Omega(y)}{|y|^n}\psi_n(|y|)dy \\ &= n v_n \Big(\int_0^\infty \frac{\psi_n(r)}{r} dr\Big)\Big(\int_{\Sigma_{n-1}} H(f,g)(x,u)\Omega(u)d\sigma(u)\Big) \\ &= \frac{2\|\psi_n\|_{L^1(\mathbb{R}^+, \frac{dr}{r})}}{\pi b_n(\Omega)} B_\Omega(f,g)(x). \end{aligned}$$

■

Let us mention the transference result we shall need later on. Let G be a l.c.a group with Haar measure m, let $R : G \to \mathcal{L}(L^p(\mu), L^p(\mu))$ be a representation of G into the space of bounded linear operators on $L^p(\mu)$ for some measure space (Ω, Σ, μ), i.e. $t \to R_t$ verifies $R_t R_s = R_{t+s}$ for $t, s \in G$, $\lim_{t\to 0} R_t f = f$ for $f \in L^p(\mu)$ and $\sup_{t\in G} \|R_t\| < \infty$. For a given $K \in L^1(G)$ with compact support we denote

$$C_K(\phi,\psi)(s) = \int_G \phi(s-t)\psi(s+t)K(t)dm(t)$$

(defined for nice functions ϕ, ψ defined on G). We consider the transferred operator by the formula

$$T_K(f,g)(w) = \int_G R_{-t}f(w)R_tg(w)K(t)dm(t)$$

where f and g are functions defined on Ω.

Theorem 1.8 *(see [4]) Let $1 \leq p_1, p_2 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$ and let R be a representation of $\mathbb{R}$ on acting $L^{p_i}(\mu)$ for $i = 1, 2$. Assume that there exists a map $S : \mathbb{R} \to \mathcal{L}(L^{p_3}(\mu), L^{p_3}(\mu))$ given by $t \to S_t$ such that S_t are invertible with $\sup_{t\in\mathbb{R}} \|S_t\| = 1$ and*

$$S_s((R_{-t}f)(R_tg)) = (R_{s-t}f)(R_{s+t}g)$$

for $s, t \in \mathbb{R}$, $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$.

If $K \in L^1(G)$ has compact support and the bilinear operator C_K is bounded from $L^{p_1}(G) \times L^{p_2}(G)$ into $L^{p_3}(G)$ with "norm" $N_{p_1,p_2}(C_K)$ then T_K is also bounded from $L^{p_1}(\mu)\times L^{p_2}(\mu)$ to $L^{p_3}(\mu)$ and with norm bounded by $CN_{p_1,p_2}(C_K)$.

For each $u \in \Sigma_{n-1}$ we can use the representation $R^u : \mathbb{R} \to \mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ given by $R^u_t f(x) = f(x - tu)$. Hence Theorem 1.8 can be applied, using $S_t = R^u_t$ together with Fubini's theorem, to obtain the following result.

Corollary 1.9 *Let $1 < p_1, p_2 < \infty, p_3 > 2/3$ and $1/p_3 = 1/p_1 + 1/p_2$. Let $\psi \in L^{p_3}(\mathbb{R}^n)$ with $\|\psi\|_{p_3} = 1$ and*

$$H_\psi(f,g)(x,y) = H(f,g)(x,y)\psi(y) \quad y \in \mathbb{R}^n \setminus \{0\}.$$

Then $H_\psi : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \to L^{p_3}(\mathbb{R}^{2n})$ is bounded with norm independent of n.

An application of Minkowski's inequality in Theorem 1.7, combined with Theorem 1.8, allows us to conclude the following boundedness result.

Theorem 1.10 *Let Ω be an odd kernel, homogeneous of degree 0, and let $1 < p_1, p_2 < \infty, p_3 \geq 1$ and $1/p_3 = 1/p_1 + 1/p_2$. Then $B_\Omega : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ with*

$$\|B_\Omega\|_{L^{p_1}\times L^{p_2}\to L^{p_3}} \leq \frac{\pi}{2}\|H\|_{L^{p_1}\times L^{p_2}\to L^{p_3}} n v_n b_n(\Omega)\|\Omega\|_{L^1(\Sigma_{n-1})}.$$

Finally combining Theorem 1.10, Proposition 1.3 and (7) one obtains our main result.

