An introduction to bilinear multipliers

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This article contains a little survey on the theory of bilinear multipliers and several techniques that are used to transfer the results stated in the real line to the corresponding ones stated in the torus or the integers.

Key Words: bilinear multipliers, bilinear Hilbert transform, transference methods, discretization

1. INTRODUCTION.

These notes are devoted to present a little survey and overview of the theory of bilinear multipliers, that was originated with the first attempts and formulation of several questions in the work by R. Coiffman and C. Meyer in the eighties and was retaken and pushed in the nineties after the celebrated result by M. Lacey and C. Thiele, solving the old standing conjecture of Calderón on the boundedness of the bilinear Hilbert transform.

This paper mainly contains the lecture notes of a course on this topic that I gave in the school "Real Analysis and its applications" organized by CIMPA at La Falda, Córdoba, Argentina in May of 2008. Some results appearing here have not appeared anywhere else, but some others are a recollection of those proved by the author (and his coauthors) and already published in several journals or books (see [2, 3, 4, 5, 6, 7, 8]).

The bilinear versions of several classical operators appearing in Harmonic Analysis, such as Hilbert transforms, Hardy-Littlewood maximal functions

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or fractional integrals, have been studied in the last decade. These operators are the starting point in our motivation of the notion of bilinear multipliers.

Given $f, g : \mathbb{R} \to \mathbb{C}$ belonging to the Schwarzt class $\mathcal{S}(\mathbb{R})$ we define the *bilinear Hilbert transform* by

$$H(f,g)(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)g(x+y)}{y} dy,$$

the *bisublinear Hardy-Littlewood maximal function* by

$$M(f,g)(x) = \sup_{\varepsilon>0} \frac{1}{2\varepsilon} \int_{|y|<\varepsilon} |f(x-y)g(x+y)| dy,$$

and the *bilinear fractional integral* by

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}} \frac{f(x-y)g(x+y)}{|y|^{1-\alpha}} dy, \quad 0 < \alpha < 1.$$

These mappings are the bilinear counterparts of the classical operators (with the same name) and their boundedness in the linear case is very well understood nowadays in most of the classical function spaces. The corresponding boundedness results on L^p-spaces for the just mentioned bilinear operators took long time and it is collected in the following theorem.

THEOREM 1. *Let $1 < p_1, p_2 < \infty$, $0 < \alpha < 1/p_1 + 1/p_2$, $1/q = 1/p_1 + 1/p_2 - \alpha$, $1/p_3 = 1/p_1 + 1/p_2$ and $2/3 < p_3 < \infty$. Then there exist constants A, B and C such that*

$$\|H(f,g)\|_{p_3} \leq A \|f\|_{p_1} \|g\|_{p_2} \textit{(Lacey-Thiele, [20, 21, 22])}, \tag{1}$$

$$\|M(f,g)\|_{p_3} \leq B \|f\|_{p_1} \|g\|_{p_2}. \textit{(Lacey, [19])}, \tag{2}$$

$$\|I_\alpha(f,g)\|_q \leq C \|f\|_{p_1} \|g\|_{p_2}. \textit{ (Kenig-Stein [18], Grafakos-Kalton [17])}. \tag{3}$$

Throughout the paper $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class on $\mathbb{R}^n$, i.e. $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \in C^\infty(\mathbb{R}^n)$ and $x^\alpha \frac{\partial^{|\beta|} f(x)}{\partial x_1^{\beta_1} ... \partial x_n^{\beta_n}}$ is bounded for any $\beta = (\beta_1, ..., \beta_n)$ and $\alpha = (\alpha_1, ..., \alpha_n)$ where $x^\alpha = x_1^{\alpha_1} ... x_n^{\alpha_n}$ and $|\beta| = \beta_1 + ... + \beta_n$, the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x, \xi \rangle} dx$ and $\mathcal{P}(\mathbb{R}^n)$ stands for the set of functions in $\mathcal{S}(\mathbb{R}^n)$ such that *supp*$\hat{f}$ is compact.

To handle the previously mentioned operators one needs to understand first the following "bilinear convolution"-type operator: For a given $K \in L^1_{loc}(\mathbb{R})$ we define

$$C_K(f,g)(x) = \int_{\mathbb{R}} f(x-y)g(x+y)K(y)dy \tag{4}$$

for f and g compactly supported continuous functions in $\mathbb{R}$. Note that a bisublinear maximal operator is now defined, for a sequence $K_j \in L^1_{loc}(\mathbb{R})$, as

$$M_{(K_j)}(f,g) = \sup_j C_{K_j}(|f|,|g|)$$

and a bilinear singular integral is defined as

$$H_K(f,g) = \lim_{\varepsilon\to 0,\delta\to\infty} C_{K_{\varepsilon,\delta}}(f,g)$$

where $K_{\varepsilon,\delta} = K\chi_{\varepsilon<|x|<\delta}$ is a Calderón-Zygmund kernel.

Let us first observe that actually the bilinear convolutions C_K are special examples of a wider class of operators.

Assume that $K \in L^1(\mathbb{R})$ and that f and $g \in \mathcal{S}(\mathbb{R})$. By writing $f(x-y) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i(x-y)\xi}d\xi$ and $g(x+y) = \int_{\mathbb{R}} \hat{g}(\eta)e^{2\pi i(x+y)\eta}d\eta$ we have

$$\begin{aligned} C_K(f,g)(x) &= \int_{\mathbb{R}} f(x-y)g(x+y)K(y)dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)K(y)e^{2\pi i(x-y)\xi}e^{2\pi i(x+y)\eta}d\xi d\eta dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} K(y)e^{-2\pi i(\xi-\eta)y}dy)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{g}(\eta)\hat{f}(\xi)\hat{K}(\xi-\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta. \end{aligned}$$

This motivates the following extension. Let $m(\xi,\eta)$ be measurable function and assume

$$B_m(f,g)(x) = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \tag{5}$$

for "nice" functions$f$ and g (for instance $f,g \in \mathcal{S}(\mathbb{R})$ when assuming $|m(\xi,\eta)| \le C(1+|\xi|^2)^N(1+|\eta|^2)^N$ for some constants $C>1$ and $N \ge 0$ or $m \in L^1_{loc}(\mathbb{R}^2)$ for $f,g \in \mathcal{P}(\mathbb{R})$.)

Given $1 \le p_1,p_2 \le \infty$ and $0 < p_3 \le \infty$ we shall say that m is a *bilinear multiplier* on $\mathbb{R}$ of type (p_1,p_2,p_3) if B_m extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3}(\mathbb{R})$.

The study of bilinear multipliers for smooth symbols (where $m(\xi, \eta)$ is a "nice" regular function) goes back to the work by R.R. Coifman and Y. Meyer in [10]. Note that actually the bilinear multipliers arising from operators C_K given by a kernel correspond to the particular case where $m(\xi, \eta) = M(\xi - \eta)$ for a measurable function M. It was only in the last decade that the cases $M_0(x) = \frac{1}{|x|^{1-\alpha}}$ was shown to define a bilinear multiplier of type (p_1, p_2, p_3) for $1/p_3 = 1/p_1 + 1/p_2 - \alpha$ for $1 < p_1, p_2 < \infty$ and $0 < \alpha < 1/p_1 + 1/p_2$ (see (3) in Theorem 1) and $M_1(x) = sign(x)$ was shown to define a bilinear multiplier of type (p_1, p_2, p_3) for $1/p_3 = 1/p_1 + 1/p_2$ for $1 < p_1, p_2 < \infty$ and $p_3 > 2/3$ (see (1) in Theorem 1).

Recall that in the linear case $\mathcal{M}_{p,q}(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, denotes the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $u * \phi \in L^q(\mathbb{R}^n)$ for all $\phi \in L^p(\mathbb{R}^n)$. Equivalently $\tilde{\mathcal{M}}_{p,q}(\mathbb{R}^n)$ stands for the space of bounded functions m such that

$$\widehat{T_m(\phi)}(\xi) = m(\xi)\hat{f}(\xi) \tag{6}$$

defines a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. We endow the space with the "norm" of the operator T_m, that is $\|m\|_{p,q} = \|T_m\|$.

We would like to mention some well known properties of the space of linear multipliers (see [1, 23]): $\mathcal{M}_{p,q}(\mathbb{R}^n) = \{0\}$ whenever $q < p$, $\mathcal{M}_{p,q}(\mathbb{R}^n) = \mathcal{M}_{q',p'}(\mathbb{R}^n)$ for $1 < p \leq q < \infty$ and for $1 \leq p \leq 2$,

$$\mathcal{M}_{1,1}(\mathbb{R}^n) \subset \mathcal{M}_{p,p}(\mathbb{R}^n) \subset \mathcal{M}_{2,2}(\mathbb{R}^n),$$

$$\tilde{\mathcal{M}}_{2,2}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),$$

$$\mathcal{M}_{1,q}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : u \in L^q(\mathbb{R}^n)\}, 1 < q < \infty,$$

$$\mathcal{M}_{1,1}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : u = \mu \in M(\mathbb{R}^n)\}.$$

In this notes we shall be dealing with their bilinear analogues.

Definition 2. Let $m(\xi, \eta)$ be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta) m(\xi, \eta) e^{2\pi i (\langle \xi + \eta, x \rangle} d\xi d\eta$$

for $f, g \in \mathcal{P}(\mathbb{R}^n)$, i.e. $supp(\hat{f}) \cup supp(\hat{g}) \subset B(0; R)$ for some $R > 0$.

Let $1 \leq p_1, p_2 \leq \infty$ and $0 < p_3 \leq \infty$. m is said to be a *bilinear multiplier* on $\mathbb{R}^n$ of type (p_1, p_2, p_3) if B_m there exists $C > 0$ such that

$$\|B_m(f,g)\|_{p_3} \leq C\|f\|_{p_1}\|g\|_{p_2}$$

for any $f, g \in \mathcal{P}(\mathbb{R}^n)$, i.e. B_m extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ (where we replace $L^\infty(\mathbb{R}^n)$ for $C_0(\mathbb{R}^n)$ in the case $p_i = \infty$ for $i = 1, 2$).

We write $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}^n)$ for the space of bilinear multipliers of type (p_1, p_2, p_3) and $\|m\|_{p_1,p_2,p_3} = \|B_m\|$.

Among them we distinguish the class of multipliers arising from operators C_K in the following way.

Definition 3. Let $1 \le p_1, p_2 \le \infty$ and $0 < p_3 \le \infty$. We denote by $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R}^n)$ the space of measurable functions $M : \mathbb{R}^n \to \mathbb{C}$ such that $m(\xi, \eta) = M(\xi - \eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, that is to say

$$B_M(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta) M(\xi - \eta) e^{2\pi i \langle \xi+\eta, x\rangle} d\xi d\eta$$

extends to a bounded bilinear map from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$. We keep the notation $\|M\|_{p_1,p_2,p_3} = \|B_M\|$.