Corollary 1.11 *Let $|a|=1$, $1<p_1,p_2<\infty, p_3\geq 1$ and $1/p_3=1/p_1+1/p_2$. Then $\sum_{k=1}^n a_k R_k$ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ with norm independent of the dimension.*

Remark 1.2 *Observe that Theorems 1.7 and 1.10 are valid for vector-valued kernels. We can consider $\bar\Omega(x)=(\Omega_1(x),...,\Omega_n(x))=\frac{x}{|x|}$ as a ℓ_2^n-valued kernel, where $\Omega_i=\Omega_{e_i}$.*

Defining

$$B_{\bar\Omega}(f,g)=(R_1(f,g),...,R_n(f,g))=b_n\int_{\mathbb{R}^n} f(x-y)g(x+y)\frac{y}{|y|}dy,$$

the previous method does not give the analogue of (3). Note that $\|\bar\Omega(x)\|_{\ell_2^n}=1$ for each $x\in\mathbb{R}^n$ gives

$$\|\bar\Omega\|_{L^1(\Sigma_{n-1},\ell_2^n)}=1$$

and now, using $b_n\leq c_n$, one can only estimate $\frac{4\pi^{\frac{n}{2}}b_n(\bar\Omega)}{\Gamma(\frac{n}{2})}\|\bar\Omega\|_{L^1(\Sigma_{n-1},\ell^2)}\leq C\sqrt{n}$.

Our aim is now to show that in spite of this observation, also the norm for the ℓ_2^n-valued formulation of the bilinear Riesz transform, at least for $p_3>1$, is independent of the dimension.

Let us select $\psi_n(r)=(2\pi)^{-\frac{n}{2}}r^{n+1}e^{-\frac{r^2}{2}}$ and $\Omega(x)=\Omega_a(x)$, $|a|=1$, in Theorem 1.7. Observe that

$$\|\psi_n\|_{L^1(\frac{dr}{r})}=(2\pi)^{-\frac{n}{2}}\int_0^\infty r^n e^{-\frac{r^2}{2}}dr=(2\pi)^{-\frac{n}{2}}2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})=\sqrt{\frac{\pi}{2}}c_n$$

which gives

$$\frac{2\|\psi_n\|_{L^1(\frac{dr}{r})}}{\pi b_n(\Omega)}=\sqrt{\frac{2}{\pi}}\frac{c_n}{b_n}.$$

In particular, denoting by $d\gamma_n(y)=(2\pi)^{-\frac{n}{2}}e^{-\frac{|y|^2}{2}}dy$ the Gaussian measure our formula (11) becomes

$$\langle H(f,g)(x,\cdot),\langle a,\cdot\rangle\rangle_{\gamma_n}=\sqrt{\frac{2}{\pi}}\frac{c_n}{b_n}B_{\Omega_a}(f,g)(x). \tag{12}$$

Observing that the coordinate functions y_k are an orthonormal system in $L^2(\gamma_n)$ and following G. Pisier ([12]) we define $\mathcal{A}_n$ to be the subspace generated by $\{y_1, ..., y_n\}$ in $L^2(\gamma_n)$ and by $Q : L^2(\gamma_n) \to \mathcal{A}_n$ the orthogonal projection, that is

$$Q(f)(y) = \sum_{k=1}^{n} (\int_{\mathbb{R}^n} f(y) y_k d\gamma_n(y)) y_k. \tag{13}$$

Hence applying (12) to this particular case one gets the following analogue to the result given in [12]

$$Q(H(f,g))(x,y) = \sqrt{\frac{2}{\pi}} \frac{c_n}{b_n} \sum_{k=1}^{n} y_k R_k(f,g)(x). \tag{14}$$

This allows us to repeat Pisier's argument ([12]) and get the following analogue of (3).

Theorem 1.12 *Let $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, $p_3 > 1$. There exists C independent of n such that*

$$\|(\sum_{k=1}^{n} |R_k(f,g)|^2)^{1/2}\|_{L^{p_3}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{15}$$

Proof. Following Pisier's proof one first uses the fact that

$$\| \sum_{k=1}^{n} \lambda_k y_k \|_{L^p(\gamma_n)} = (\sum_{k=1}^{n} |\lambda_k|^2)^{1/2} \gamma(p) \tag{16}$$

where $\gamma(p) = (\int_{\mathbb{R}} |t|^p e^{-\frac{t^2}{2}} \frac{dt}{\sqrt{2\pi}})^{1/p}$.

$$\begin{aligned} & \|(\sum_{k=1}^{n} |R_k(f,g)|^2)^{1/2}\|^{p_3}_{L^{p_3}(\mathbb{R}^n)} \\ = \;& \gamma(p_3)^{-p_3} \| \sum_{k=1}^{n} y_k R_k(f,g)\|^{p_3}_{L^{p_3}(\mathbb{R}^n \times \gamma_n)} \\ \leq \;& C \frac{b_n}{c_n} \|Q(H(f,g))\|^{p_3}_{L^{p_3}(\mathbb{R}^n \times \gamma_n)} \\ \leq \;& C\|Q\|^{p_3}_{L^{p_3}(\gamma_n) \to L^{p_3}(\gamma_n)} \|H(f,g)\|^{p_3}_{L^{p_3}(\mathbb{R}^n \times \gamma_n)} \\ \leq \;& C\|Q\|^{p_3}_{L^{p_3}(\gamma_n) \to L^{p_3}(\gamma_n)} \|f\|^{p_3}_{L^{p_1}(\mathbb{R}^n)} \|g\|^{p_3}_{L^{p_2}(\mathbb{R}^n)}. \end{aligned}$$

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