Our objective is to study the basic properties of the classes $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\mathcal{M}_{p_1,p_2,p_3}(\mathbb{R})$, to find examples of bilinear multipliers in these classes, and get methods to produce new ones. We shall present some transference methods and discretization techniques which will allow to show the boundedness of the analogue formulations of the bilinear multipliers considered in Theorem 1 when defined on other groups such as $\mathbb{T}$ or $\mathbb{Z}$.

As usual, if $f \in L^1(\mathbb{R}^n)$ we denote the translation by $\tau_x f(y) = f(y - x)$ for $x \in \mathbb{R}^n$, the modulation by $M_x f(y) = e^{2\pi i \langle x,y\rangle} f(y)$ and, for each $0 < p, t < \infty$ the dilation $D_t^p f(x) = t^{-n/p} f(\frac{x}{t})$.

With this notation out of the way one has, for $1/p + 1/p' = 1, 1 \le p \le \infty$,

$$\widehat{(\tau_x f)}(\xi) = M_{-x}\hat{f}(\xi), \quad \widehat{(M_x f)}(\xi) = \tau_x \hat{f}(\xi), \quad \widehat{(D_t^p f)}(\xi) = D_{t^{-1}}^{p'} \hat{f}(\xi). \quad (7)$$

Clearly τ_x, M_x and D_t^p are isometries on $L^p(\mathbb{R}^n)$ for any $0 < p \le \infty$.

It is elementary to see that if $\phi, \psi \in L^1(\mathbb{R}^n)$ and $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{s}$ then

$$D_t^p \phi * \psi = D_t^s(\phi * D_{t^{-1}}^q \psi), \quad t > 0. \quad (8)$$

Although most of the results presented in what follows have a formulation in $n \ge 1$ we shall restrict ourselves to the case $n = 1$ for simplicity.

2. BILINEAR MULTIPLIERS: THE BASICS

Let us start with some elementary properties of the bilinear multipliers when composing with translations, modulations and dilations.

Proposition 4. *Let $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.*

(a) *If $m_1 \in \tilde{\mathcal{M}}_{s_1,p_1}(\mathbb{R})$ and $m_2 \in \tilde{\mathcal{M}}_{s_2,p_2}(\mathbb{R})$ then $m_1(\xi)m(\xi,\eta)m_2(\eta) \in \mathcal{BM}_{(s_1,s_2,p_3)}(\mathbb{R})$. Moreover*

$$\|m_1 m m_2\|_{s_1,s_2,p_3} \leq \|m_1\|_{s_1,p_1}\|m\|_{p_1,p_2,p_3}\|m_2\|_{s_2,p_2}$$

(b) *$\tau_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0) \in \mathbb{R}^2$ and*

$$\|\tau_{(\xi_0,\eta_0)}m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}.$$

(c) *$M_{(\xi_0,\eta_0)}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ for each $(\xi_0,\eta_0) \in \mathbb{R}^2$ and*

$$\|M_{(\xi_0,\eta_0)}m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}$$

(d) *If $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ and $0 < t < \infty$ then $D_t^q m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and*

$$\|D_t^q m\|_{p_1,p_2,p_3} = \|m\|_{p_1,p_2,p_3}.$$

Proof. Use (7) to deduce the following formulas

$$B_{m_1 m m_2}(f,g) = B_m(T_{m_1}f, T_{m_2}g). \tag{9}$$

$$B_{\tau_{(\xi_0,\eta_0)}m}(f,g) = M_{\xi_0+\eta_0}B_m(M_{-\xi_0}f, M_{-\eta_0}g). \tag{10}$$

$$B_{M_{(\xi_0,\eta_0)}m}(f,g) = B_m(\tau_{-\xi_0}f, \tau_{-\eta_0}g). \tag{11}$$

$$B_m(D_t^{p_1}f, D_t^{p_2}g) = D_t^{p_3}B_{D_t^q m}(f,g). \tag{12}$$

Let us check only the validity of last one. The other ones follow easily from the previous facts.

$$\begin{aligned} B_m(D_t^{p_1}f, D_t^{p_2}g)(x) &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1'}}\hat{f}(t\xi)t^{\frac{1}{p_2'}}\hat{g}(t\eta)m(\xi,\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} t^{\frac{1}{p_1'}}\hat{f}(\xi)t^{\frac{1}{p_2'}}\hat{g}(\eta)m(\frac{\xi}{t},\frac{\eta}{t})e^{2\pi i(\xi+\eta)\frac{x}{t}}t^{-2}d\xi d\eta \\ &= t^{-\frac{1}{p_3}}\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)t^{-\frac{1}{p_1}-\frac{1}{p_2}+\frac{1}{p_3}}m(\frac{\xi}{t},\frac{\eta}{t})e^{2\pi i(\xi+\eta)\frac{x}{t}}d\xi d\eta \\ &= D_t^{p_3}B_{D_t^q m}(f,g)(x). \end{aligned}$$

∎

Let us combine the previous results to get new bilinear multipliers from a given one.

PROPOSITION 5. *Let* $p_3 \geq 1$ *and* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.

(a) *If* $Q = [a,b]\times[c,d]$ *and* $1 < p_1, p_2 < \infty$ *then* $m\chi_Q \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|m\chi_Q\|_{p_1,p_2,p_3} \leq C\|m\|_{p_1,p_2,p_3}$.

(b) *If* $\Phi \in L^1(\mathbb{R}^2)$ *then* $\Phi * m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|\Phi * m\|_{p_1,p_2,p_3} \leq \|\Phi\|_1\|m\|_{p_1,p_2,p_3}$.

(c) *If* $\Phi \in L^1(\mathbb{R}^2)$ *then* $\hat{\Phi}m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|\hat{\Phi}m\|_{p_1,p_2,p_3} \leq \|\Phi\|_1\|m\|_{p_1,p_2,p_3}$.

(d) *If* $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3}-(\frac{1}{p_1}+\frac{1}{p_2})})$ *then* $m_\psi(\xi,\eta) = \int_0^\infty m(t\xi,t\eta)\psi(t)dt \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. *Moreover* $\|m_\psi\|_{p_1,p_2,p_3} \leq \|\psi\|_1\|m\|_{p_1,p_2,p_3}$.

Proof. (a) Use that $\chi_{[a,b]} \in \tilde{\mathcal{M}}_{p_1,p_1}$ for $1 < p_1 < \infty$ and $\chi_{[c,d]} \in \tilde{\mathcal{M}}_{p_2,p_2}$ for $1 < p_2 < \infty$ together with Proposition 4 part (a).

(b) Note that

$$\begin{aligned} B_{\Phi*m}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\Big(\int_{\mathbb{R}^2} m(\xi-u,\eta-v)\Phi(u,v)dudv\Big)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)m(\xi-u,\eta-v)e^{2\pi i(\xi+\eta)x}d\xi d\eta\Big)\Phi(u,v)dudv \\ &= \int_{\mathbb{R}^2} B_{\tau_{(u,v)}m}(f,g)(x)\Phi(u,v)dudv. \end{aligned}$$

From the vector-valued Minkowski inequality and Proposition 4 part (b), we have

$$\begin{aligned} \|B_{\Phi*m}(f,g)\|_{p_3} &\leq \int_{\mathbb{R}^2} \|B_{\tau_{(u,v)}m}(f,g)\|_{p_3}|\Phi(u,v)|dudv \\ &\leq \|m\|_{p_1,p_2,p_3}\|f\|_{p_1}\|g\|_{p_2}\|\Phi\|_1. \end{aligned}$$

(c) Observe that

$$\begin{aligned} B_{\hat{\Phi}m}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\Big(\int_{\mathbb{R}^2} M_{(-u,-v)}m(\xi,\eta)\Phi(u,v)dudv\Big)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_{(-u,-v)}m}(f,g)(x)\Phi(u,v)dudv. \end{aligned}$$

Argue as above, using now Proposition 4 part (c), to conclude the result.

(d) Use now Proposition 4 part (d), for $\frac{1}{p_3} - (\frac{1}{p_1} + \frac{1}{p_2}) = -\frac{2}{q}$,

$$\begin{aligned} B_{m_\psi}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_0^\infty D^q_{t^{-1}} m(\xi,\eta) t^{-2/q}\psi(t)dt) e^{2\pi i(\xi+\eta)x} d\xi d\eta \\ &= \int_0^\infty B_{D^q_{t^{-1}} m}(f,g)(x) t^{-2/q}\psi(t)dt. \end{aligned}$$

▮

With all these procedures we have several useful methods to produce multipliers in $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. Let us mention one application of each of them.

Example 1.

(1) *If* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $m_1 \in \tilde{\mathcal{M}}_{(p_1,p_1)}$ *and* $m_2 \in \tilde{\mathcal{M}}_{(p_2,p_2)}$ *then* $m(\xi,\eta) = m_1(\xi)m_2(\eta) \in \mathcal{BM}_{p_1,p_2,p_3}$.

(2) *If* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, $p_3 \geq 1$ *and* Q_1, Q_2 *are bounded measurable sets in* $\mathbb{R}$ *then*

$$\frac{1}{|Q_1||Q_2|} \int_{Q_1\times Q_2} m(\xi+u,\eta+v)dudv \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}).$$

(3) *If* $\Phi \in L^1(\mathbb{R}^2)$ *then* $\hat{\Phi} \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *for* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$, $p_3 \geq 1$.

(4) *If* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, $0 \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} < 1$ *then*

$$m_1(\xi,\eta) = \int_0^\infty m(t\xi,t\eta)\frac{dt}{1+t^2} \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R}).$$

A combination of the previous results gives the following examples of bilinear multipliers in $\mathcal{B}M_{(1,1,p_3)}(\mathbb{R})$ whose proof is left to the reader.

Corollary 6. *Let* $\Phi \in L^1(\mathbb{R}^2)$, $\psi_1 \in L^{p_1}(\mathbb{R})$ *and* $\psi_2 \in L^{p_2}(\mathbb{R})$ *and* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} \leq 1$ *then*

$$m(\xi,\eta) = \hat{\psi}_1(\xi)\hat{\Phi}(\xi,\eta)\hat{\psi}_2(\eta) \in \mathcal{B}M_{(1,1,p_3)}(\mathbb{R}).$$

Let us point out a characterization, for $p_3 \geq 1$, in terms of the duality.

Proposition 7. *Let* $1 \leq p_3 \leq \infty$. $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *if and only if there exists* $C > 0$ *such that*

$$|\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta| \leq C\|f\|_{p_1}\|g\|_{p_2}\|h\|_{p_3'}$$

for all $f, g, h \in \mathcal{P}(\mathbb{R})$.

Let us use the previous duality result and interpolation to get a sufficient integrability condition to guarantee that $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.

THEOREM 8.
Let $1 \le p_1, p_2 \le p \le 2$ *and* $p_3 \ge p'$ *such that* $\frac{1}{p_1} + \frac{1}{p_2} - \frac{2}{p} = \frac{1}{p_3}$. *If* $m \in L^p(\mathbb{R}^2)$ *then* $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.

Proof. Let us show first that $m \in \mathcal{BM}_{(p,p,\infty)}(\mathbb{R})$. Let $f \in L^p(\mathbb{R})$, $g \in L^p(\mathbb{R})$ and $h \in L^1(\mathbb{R})$. Using Hölder and Hausdorff-Young's one gets

$$\begin{aligned} |\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta| &\le \|m\|_{L^p(\mathbb{R}^2)}\|\hat{h}\|_\infty\|\hat{f}\|_{p'}\|\hat{g}\|_{p'} \\ &\le \|m\|_{L^p(\mathbb{R}^2)}\|h\|_1\|f\|_p\|g\|_p. \end{aligned}$$

Similarly, changing the variables $\xi + \eta = u$, $\xi = -v$, one has

$$\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)\hat{h}(\xi+\eta)m(\xi,\eta)d\xi d\eta = \int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v,u+v)dvdu.$$

An argument as above gives also the estimate

$$|\int_{\mathbb{R}^2} \hat{f}(-v)\hat{g}(u+v)\hat{h}(u)m(-v,u+v)dvdu| \le \|m\|_{L^p(\mathbb{R}^2)}\|g\|_1\|f\|_p\|h\|_p.$$

This shows that $m \in \mathcal{BM}_{(p,1,p')}(\mathbb{R})$ and similarly $m \in \mathcal{BM}_{(1,p,p')}(\mathbb{R})$.

Given $1 \le \tilde{p}_1 \le p$ and $p' \le \tilde{p}_3 \le \infty$ with $\frac{1}{\tilde{p}_1} - \frac{1}{\tilde{p}_3} = \frac{1}{p}$ we have $0 \le \theta \le 1$ such that $\frac{1}{\tilde{p}_1} = \frac{1-\theta}{p} + \frac{\theta}{1}$ and $\frac{1}{\tilde{p}_3} = \frac{1-\theta}{\infty} + \frac{\theta}{p'}$. Hence, by interpolation, $m \in \mathcal{BM}_{(\tilde{p}_1,p,\tilde{p}_3)}(\mathbb{R})$.

Similar argument shows that $m \in \mathcal{BM}_{(p,\tilde{p}_2,\tilde{q}_3)}(\mathbb{R})$ whenever $1 \le \tilde{p}_2 \le p$ and $p' \le \tilde{q}_3 \le \infty$ with $\frac{1}{\tilde{p}_2} - \frac{1}{\tilde{q}_3} = \frac{1}{p}$.

To finish the proof we observe that if $1 < p_1 < p$ and $1 < p_2 < p$ then for each $0 < \theta < 1$ there exist $1 \le \tilde{p}_1 \le p_1 < p$ and $1 \le \tilde{p}_2 \le p_2 < p$ such that

$$\frac{1}{p_1} - \frac{1}{p} = (1-\theta)(\frac{1}{\tilde{p}_1} - \frac{1}{p}), \quad \frac{1}{p_2} - \frac{1}{p} = \theta(\frac{1}{\tilde{p}_2} - \frac{1}{p}).$$

Denoting $\tilde{p}_3, \tilde{q}_3$ the values such that $\frac{1}{\tilde{p}_2} - \frac{1}{p} = \frac{1}{\tilde{p}_3}$ and $\frac{1}{\tilde{p}_2} - \frac{1}{p} = \frac{1}{\tilde{q}_3}$ one obtains that

$$\frac{1}{p_1} = \frac{(1-\theta)}{\tilde{p}_1} + \frac{\theta}{p}, \quad \frac{1}{p_2} = \frac{(1-\theta)}{p} + \frac{\theta}{\tilde{p}_1}, \quad \frac{1}{p_3} = \frac{(1-\theta)}{\tilde{p}_3} + \frac{\theta}{\tilde{q}_3}.$$

Hence the result follows again from interpolation between the just mentioned ones. ▮

3. THE CASE $\frac{1}{P_1}+\frac{1}{P_2}=\frac{1}{P_3}\leq 1$.

We start presenting an elementary example of bilinear multipliers. If μ is a Borel regular measure in $\mathbb{R}$ we denote $\hat{\mu}(\xi)=\int_{\mathbb{R}} e^{-2\pi i x\xi}d\mu(x)$ its Fourier transform.

Proposition 9. *Let $p_3\geq 1$ and $1/p_1+1/p_2=1/p_3$ and let $m(\xi,\eta)=\hat{\mu}(\alpha\xi+\beta\eta)$ where μ is a Borel regular measure in $\mathbb{R}$ and $(\alpha,\beta)\in\mathbb{R}^2$. Then $m\in\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|m\|_{p_1,p_2,p_3}\leq\|\mu\|_1$.*

Proof. Let us first rewrite the value $B_m(f,g)$ as follows:

$$\begin{aligned}
B_m(f,g)(x) &= \int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)\hat{\mu}(\alpha\xi+\beta\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta\\
&= \int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} e^{-2\pi i(\alpha\xi+\beta\eta)t}d\mu(t))e^{2\pi i(\xi+\eta)x}d\xi d\eta\\
&= \int_{\mathbb{R}}(\int_{\mathbb{R}^2}\hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-\alpha t)\xi}e^{2\pi i(x-\beta t)\eta}d\xi d\eta)d\mu(t)\\
&= \int_{\mathbb{R}} f(x-\alpha t)g(x-\beta t)d\mu(t).
\end{aligned}$$

Hence, using Minkowski's inequality, one has

$$\begin{aligned}
\|B_m(f,g)\|_{p_3} &\leq \int_{\mathbb{R}}\|f(\cdot-\alpha t)g(\cdot-\beta t)\|_{p_3}d|\mu|(t)\\
&\leq \int_{\mathbb{R}}\|f(\cdot-\alpha t)\|_{p_1}\|g(\cdot-\beta t)\|_{p_2}d|\mu|(t)\\
&= \|f\|_{p_1}\|g\|_{p_2}\int_{\mathbb{R}}d|\mu|(t)=\|\mu\|_1\|f\|_{p_1}\|g\|_{p_2}.
\end{aligned}$$

▮

This condition is also connected to the homogeneity of the symbol.

Proposition 10. *Let $m\in\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ such that $m(t\xi,t\eta)=m(\xi,\eta)$ for any $t>0$. Then $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$.*

Proof. From assumption $D_t^\infty m = m$. Using now Proposition 4 we have $B_m(D_t^{p_1} f, D_t^{p_2} g) = t^{1/p_3-(1/p_1+1/p_2)} D_t^{p_3} B_m(f,g)$ and therefore

$$\begin{aligned}\|B_m(f,g)\|_{p_3} &= \|D_t^{p_3} B_m(f,g)\|_{p_3} \\ &= t^{-1/p_3+(1/p_1+1/p_2)} \|B_m(D^{p_1} f, D_t^{p_2} g)\|_{p_3} \\ &\leq t^{-1/p_3+(1/p_1+1/p_2)} \|B_m\| \|f\|_{p_1} \|g\|_{p_2}.\end{aligned}$$

For this to hold for any $0 < t < \infty$ one needs $1/p_1 + 1/p_2 = 1/p_3$. ▮

Let us now get a new characterization of multipliers following DeLeeuw ideas (see [12]). It is well known that in the case that $\mu \in M(\mathbb{R})$ is supported on a finite set then $\hat{\mu}$ is almost periodic and bounded. For an almost periodic function g we denote $\|g\|_{B_p} = \lim_{T\to\infty} (\frac{1}{2T} \int_{-T}^{T} |g(t)|^p dt)^{1/p}$. To simplify the notation we use $\phi_t(x) = D_t^1 \phi(x) = \frac{1}{t}\phi(\frac{x}{t})$.

Lemma 2. *Let $1 \leq p < \infty$, $\phi \in \mathcal{S}(\mathbb{R})$ be a non-negative, radial, non-increasing function. If g is almost periodic and bounded in $\mathbb{R}$ then*

$$\|g\|_{B_p}^p \approx \limsup_{T\to\infty} \int_{\mathbb{R}} |g(x)|^p \phi_T(x) dx.$$

Proof. Denote $C_1 = \min\{\phi(u) : |u| \leq 1\}$ and $C_2 = \max\{\phi(u) : |u| \leq 1\}$. For $T > 1$

$$\frac{1}{2T} \int_{-T}^{T} |g(x)|^p dx \leq \frac{1}{2C_1} \int_{\mathbb{R}} |g(x)|^p \phi_T(x) dx$$

and

$$\begin{aligned}\int_{\mathbb{R}} |g(x)|^p \phi_T(x) dx &\leq \int_{|x|\leq T} |g(x)|^p \phi_T(x) dx + \frac{\|g\|_\infty^p}{T} \int_{|x|>T} \phi(x) dx \\ &\leq \frac{2C_2}{2T} \int_{-T}^{T} |g(x)|^p dx + \frac{\|\phi\|_1 \|g\|_\infty^p}{T}.\end{aligned}$$

Taking limits as $T \to \infty$ one gets the result. ▮

Recall that a function m is called regulated if

$$\lim_{\varepsilon\to 0} \frac{1}{4\varepsilon^2} \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} m(x-s, y-t) ds dt = m(x,y)$$

for all $(x,y) \in \mathbb{R}^2$.

Theorem 11. *(see [2]) Let $p_3 \geq 1$ and $1/p_1 + 1/p_2 = 1/p_3$ and let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R} \times \mathbb{R}$. The following are equivalent:*

(i) $m \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$.
(ii) There exists a constant K *so that*

$$|\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})| \leq K\|\hat{\mu}\|_{B_{p_1}}\|\hat{\nu}\|_{B_{p_2}}\|\hat{\lambda}\|_{B_{p'_3}}$$

for all measures μ,ν,λ *supported on a finite number of points.*

Proof. It suffices to show the result for continuous and bounded function m. The general case follows from this one and Proposition 5 and Proposition 7.

(i) $\Rightarrow$ (ii) Denote by ϕ the Gaussian function $\phi(x) = e^{-x^2/2}$. Then for any $\alpha > 0$ and $a \in \mathbb{R}$

$$(\tau_a\phi)^\alpha_\varepsilon(\xi) = (\frac{1}{\varepsilon})^\alpha\phi^\alpha(\frac{\xi-a}{\varepsilon}) = \delta_a * (\phi_\varepsilon)^\alpha(\xi) \tag{13}$$

Now choose $0 < \alpha,\beta,\gamma$ such that $\alpha+\beta+\gamma = 2$, and $\mu = \delta_a$, $\nu = \delta_b$ and $\lambda = \delta_c$ for $a,b,c \in \mathbb{R}$. It is easily checked that

$$\int_{\mathbb{R}^2}\frac{1}{\varepsilon^2}\phi^\alpha(\frac{\xi-a}{\varepsilon})\phi^\beta(\frac{\eta-b}{\varepsilon})\phi^\gamma(\frac{\xi+\eta-c}{\varepsilon})m(\xi,\eta)d\xi d\eta =$$

$$= \int_{\mathbb{R}^2}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta+\frac{a+b-c}{\varepsilon})m(a+\varepsilon\xi, b+\varepsilon\eta)d\xi d\eta =$$

$$= \int_{\mathbb{R}^2}\mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta.$$

Since

$$\lim_{\varepsilon\to 0}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta+\frac{a+b-c}{\varepsilon})m(a+\varepsilon\xi,b+\varepsilon\eta) =$$

$$\delta_c(a+b)\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)m(a,b),$$

the Lebesgue convergence theorem implies that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\frac{1}{\varepsilon^2}\phi^\alpha(\frac{\xi-a}{\varepsilon})\phi^\beta(\frac{\eta-b}{\varepsilon})\phi^\gamma(\frac{\xi+\eta-c}{\varepsilon})m(\xi,\eta)d\xi d\eta$$

$$= Cm(a,b)\delta_c(a+b) = Cm(a,b)\mu(\{a\})\nu(\{b\})\lambda(\{a+b\}).$$

where $C = \int_{\mathbb{R}^2}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)d\xi d\eta$.

Therefore we have that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^2}\mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta$$

$$=C\sum_{t\in\mathbb{R}}\sum_{s\in\mathbb{R}}m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{(t+s)\})$$

for all measures μ,ν,λ having their supports on finite sets of points.

On the other hand, from (i) and Proposition 7 we have

$$|\int_{\mathbb{R}^2}\mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta|$$

$$\leq K\|\widehat{\hat\mu(\phi_\varepsilon)^\alpha}\|_{p_1}\|\widehat{\hat\nu(\phi_\varepsilon)^\beta}\|_{p_2}\|\widehat{\hat\lambda(\phi_\varepsilon)^\gamma}\|_{p_3'}.$$

Let us now choose $\alpha=\frac{1}{p_1'}$, $\beta=\frac{1}{p_2'}$ and $\gamma=\frac{1}{p_3}$. Since $(\phi_\varepsilon)^\alpha=\frac{\varepsilon^{1-\alpha}}{\alpha^{1/2}}\phi_{\varepsilon\alpha^{-1/2}}$, we get $\widehat{(\phi_\varepsilon)^\alpha}(\xi)=C_\alpha\varepsilon^{1/p_1}e^{-\frac{\varepsilon^2\xi^2}{2\alpha}}$, $\widehat{(\phi_\varepsilon)^\beta}(\xi)=C_\beta\varepsilon^{1/p_2}e^{-\frac{\varepsilon^2\xi^2}{2\beta}}$ and $\widehat{(\phi_\varepsilon)^\gamma}(\xi)=C_\gamma\varepsilon^{1/p_3}e^{-\frac{\varepsilon^2\xi^2}{2\gamma}}$ for some constants C_α, C_β and C_γ .

Now taking into account that $\int_{\mathbb{R}}e^{-\frac{\varepsilon^2p_1\xi^2}{2\alpha}}d\xi=C'_\alpha\varepsilon^{-1}$ we have that

$$\|\widehat{\hat\mu(\phi_\varepsilon)^\alpha}\|_{p_1}=C\varepsilon\int_{\mathbb{R}}|\hat\mu(\xi)|^{p_1}e^{-\frac{p_1\varepsilon^2\xi^2}{2\alpha}}d\xi)^{1/p_1}.$$

Hence, from Lemma 10, $\limsup_{\varepsilon\to 0}\|\hat\mu\hat\phi_\varepsilon^\alpha\|_{p_1}\leq C\|\hat\mu\|_{B_{p_1}}$.

Applying similar procedure for ν and λ we finish this implication.

(ii) $\Rightarrow$(i) From (ii) we can get that the inequality holds for all finite measures μ,ν,λ with countable support. Let us take ϕ,ψ and ρ such that $\hat\phi,\hat\psi$ and $\hat\rho$ have compact support contained in $[-N/2,N/2]$ for N big enough. Now consider μ_N, ν_N and λ_N the measures with support in $(1/N)\mathbb{Z}$ whose Fourier transform coincide with the periodic extensions of $\hat\phi,\hat\psi$ and $\hat\rho$. In particular we have

$$\mu_N(\{\frac{n}{N}\})=\frac{1}{N}\phi(\frac{n}{N}),\nu_N(\{\frac{n}{N}\})=\frac{1}{N}\psi(\frac{n}{N})\text{ and }\lambda_N(\{\frac{n}{N}\})=\frac{1}{N}\rho(\frac{n}{N}).$$

Therefore we have

$$\begin{aligned}&\lim_{N\to\infty}N\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}}m(t,s)\mu_N(\{t\})\nu_N(\{s\})\lambda_N(\{t+s\})\\
=&\lim_{N\to\infty}\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}}m(\frac{n}{N},\frac{m}{N})\phi(\frac{n}{N})\psi(\frac{m}{N})\rho(\frac{n+m}{N})\frac{1}{N^2}\\
=&\int_{\mathbb{R}^2}m(\xi,\nu)\phi(\xi)\psi(\eta)\rho(\xi+\eta)d\xi d\eta.\end{aligned}$$

Now observe that $\|\hat{\mu}_N\|_{B_{p_1}} = (\frac{1}{2N}\int_{-N}^{N}|\hat{\phi}(\xi)|^{p_1}d\xi)^{1/p_1} = (\frac{1}{2N})^{1/p_1}\|\hat{\phi}\|_{p_1}$ and the same for the others.

Using that $\|\hat{\mu}_N\|_{B_{p_1}}.\|\hat{\nu}_N\|_{B_{p_2}}\|\hat{\lambda}_N\|_{B_{p'_3}} = \frac{1}{2N}$ and passing to the limit we get the result. ▌

REMARK 3. *Condition (ii) in Theorem 11 means that m defines a bounded operator from $L^{p_1}(\mathbb{D}) \times L^{p_2}(\mathbb{D})$ into $L^{p_3}(\mathbb{D})$ where $\mathbb{D}$ is the group $\mathbb{R}$ with the discrete topology.*

4. A SPECIAL CLASS OF BILINEAR MULTIPLIERS

Let us restrict ourselves to a smaller family of multipliers where $m(\xi,\eta) = M(\xi-\eta)$ for some M defined in $\mathbb{R}$. As in the introduction we use the notation $\tilde{\mathcal{M}}_{p_1,p_2,p_3}(\mathbb{R})$ for the space of functions $M : \mathbb{R} \to \mathbb{C}$ such that $m(\xi,\eta) = M(\xi-\eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, that is to say

$$B_M(f,g)(x) = \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta,$$

defined for $\hat{f}$ and $\hat{g}$ compactly supported, extends to a bounded bilinear map from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$. We keep the notation $\|M\|_{p_1,p_2,p_3} = \|B_M\|$.

The case $M(x) = \frac{1}{|x|^{1-\alpha}}$ (and even the n-dimensional case) corresponds to the bilinear fractional integral and it was first shown by C. Kenig and E. Stein in [18] to belong to $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any $1 < p_1, p_2 < \infty$, $0 < \alpha < 1/p_1 + 1/p_2$ and $1/p_1 + 1/p_2 = 1/p_3 - \alpha$. Another very important and non trivial example is the bilinear Hilbert transform, given by $M(x) = -isign(x)$, which was shown by M. Lacey and C.Thiele in [20, 21, 22] to belong to $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_3$ and $p_3 > 2/3$. These results were extended to other cases in [17] and [14, 15] respectively. We do not pretend to give a proof of these results here. We shall concentrate simply in analyzing some properties and results on the space $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ that will allow to show new conditions to get multipliers belonging to this class and to generate more examples from the known ones.

The reader should be aware that the starting assumption on the function M is only relevant for the definition of the bilinear mapping to make sense when acting on certain classes of "nice" functions. Then a density argument allows to extend functions belonging to Lebesgue spaces. We would like to point out the following observation.

REMARK 4. *If $M_n \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ are functions such that $M_n(x) \to M(x)$ a.e and $\sup_n \|M_n\| < \infty$ then $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|M\|_{p_1,p_2,p_3} \leq$*

$\sup_n \|M_n\|_{p_1,p_2,p_3}$. *This follows from Fatou's lemma, since*

$$\|B_M(f,g)\|_{p_3} \leq \liminf \|B_{M_n}(f,g)\|_{p_3} \leq \sup_n \|M_n\|_{p_1,p_2,p_3} \|f\|_{p_1} \|g\|_{p_2}.$$

We start reformulating the definition of these bilinear multipliers.

PROPOSITION 12. *Let $M \in L^1_{loc}(\mathbb{R})$, $f, g \in \mathcal{P}(\mathbb{R})$. Then*

$$B_M(f,g)(x) = \frac{1}{2} \int_{\mathbb{R}^2} \hat{f}(\frac{u+v}{2}) \hat{g}(\frac{u-v}{2}) M(v) e^{2\pi i u x} du dv \tag{14}$$

$$B_M(f,g)(-x) = \int_{\mathbb{R}} (\widehat{\tau_x g} * M)(\xi) \widehat{\tau_x f}(\xi) d\xi. \tag{15}$$

$$\widehat{B_M(f,g)}(x) = \frac{1}{2} C_M(\widehat{D^1_{1/2} f}, \widehat{D^1_{1/2} g})(x). \tag{16}$$

Proof. (14) follows changing variables.
To show (15) observe that

$$\begin{aligned} B_M(f,g)(-x) &= \int_{\mathbb{R}^2} \widehat{\tau_x f}(\xi) \widehat{\tau_x g}(\eta) M(\xi - \eta) d\xi d\eta \\ &= \int_{\mathbb{R}} (\int_{\mathbb{R}} \widehat{\tau_x g}(\eta) M(\xi - \eta) d\eta) \widehat{\tau_x f}(\xi) d\xi \\ &= \int_{\mathbb{R}} (\widehat{\tau_x g} * M)(\xi) \widehat{\tau_x f}(\xi) d\xi \end{aligned}$$

Finally, using (14), we have

$$\begin{aligned} B_M(f,g)(x) &= \frac{1}{2} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \hat{f}(\frac{u+v}{2}) \hat{g}(\frac{u-v}{2}) M(v) dv \Big) e^{2\pi i u x} dv \\ &= \frac{1}{2} \int_{\mathbb{R}} C_M(D^\infty_{1/2} \hat{f}, D^\infty_{1/2} \hat{g})(u) e^{2\pi i u x} du. \end{aligned}$$

This implies (16). ∎

For symbols M which are integrable we can write B_M in terms of a kernel C_K.

PROPOSITION 13. *Let $M \in L^1(\mathbb{R})$ and set $K(t) = \hat{M}(-t)$. Then $B_M = C_K$, i.e*

$$B_M(f,g) = \int_{\mathbb{R}} f(x-t) g(x+t) K(t) dt$$

Proof.

$$\begin{aligned}
C_K(f,g)(x) &= \int_{\mathbb{R}} f(x-t)g(x+t)\hat{M}(-t)dt \\
&= \int_{\mathbb{R}} (\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)e^{2\pi i(x-t)\xi}e^{2\pi i(x+t)\eta}d\xi d\eta)\hat{M}(-t)dt \\
&= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} \hat{M}(t)e^{2\pi i(\xi-\eta)t}dt)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\
&= B_M(f,g)(x).
\end{aligned}$$

▮

This class does have much richer properties than $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$. As above use the notation $f_t(x) = D_t^1 f(x) = \frac{1}{t}f(\frac{x}{t})$ for a function f defined in $\mathbb{R}$. The following facts are immediate.

$$\tau_y B_M(f,g) = B_M(\tau_y f, \tau_y g), y \in \mathbb{R}. \tag{17}$$

$$M_{2y} B_M(f,g) = B_M(M_y f, M_y g), y \in \mathbb{R}. \tag{18}$$

$$(B_M(f,g))_t = B_{D^1_{t^{-1}}M}(f_t, g_t), t > 0. \tag{19}$$

When specializing the properties obtained for $m(\xi,\eta)$ to the case $M(\xi-\eta)$ we get the following facts:

$$B_M(\tau_{-y}f, \tau_y g) = B_{M_y M}(f,g), y \in \mathbb{R}. \tag{20}$$

$$B_M(M_y f, M_{-y} g) = B_{\tau_{2y} M}(f,g), y \in \mathbb{R}. \tag{21}$$

For $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ we have

$$B_M(D_t^{p_1} f, D_t^{p_2} g) = D_t^{p_3} B_{D_t^q M}(f,g), t > 0. \tag{22}$$

As in the previous section we can generate new multipliers in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$.

PROPOSITION 14. *Let $p_3 \geq 1$, $\phi \in L^1(\mathbb{R})$ and $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Then*

(a)$\phi * M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|\phi * M\|_{p_1,p_2,p_3} \leq \|\phi\|_1 \|M\|_{p_1,p_2,p_3}$.
(b)$\hat{\phi} M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|\hat{\phi} M\|_{p_1,p_2,p_3} \leq \|\phi\|_1 \|M\|_{p_1,p_2,p_3}$.

(c)*If* $\psi \in L^1(\mathbb{R}^+, t^{\frac{1}{p_3}-(\frac{1}{p_1}+\frac{1}{p_2})})$ *then* $M_\psi(\xi) = \int_0^\infty M(t\xi)\psi(t)dt \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. *Moreover* $\|M_\psi\|_{p_1,p_2,p_3} \leq \|\psi\|_1 \|M\|_{p_1,p_2,p_3}$.

Proof. (a) Apply Minkowski's inequality to the following fact:

$$\begin{aligned} B_{\phi * M}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} M(\xi-\eta-u)\phi(u)du)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}}(\int_{\mathbb{R}^2} \widehat{M_{-u}f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)x}d\xi d\eta)e^{2\pi iux}\phi(u)du \\ &= \int_{\mathbb{R}} M_u B_M(M_{-u}f,g)(x)\phi(u)du. \end{aligned}$$

(b) Observe that

$$\begin{aligned} B_{\hat{\phi}m}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_{\mathbb{R}} (M_{-u}m)(\xi-\eta)\phi(u)du)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}^2} B_{M_{-u}m}(f,g)(x)\phi(u)du. \end{aligned}$$

Use now Minkowski's again and (20).
(c) Write $\frac{1}{p_3} - (\frac{1}{p_1}+\frac{1}{p_2}) = -\frac{1}{q}$,

$$\begin{aligned} B_{M_\psi}(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)(\int_0^\infty D^q_{t^{-1}}M(\xi)t^{-1/q}\psi(t)dt)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_0^\infty B_{D^q_{t^{-1}}M}(f,g)(x)t^{-1/q}\psi(t)dt. \end{aligned}$$

The result follows from (22) and Minkowski's again. ∎

Proposition 15. *Let* $p_3 \geq 1$, $\phi \in L^1(\mathbb{R})$ *and* $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. *Then* $m(\xi,\eta) = M(\xi-\eta)\hat{\phi}(\xi+\eta) \in \mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$ *and* $\|m\|_{p_1,p_2,p_3} \leq \|\phi\|_1\|M\|_{p_1,p_2,p_3}$.

Proof. Apply Young's inequality to the following fact:

$$\begin{aligned} B_m(f,g)(x) &= \int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)(\int_{\mathbb{R}} \phi(y)e^{-2\pi i(\xi+\eta)y}dy)e^{2\pi i(\xi+\eta)x}d\xi d\eta \\ &= \int_{\mathbb{R}}(\int_{\mathbb{R}^2} \hat{f}(\xi)\hat{g}(\eta)M(\xi-\eta)e^{2\pi i(\xi+\eta)(x-y)}d\xi d\eta)\phi(y)dy \\ &= \phi * B_M(f,g)(x). \end{aligned}$$

∎

Let us show that the classes $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ are reduced to $\{0\}$ for some values of the parameters.

THEOREM 16. *Let $p_3 \geq 1$ such that $\frac{1}{p_1}+\frac{1}{p_2} < \frac{1}{p_3}$. Then $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R}) = \{0\}$.*

Proof. Let $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Using Proposition 14 we have that $\phi * M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any ϕ continuous with compact support. Hence we may assume that $M \in L^1(\mathbb{R})$. Using Proposition 13 one has that

$$B_M(f,g)(x) = \int_{(x+B_R)\cap(-x+B_R)} f(x-t)g(x+t)\hat{M}(-t)dt$$

for any f and g continuous functions supported in a ball $B_R = \{|x| \leq R\}$. Therefore one concludes that $supp(B_M(f,g)) \subset B_{2R}$ in such a case. On the other hand for any compactly supported function h, $0 < p < \infty$ and y big enough one can say that $\|h \pm \tau_y f\|_p = 2^{1/p}\|f\|_p$.

Consider $\{r_k\}$ the Rademacher system in $[0,1]$ and observe that, for each $N \in \mathbb{N}$ and $y \in \mathbb{R}$, the orthonormality of the system gives

$$\int_0^1 B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}f)dt = \sum_{k=0}^N B_M(\tau_{ky}f, \tau_{ky}g)$$

Therefore, since $\sum_{k=0}^N B_M(\tau_{ky}f, \tau_{ky}g) = \sum_{k=0}^N \tau_{ky}B_M(f,g)$, we conclude that for y big enough

$$\|\sum_{k=0}^N \tau_{ky}B_M(f,g)\|_{p_3}^{p_3} = (N+1)\|B_M(f,g)\|_{p_3}^{p_3}.$$

On the other hand, for $p_3 \geq 1$,

$$\begin{aligned}
&\|\int_0^1 B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}g)dt\|_{p_3} \\
\leq\ & \int_0^1 \|B_M(\sum_{k=0}^N r_k(t)\tau_{ky}f, \sum_{k=0}^N r_k(t)\tau_{ky}g)\|_{p_3}dt \\
\leq\ & \int \|B_M\|\|\sum_{k=0}^N r_k(t)\tau_{ky}f\|_{p_1}\|\sum_{k=0}^N r_k(t)\tau_{ky}g)\|_{p_2}dt \\
\leq\ & \|B_M\| \sup_{0<t<1} \|\sum_{k=0}^N r_k(t)\tau_{ky}f\|_{p_1} \sup_{0<t<1} \|\sum_{k=0}^N r_k(t)\tau_{ky}g\|_{p_2} \\
\leq\ & \|B_M\|(N+1)^{1/p_1}\|f\|_{p_1}(N+1)^{1/p_2}\|g\|_{p_2}.
\end{aligned}$$

This implies that $(N+1)^{1/p_3}\|B_M(f,g)\|^{p_3} \leq C(N+1)^{1/p_1+1/p_2}\|f\|_{p_1}\|g\|_{p_2}$. Hence $1/p_1+1/p_2 \geq 1/p_3$. ∎

The following elementary lemma is quite useful to get necessary conditions on multipliers.

LEMMA 5. *Let $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. If $\frac{1}{q} = \frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}$ then there exists $C>0$ such that*

$$|\int_{\mathbb{R}} e^{-\lambda^2\xi^2} M(\xi)d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}$$

for any $\lambda > 0$.

Proof. Let $\lambda>0$ and denote G_λ such that $\hat{G}_\lambda(\xi) = e^{-2\lambda^2\xi^2}$. Using (14) one concludes that

$$\begin{aligned} B_M(G_\lambda,G_\lambda)(x) &= \frac{1}{2}\int_{\mathbb{R}^2} e^{-\lambda^2v^2}e^{-\lambda^2u^2}M(v)e^{2\pi iux}dudv \\ &= \frac{1}{2}(\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv)(\frac{1}{\lambda}\int_{\mathbb{R}} e^{-u^2}e^{2\pi iu\frac{x}{\lambda}}du) \\ &= C\frac{1}{\lambda}e^{-\pi^2\frac{x^2}{\lambda^2}}(\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv). \end{aligned}$$

Since $\|G_\lambda\|_p = C_p\lambda^{\frac{1}{p}-1}$ and $M \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ one gets that

$$\|B_M(G_\lambda,G_\lambda)\|_{p_3} = C|\int_{\mathbb{R}} e^{-\lambda^2v^2}M(v)dv|\lambda^{\frac{1}{p_3}-1} \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{p_1}-1}\lambda^{\frac{1}{p_2}-1}.$$

Therefore $|\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi)d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}$. ∎

THEOREM 17. *If there exists a non-zero continuous and integrable function M belonging to $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ then*

$$\frac{1}{p_3} \leq \frac{1}{p_1}+\frac{1}{p_2} \leq \frac{1}{p_3}+1.$$

Proof. Assume first that $\frac{1}{p_1}+\frac{1}{p_2} < \frac{1}{p_3}$. Use Lemma 16 applied to $\tau_{-2y}M$ for any $y \in \mathbb{R}$ together with (20) to obtain

$$|\lambda\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi+2y)d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}}.$$

Therefore, using the continuity of M and $q<0$ one gets

$$\lim_{\lambda\to\infty}|\lambda\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi+2y)d\xi| = |M(2y)| = 0.$$

Hence $M=0$.

Assume now that $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}>1$. Using again Lemma 16, applied to M_yM, together with (21) we obtain

$$|\int_{\mathbb{R}} e^{-\lambda^2\xi^2}M(\xi)e^{2\pi iy\xi}d\xi| \leq C\|M\|_{p_1,p_2,p_3}\lambda^{\frac{1}{q}-1}.$$

Therefore, taking limits again as $\lambda\to 0$, since $1/q-1>0$ we get $|\hat{M}(y)|=0$. Hence $M=0$. ∎

Corollary 18. *(see [25, Prop 3.1]) Let $p_3\geq 1$ such that $\frac{1}{p_1}+\frac{1}{p_2}<\frac{1}{p_3}$ or $\frac{1}{p_1}+\frac{1}{p_2}>\frac{1}{p_3}+1$. Then $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})=\{0\}$.*

Proof. Let $M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. From Proposition 14 we have that $\phi*M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for any ϕ compactly supported and continuous. Now use Theorem 17 to conclude that $\phi*M=0$ for any compactly supported and continuous ϕ. This implies that $M=0$. ∎

Let us now use some interpolation methods to get more examples of multipliers in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. First note that, selecting $\alpha=1$ and $\beta=-1$ in Proposition 9 we obtain the following simple example.

Proposition 19. *If $\mu\in M(\mathbb{R})$ then $M=\hat{\mu}\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}\leq 1$ and $\|M\|\leq\|\mu\|_1$.*

Theorem 20. *Let $\frac{1}{p_3}\leq\frac{1}{p_1}+\frac{1}{p_2}\leq\min\{2,\frac{1}{p_3}+1\}$. If $M\in L^1(\mathbb{R})$ and $M=\hat{K}$ for some $K\in L^q(\mathbb{R})$ where $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=1-\frac{1}{q}$ then $M\in\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ with $\|M\|_{p_1,p_2,p_3}\leq C\|K\|_q$.*

Proof. Consider the trilinear form

$$T(K,f,g)=\int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt.$$

From Proposition 13 we have $B_M(f,g)=T(K,f,g)$ for $M=\hat{K}$. Now use Proposition 19 to conclude that T is bounded in $L^1(\mathbb{R})\times L^{q_1}(\mathbb{R})\times L^{q_2}(\mathbb{R})\to L^{s_1}(\mathbb{R})$ where $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{s_1}\leq 1$ and it has norm bounded by 1.

Assume first that $\frac{1}{p_1}+\frac{1}{p_2}\leq 1$. Hence T is bounded in $L^1(\mathbb{R})\times L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})\to L^p(\mathbb{R})$ for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$.

On the other hand, using Hölder's inequality

$$\sup_x |\int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt| \leq \|f\|_{p_1}\|g\|_{p_2}\|K\|_{p'}.$$

This shows that T is also bounded in $L^{p'}(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^\infty(\mathbb{R})$. Therefore, by interpolation, selecting $0 < \theta < 1$ such that $\frac{1}{p_3} = \frac{1-\theta}{p}$, one obtains that T is bounded in $L^q(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ for $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = 1 - \frac{1}{q}$.

Assume now that $1 < \frac{1}{p_1} + \frac{1}{p_2} \leq 2$.

Using that $\int_{\mathbb{R}} f(x-t)g(x+t)dt = f * g(2x)$, Young's inequality implies that

$$\|\int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt\|_{r_3} \leq \|K\|_\infty \|D^\infty_{1/2}(|f|*|g|)\|_{r_3} \leq C\|f\|_{r_1}\|g\|_{r_2}\|K\|_\infty$$

whenever $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ and $\frac{1}{r_1} + \frac{1}{r_2} - 1 = \frac{1}{r_3}$.

Hence T is bounded in $L^\infty(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^p(\mathbb{R})$ where $\frac{1}{p_1} + \frac{1}{p_2} - 1 = \frac{1}{p} \leq 1$.

Using duality, $\langle T(K,f,g),h\rangle = \langle T(h,\bar{f},g),K\rangle$, where $\bar{f}(x) = f(-x$, that is

$$\int_{\mathbb{R}^2} f(x-t)g(x+t)K(t)h(x)dtdx = \int_{\mathbb{R}}(\int_{\mathbb{R}} \bar{f}(t-x)g(x+t)h(x)dx)K(t)dt.$$

Therefore T is also bounded in $L^{p'}(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^1(\mathbb{R})$.

Select $0 \leq \theta \leq 1$ such that $\frac{1}{p_3} = \frac{1}{p} + \frac{\theta}{p'}$. Now using interpolation T will be bounded in $L^q(\mathbb{R}) \times L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ for $\frac{1}{q} = \frac{\theta}{p'} = \frac{1}{p_3} - \frac{1}{p} = \frac{1}{p_3} - \frac{1}{p_1} - \frac{1}{p_2} + 1$.

∎

5. COIFFMAN-WEISS BILINEAR TRANSFERENCE METHOD

There are several procedures to transfer results from $\mathbb{R}$ to $\mathbb{T}$ and $\mathbb{Z}$. A method, which applies to multipliers in $\mathcal{BM}_{(p_1,p_2,p_3)}(\mathbb{R})$, following the DeLeeuw approach ([12]), was obtained by the author in [2] but it will not be considered here. We would like to present a Coifman-Weiss type transference method in the bilinear setting that was developed recently (see [7, 5, 6]) and which applies to multipliers in $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$.

Our aim is to consider the analogues of the bilinear Hilbert transform in the periodic or the discrete case and to analyze their boundedness in the corresponding L^p-spaces.

For the periodic case one defines the *bilinear conjugate function* as

$$H_{\mathbb{T}}(F,G)(e^{it}) = \int_0^1 F(t-s)G(t+s)\cot(\pi s)ds$$

where F and G are polynomials on $\mathbb{T}$.

Using Fourier series expansion of the polynomials, it can also be written as

$$H_{\mathbb{T}}(F,G)(e^{it}) = -i\sum_k\Big(\sum_{n+m=k} sign(n-m)\hat{F}(n)\hat{G}(m)\Big)e^{2\pi ikt}$$

where $F(t) = \sum_{-N}^{N}\hat{F}(n)e^{2\pi int}$ and $G(t) = \sum_{-M}^{M}\hat{G}(m)e^{2\pi imt}$.

Question 1: *Is the bilinear conjugate transform bounded from $L^{p_1}(\mathbb{T})\times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ for some values of p_1, p_2, p_3?.*

As far as I know there is no way to adapt the proof in the real line, where the dilation plays a very important role, to the periodic situation. However this result was first observed by A. Bonami and J. Bruna ([9]) to transfer to the torus from the real line in the case of the bilinear Hilbert transform and later shown by D. Fan and F. Sato ([13]) using certain transference techniques for more general multipliers. We will be able to show the result by representing the group $\mathbb{R}$ into the space of bounded linear operators $\mathcal{L}(L^p(\mathbb{T}), L^p(\mathbb{T}))$, for any $1\le p\le\infty$ by the action $u\to R_u(f)(e^{it}) = f(e^{i(t-u)})$ and then using some the bilinear version of the Coiffman-Weiss transference methods.

We shall formulate the abstract method to be applied in a general setting. Let G be a l.c.a group with Haar measure m, let (Ω,Σ,μ) be a measure space and let R_u be a representation of G in the space of bounded linear operators on $L^p(\mu)$, i.e. $R: G\to\mathcal{L}(L^p(\mu), L^p(\mu))$ such that $u\to R_u$ verifies

- $R_uR_v = R_{uv}$ for $u, v\in G$,
- $\lim_{u\to 0} R_uf = f$ for $f\in L^p(\mu)$,
- $\sup_{u\in G}\|R_u\| < \infty$.

Given $K\in L^1(G)$ with compact support we also use the notation

$$C_K(\phi,\psi)(v) = \int_G \phi(v-u)\psi(v+u)K(u)dm(u)$$

for ϕ,ψ simple functions defined on G, and assume that, for $1\le p_1, p_2<\infty$ and $1/p_1 + 1/p_2 = 1/p_3$, the bilinear operator C_K is bounded from $L^{p_1}(G)\times L^{p_2}(G)$ to $L^{p_3}(G)$ with "norm" $N_{p_1,p_2}(C_K)$.

We now consider the transferred operator by the formula

$$T_K(f,g)(w) = \int_G R_{-u}f(w)R_ug(w)K(u)dm(u)$$

for $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$.

The reader should be aware that the assumptions in next result can be weakened (see [7, 5, 6]) but we restrict ourselves to this case for simplicity.

THEOREM 21. *([7]) Let $G = \mathbb{R}$, (Ω, Σ, μ) a measure space, $1 \le p_1, p_2 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. Let R be a representation of $\mathbb{R}$ on acting $L^{p_i}(\mu)$ for $i = 1, 2$ with $\sup_{u\in\mathbb{R}} \|R_u\|_{\mathcal{L}(L^{p_i},L^{p_i})} = 1$ for $i = 1, 2$*

Assume that there exists a map $u \to \mathcal{L}(L^{p_3}(\mu), L^{p_3}(\mu))$ given by $u \to S_u$ such that S_u are invertible with $\sup_{u\in G} \|S_u^{-1}\| = 1$ and

$$S_v((R_{-u}f)(R_u g)) = (R_{v-u}f)(R_{v+u}g)$$

for $u, v \in \mathbb{R}$, $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$.

Assume that $K \in L^1(\mathbb{R})$, $supp(K) \subset [-A.A]$ and the bilinear map C_K is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^{p_3}(\mathbb{R})$ with norm $N_{p_1,p_2}(C_K)$.

Then T_K is also bounded from $L^{p_1}(\mu) \times L^{p_2}(\mu)$ to $L^{p_3}(\mu)$ and with norm bounded by $N_{p_1,p_2}(C_K)$.

Proof. Write, for each $v \in \mathbb{R}$,

$$\begin{aligned} T_K(f,g) &= S_v^{-1}(S_v \int_{\mathbb{R}} R_{-u}fR_u g K(u)du) \\ &= S_v^{-1}(\int_{\mathbb{R}} S_v(R_{-u}fR_u g)K(u)du) \\ &= S_v^{-1}(\int_{\mathbb{R}} (R_{v-u}f)(R_{v+u}g)K(u)du) \end{aligned}$$

Hence

$$\|T_K(f,g)\|^{p_3}_{L^{p_3}(\mu)} \le \| \int_{\mathbb{R}} (R_{v-u}f)(R_{v+u}g)K(u)du\|^{p_3}_{L^{p_3}(\mu)}$$

Given $N \in \mathbb{N}$, integrating over $v \in [-N, N]$,

$$2N\|T_K(f,g)\|^{p_3}_{L^{p_3}(\mu)} \le \int_{-N}^{N} \| \int_{\mathbb{R}} (R_{v-u}f)(R_{v+u}g)K(u)du\|^{p_3}_{L^{p_3}(\mu)} dm(v).$$

Therefore, denoting $\chi_{[-A-N,A+N]} = \chi_{A,N}$ we have

$$
\begin{aligned}
2N \quad & \|T_K(f,g)\|_{L^{p_3}(\mu)}^{p_3} \leq \int_{-N}^{N}\int_\Omega |\int_{\mathbb{R}} R_{v-u}f(w)R_{v+u}g(w)K(u)du|^{p_3}d\mu(w)dv \\
= \quad & \int_\Omega (\int_{-N}^{N} |\int_{-A}^{A} R_{v-u}f(w)R_{v+u}g(w)K(u)du|^{p_3}dv)d\mu(w) \\
= \quad & \int_\Omega (\int_{\mathbb{R}} |\int_{\mathbb{R}} R_{v-u}f(w)\chi_{A,N}(v-u)R_{v+u}g(w)\chi_{A,N}(v+u)K(u)du|^{p_3}dv)d\mu(w) \\
= \quad & \int_\Omega (\int_{\mathbb{R}} |C_K(R_uf(w)\chi_{A,N}, R_ug(w)\chi_{A,N})(v)|^{p_3}dv)d\mu(w) \\
= \quad & \int_\Omega \|C_K(R_uf(w)\chi_{[-A-N,A+N]}, R_ug(w)\chi_{A,N})\|_{L^{p_3}(\mathbb{R})}^{p_3}d\mu(w) \\
\leq \quad & N_{p_1,p_2}(C_K)^{p_3}\int_\Omega \|R_uf(w)\chi_{A,N}\|_{L^{p_1}(\mathbb{R})}^{p_3}\|R_ug(w)\chi_{A,N}\|_{L^{p_2}(\mathbb{R})}^{p_3}d\mu(w) \\
\leq \quad & N_{p_1,p_2}(C_K)^{p_3}(\int_\Omega \|R_uf(w)\chi_{A,N}\|_{L^{p_1}(\mathbb{R})}^{p_1}d\mu(w))^{p_3/p_1} \\
\times \quad & (\int_\Omega \|R_ug(w)\chi_{A,N}\|_{L^{p_2}(\mathbb{R})}^{p_2}d\mu(w))^{p_3/p_2} \\
= \quad & N_{p_1,p_2}(C_K)^{p_3}(\int_{-(A+N)}^{A+N} \|R_uf\|_{L^{p_1}(\mu)}^{p_1}du)^{p_3/p_1} \\
\times \quad & (\int_{-(A+N)}^{A+N} \|R_ug\|_{L^{p_2}(\mu)}^{p_2}du)^{p_3/p_2} \\
= \quad & N_{p_1,p_2}(C_K)^{p_3}(\int_{-(A+N)}^{A+N} \|f\|_{L^{p_1}(\mu)}^{p_1}du)^{p_3/p_1} \\
\times \quad & (\int_{-(A+N)}^{A+N} \|g\|_{L^{p_2}(\mu)}^{p_2}du)^{p_3/p_2} \\
\leq \quad & N_{p_1,p_2}(C_K)^{p_3}(2(A+N))\|f\|_{L^{p_1}(\mu)}^{p_3}\|g\|_{L^{p_2}(\mu)}^{p_3}.
\end{aligned}
$$

Therefore

$$
\|T_K(f,g\|_{L^{p_3}(\mu)} \leq (\frac{A+N}{N})^{1/p_3}N_{p_1,p_2}(C_K)\|f\|_{L^{p_1}(\mu)}^{p_3}\|g\|_{L^{p_2}(\mu)}^{p_3}.
$$

▮

Note that, in particular, the assumptions in the previous theorem hold for multiplicative representations, i.e. $R_u(fg) = (R_uf)(R_ug)$, selecting $S_u = R_u$.

We shall apply our transference method answer Question 1, and to produce another proof of the boundedness of the bilinear Hilbert transform on

$\mathbb{T}$ (first shown in [13]). For such a purpose take $G = \mathbb{R}$ with the Lebesgue measure, (Ω, Σ, μ) the measure space $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m)$ the Lebesgue measure on $\mathbb{T}$ and

$$(R_u f)(e^{i\theta}) = f(e^{i(\theta-u)}).$$

Recall that a function $M \in L^\infty(\mathbb{R})$ is said to be "regulated" (or "normalized") if $M_n = \hat{\phi}_n * M$ is pointwise convergent to M where $\phi_n(x) = \frac{1}{2n}\chi_{[-n,n]} * \chi_{[-n,n]}$.

THEOREM 22. *([7]) Let $1 \leq p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_3 \leq 1$ and let $M(\xi)$ be a bounded regulated function belonging to $\tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$. Denote*

$$\tilde{C}_K(P,Q)(x) = \sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} \hat{P}(k)\hat{Q}(k)M(k-k')e^{2\pi i x(k+k')},$$

for P and Q trigonometric polynomials.

Then

$$\tilde{C}_K : L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$$

is bounded with $\|\tilde{C}_K\| \leq C\|M\|_{p_1,p_2,p_3}$.

Proof. As in Lemma 3.5 of [11], let us take $\psi \in L^2(\mathbb{R})$ with compact support such that $\hat{\psi}(0) = 1$ and let us define $K_n(x) = (M_n\hat{h}_n)\check{}(x)$ where $h_n(x) = n\psi(nx)$. That is to say

$$\hat{K}_n = (\phi_n * M)\hat{h}_n.$$

Then $K_n \in L^1(\mathbb{R})$, it has compact support and $\hat{K}_n(x) \to M(x)$ for all $x \in \mathbb{R}$.

From Proposition 14 one has that $\hat{K}_n \in \tilde{\mathcal{M}}_{(p_1,p_2,p_3)}(\mathbb{R})$ and $\|\hat{K}_n\|_{p_1,p_2,p_3} \leq \|\psi\|_1$. Denoting $T_n = C_{K_n}$, i.e.

$$T_n(f,g)(x) = \int_{\mathbb{R}} f(x-u)g(x+u)K_n(u)du$$

we obtain that $T_n : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ are bounded and $\sup_{n\in\mathbb{N}} ||T_n|| < \infty$.

Observe that

$$\begin{aligned} T_{K_n}(P,Q)(\theta) &= \int_{\mathbb{R}} P(\theta-u)Q(\theta+u)K_n(u)du \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} P(\theta-u)Q(\theta+u)K_n(u)du \\ &= \int_0^1 P(\theta-u)Q(\theta+u)(\sum_{m\in\mathbb{Z}} K_n(u))du. \end{aligned}$$

Hence, we can apply Theorem 21 with $R_uP(\theta) = P(\theta - u)$, to get that the transferred bilinear operator

$$\tilde{T}_n(P,Q)(\theta) = \int_{\mathbb{T}} P(\theta - u)Q(\theta + u)\tilde{K}_n(u)du,$$

where $\tilde{K}_n(u) = \sum_{m\in\mathbb{Z}} K_n(m+u)$, is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T}) \to L^{p_3}(\mathbb{T})$ and the norms are uniformly bounded for $n \in \mathbb{N}$.

To finish the proof observe that if $e_k(\theta) = e^{2\pi ik\theta}$ then

$$\tilde{T}_n(e_k, e_{k'}) = e_k e_{k'} \int_{\mathbb{R}} K_n(u)e^{2\pi iu(k'-k)}du = e_{k+k'}M_n(k-k')\hat{h}_n(k-k'),$$

and hence,

$$\lim_{n\to\infty} T_n(e_k, e_{k'}) = e_{k+k'}M(k-k') = \tilde{C}_K(e_k, e_{k'}).$$

Therefore, by linearity, density and Fatou's lemma, we obtain the result. ▮

The interested reader is referred to [7] for the details which are left to cover the case $p_3 < 1$ and to obtain the complete proof of the following corollary.

Corollary 23. *The bilinear Hilbert transform on the torus*

$$H_{\mathbb{T}}(f,g)(x) = \int_{\mathbb{T}} f(x-y)g(x+y)\cot(\pi y)dy,$$

is bounded from $L^{p_1}(\mathbb{T}) \times L^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ whenever $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$.

6. DISCRETIZATION THECNIQUES

Our next objective is to analyze the discrete bilinear Hilbert transform.

For each $N \in \mathbb{N}$, we define the *truncated discrete bilinear Hilbert transform* by

$$H_{\mathbb{Z},N}(a,b)(m) = \sum_{k\neq 0, |n|\leq N} \frac{a_{m-n}b_{m+n}}{n}.$$

Question 2 *Are the discrete bilinear Hilbert transforms bounded uniformly in N from $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \to \ell^{p_3}(\mathbb{Z})$ for some values of p_1, p_2, p_3?.*

There are different techniques to handle the discrete case(see [2, 5, 6, 7]). We shall use here a "discretization" method initiated in [7] and developed in [5]. The techniques in [7] did not cover the case $p_3 < 1$ and we present here an approach that takes care also of this case.

Throughout this section $0 < p < \infty$, $I = [-1/4, 1/4]$, $C(I)$ stands for the space of continuous functions supported in I and we denote $I_n = n + I$ for $n \in \mathbb{Z}$.

Definition 24. We define the vector space

$$\mathcal{A} = \{f(x) = \sum_{m\in\mathbb{Z}} \psi_m(x-m) : \psi_m \in C(I)\}.$$

For each $\phi \in C(I)$ we define

$$\mathcal{A}_\phi = \{f(x) = \sum_{m\in\mathbb{Z}} a_m\phi(x-m) : (a_m) \subset \mathbb{C}\} \subset \mathcal{A}.$$

We shall use the notation $\mathcal{A}_0$ for the case $\phi = \chi_I$.

Note that if $f = \sum_{m\in\mathbb{Z}} \psi_m(x-m) \in \mathcal{A}$ then

$$\|f\|_p = (\sum_{m\in\mathbb{Z}} \|\psi_m\|^p_{L^p(I)})^{1/p}. \tag{23}$$

In particular, if $\phi \in C(I)$ then

$$\|\sum_{m\in\mathbb{Z}} a_m\phi(x-m)\|_p = \|\phi\|_p(\sum_{m\in\mathbb{Z}} |a_m|^p)^{1/p}.$$

Definition 25. For a given $\phi \in C(I)$ we define P_ϕ the map

$$(a_m)_{m\in\mathbb{Z}} \to P_\phi(a)(x) = \sum_{m\in\mathbb{Z}} a_m\phi(x-m),$$

for finite sequences $a = (a_m)_{m\in\mathbb{Z}}$.

We shall denote $P(a) = P_\phi(a)$ in the case $\phi = \chi_I$.

Of course P_ϕ is an isometric embedding of $\ell^p(\mathbb{Z})$ into $L^p(\mathbb{R})$ if $\|\phi\|_p = 1$.

Definition 26. Given a bounded sequence $(A_n)_{n\in\mathbb{Z}}$, let us denote by $T_{(A_n)}$ the bilinear map

$$T_{(A_n)}(a,b)(m) = \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}A_n,$$

defined for finite sequences a and b.

In the case $A_n = K(n)$ for a given continuous function K defined in $\mathbb{R}$ we write $K_n = K(n)$ and $T_{(K_n)}$ is said to be the *discretization* of C_K given by

$$C_K(f,g)(x) = \int_{\mathbb{R}} f(x-t)g(x+t)K(t)dt.$$

Our objective is to deduce the boundedness of the discretization of $T_{(K_n)}$ on the spaces $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \to \ell^{p_3}(\mathbb{Z})$ from the boundedness of C_K on the spaces $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$.

Let us start first with the following elementary observation.

LEMMA 6. *Let $(A_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive numbers, $0 < p_2, p_2 < \infty$ and $1/q_3 \leq 1/p_1 + 1/p_2 = 1/p_3$. If*

$$C = (\sum_{n\in\mathbb{Z}} A_n^{\min\{p_3,1\}})^{1/\min\{p_3,1\}} < \infty \tag{24}$$

then $T_{(A_n)}$ is bounded from $\ell^{p_1}(\mathbb{Z})\times\ell^{p_2}(\mathbb{Z}) \to \ell^{q_3}(\mathbb{Z})$ and with norm bounded by C.

Proof. The case $q_3 \geq 1$ follows from the vector-valued Minkowski's inequality that

$$\begin{aligned}
\|(T_{(A_n)}(a,b))\|_{q_3} &\leq (\sum_{m\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} |a_{m+n}||b_{m-n}|A_n\Big)^{q_3})^{1/q_3} \\
&\leq \sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |a_{m+n}|^{q_3}|b_{m-n}|^{q_3})^{1/q_3} A_n \\
&\leq \sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |a_{m+n}|^{p_3}|b_{m-n}|^{p_3})^{1/p_3} A_n \\
&\leq \sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |a_{m+n}|^{p_1})^{1/p_1} (\sum_{m\in\mathbb{Z}} |b_{m-n}|^{p_2})^{1/p_2}) A_n \\
&= \|a\|_{p_1}\|b\|_{p_2} \sum_{n\in\mathbb{Z}} A_n.
\end{aligned}$$

Notice that for $q_3 \leq 1$ and selecting s such that $q_3 = sp_3$ we have $s \geq 1$ and $p_3 \leq 1$. Hence

$$\begin{aligned}
\|(T_{(A_n)}(a,b)\|_{q_3}^{q_3} &\leq \sum_{m\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} |a_{m+n}||b_{m-n}|A_n\Big)^{sp_3} \\
&\leq (\sum_{m\in\mathbb{Z}} \Big(\sum_{n\in\mathbb{Z}} |a_{m+n}||b_{m-n}|A_n\Big)^{p_3})^s \\
&\leq (\sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |a_{m+n}|^{p_3}|b_{m-n}|^{p_3} A_n^{p_3})^s \\
&\leq (\sum_{n\in\mathbb{Z}} (\sum_{m\in\mathbb{Z}} |a_{m+n}|^{p_1})^{p_3/p_1} (\sum_{m\in\mathbb{Z}} |b_{m-n}|^{p_2})^{p_3/p_2}) A_n^{p_3})^s \\
&= \|a\|_{p_1}^{q_3} \|b\|_{p_2}^{q_3} (\sum_{n\in\mathbb{Z}} A_n^{p_3})^s.
\end{aligned}$$

∎

Corollary 27. *Taking* $A_n = \frac{1}{|n|^{1+\alpha}}$ *one gets*

$$I_\alpha(a,b)(m) = \sum_{n\in\mathbb{N}} \frac{a_{m+n}b_{m-n}}{n^{1+\alpha}}$$

defines a bounded operator from $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \to \ell^{p_3}(\mathbb{Z})$ *for* $0 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$ *whenever* $\frac{1}{1+\alpha} < p_3$.

Our main contribution is the observation that $C_K(P(a), P(b)) \in \mathcal{A}$ for finite sequences a, b. This will allows us to estimate the norms in $\ell^p(\mathbb{Z})$ of the discretization operators.

Theorem 28. *Let* $K \in L^1_{loc}(\mathbb{R})$ *and let* a, b *be finite sequences. Then*

$$C_K(P(a), P(b))(x) = \sum_{m\in\mathbb{Z}} H_m(a,b)(x-m),$$

where

$$H_m(a,b)(u) = \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n} \int_{-1/4+|u|}^{1/4-|u|} K(n+y)dy \in C(I).$$

In particular $C_K(P(a), P(b)) \in \mathcal{A}$.

Proof.

$$\begin{aligned} C_K(P(a),P(b))(x) &= \sum_{i,j\in\mathbb{Z}} a_i b_j C_K(\chi_{I_i},\chi_{I_j})(x) \\ &= \sum_{i,j\in\mathbb{Z}} a_i b_j \int_{(x-i+I)\cap(-x+j+I)} K(y)dy \\ &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}\int_{(x-m+I)\cap(-x+m+I)} K(n+y)dy \end{aligned}$$

Note that if $(x-m+I)\cap(-x+m+I)=\emptyset$ then

$$\int_{(x-m+I)\cap(-x+m+I)} K(n+y)dy = 0.$$

Therefore, denoting

$$\Phi_n(u) = \int_{(u+I)\cap(-u+I)} K(n+y)dy = \int_{-1/4+|u|}^{1/4-|u|} K(n+y)dy,$$

one has that $supp\Phi_n \subset I$. Indeed, $(u+I)\cap(-u+I)\neq\emptyset$ implies $u+u_1 = -u+u_2$ for $u_1,u_2\in I$ and, hence $|u|\le 1/4$.

We have shown that

$$C_K(P(a),P(b))(x) = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}\Phi_n(x-m).$$

Define $H_m(a,b)(u) = \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}\Phi_n(u)$.

Hence $H_m(a,b)(u) = H_m(a,b)(-u)$, $suppH_m(a,b)\subset I$, $H_m(a,b)\in C(I)$ and the result is complete. ▌

Lemma 7. *Let $K\in C^1(\mathbb{R})$ and compactly supported and let a,b be finite sequences. Denote*

$$A_n = A_n(K) = \sup_{\xi\in I_n} |K'(\xi)|,$$

$\phi_0(u) = 2(1/4-|u|)\chi_I(u)$ *and* $\phi_1(u) = (1/4-|u|)^2\chi_I(u)$. *Then*

$$|P_{\phi_0}(T_{(K_n)}(a,b))| \le |C_K(P(a),P(b))| + |P_{\phi_1}(T_{(A_n)}(|a|,|b|)|. \qquad (25)$$

Proof. Write for $m \in \mathbb{Z}$ and $u \in I$

$$\begin{aligned} H_m(a,b)(u) &= \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}\int_{-1/4+|u|}^{1/4-|u|} K(n+y)dy \\ &= \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}K(n)\phi_0(u) \\ &+ \sum_{n\in\mathbb{Z}} a_{m+n}b_{m-n}\int_{-1/4+|u|}^{1/4-|u|}[K(n+y)-K(n)]dy \\ &= T_{(K_n)}(a,b)(m)\phi_0(u) + \tilde{H}_m(a,b)(u) \end{aligned}$$

Using the Mean Value Theorem we write

$$\begin{aligned} |\tilde{H}_m(a,b)(u)| &\leq \sum_{n\in\mathbb{N}} |a_{m+n}||b_{m-n}|\int_{-1/4+|u|}^{1/4-|u|} |K(n+y)-K(n)|dy \\ &\leq \sum_{n\in\mathbb{Z}} |a_{m+n}||b_{m-n}|A_n(1/4-|u|)^2 \\ &= T_{(A_n)}(|a|,|b|)(m)\phi_1(u). \end{aligned}$$

From Theorem 28

$$\begin{aligned} C_K(P(a),P(b))(x) &= \sum_{m\in\mathbb{Z}} H_m(a,b)(x-m) \\ &= \sum_{m\in\mathbb{Z}} T_{(K_n)}(a,b)(m)\phi_0(x-m) + \sum_{m\in\mathbb{Z}} \tilde{H}_m(a,b)(x-m) \\ &= P_{\phi_0}(T_{(K_n)}(a,b))(x) + \sum_{m\in\mathbb{Z}} \tilde{H}_m(a,b)(x-m) \end{aligned}$$

From this and the previous estimate one gets the result. ∎

We are now ready to present the main result of this section.

THEOREM 29. *Let $0 < p_1, p_2, q_3 < \infty$ and $1/q_3 \leq 1/p_1 + 1/p_2 = 1/p_3$. Assume that $M = \hat{K} \in \mathcal{M}_{(p_1,p_2,q_3)}$ for some $K \in C^1(\mathbb{R}\setminus\{0\})$ with compact support and*

$$C = (\sum_{n\neq 0} A_n(K)^{\min\{p_3,1\}})^{1/\min\{p_3,1\}} < \infty$$

where $A_n(K) = \sup_{\xi \in I_n} |K'(\xi)|$.

Then $T_{(K_n)}$ *is bounded from* $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \to \ell^{q_3}(\mathbb{Z})$ *with norm bounded by* $C'(\|C_K\| + C)$ *for some constant* $C' > 0$ *.*

Proof. Using (25) for finite sequences a, b and the triangular inequality in $L^{q_3}(\mathbb{R})$ (denoting $C(q_3) = 1$ if $q_3 \geq 1$ and $C(q_3) = 2^{1/q_3}$) implies

$$\begin{aligned} \|\phi_0\|_{q_3} \|T_{(K_n)}(a,b)\|_{q_3} &= \|P_{\phi_0}(T_{(K_n)}(a,b))\|_{q_3} \\ &\leq C(q_3)(\|C_K(P(a),P(b))\|_{q_3} + \|P_{\phi_1}(T_{(A_n)}(|a|,|b|)\|_{q_3}) \\ &\leq C(q_3)(\|C_K\| \|P(a)\|_{p_1} \|P(b)\|_{p_2} + \|P_{\phi_1}\|_{q_3} \|(T_{(A_n)}(|a|,|b|)\|_{q_3}). \end{aligned}$$

Now apply Lemma 26 to conclude the result. ▮

We can now give the following discrete version, which extends the trivial estimates given in Corollary 27.

Corollary 30. *Let* $1 < p_1, p_2 < \infty$, $0 < \alpha < 1$ *and* $\frac{1}{\alpha+2} < p_3 < 1$. *Then*

$$I_\alpha(a,b)(m) = \sum_{n \in \mathbb{N}} \frac{a_{m+n} b_{m-n}}{n^{1+\alpha}}$$

maps boundedly $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z}) \to \ell^{q_3}(\mathbb{Z})$ *for* $1/q_3 = 1/p_1 + 1/p_2 - \alpha$.

Proof. Consider $K(t) = \frac{1}{|t|^{1+\alpha}}$ for $t \neq 0$. Observe that $A_n(K) \leq C \frac{1}{|n|^{2+\alpha}}$. Hence the assumption $\frac{1}{\alpha+2} < p_3 < 1$ allows us to conclude the result invoking Theorem 29. ▮

Corollary 31. *The bilinear discrete Hilbert transforms* $H_{\mathbb{Z},N}$ *are bounded from* $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ *to* $\ell^{p_3}(\mathbb{Z})$ *whenever* $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_3$ *and* $p_3 > 2/3$. *with the norm bounded by a constant independent of* N.

Proof. Consider $K(t) = \frac{1}{t}$ for $t \neq 0$. Observe that $A_n(K) \leq C \frac{1}{n^2}$. Hence the assumption $2/3 < p_3$ allows us to conclude the result invoking again Theorem 29. ▮

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