Fourier Analysis for vector-measures.

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Abstract

We analyze the Fourier transform of vector measures ν as well as the convolution between scalar and vector-valued regular measures defined on the Borel sets of a compact abelian group. We make special emphasis on the Riemann-Lebesgue lemma and Young's convolution type results in this setting. Applications to Fourier transform and convolutions between functions in $L^p(\nu)$ and, particularly, for translation invariant type measures ν are given.

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1 Introduction

Let G be a compact abelian group,we write $\mathcal{B}(G)$ for the Borel σ-algebra of G and m_G for the Haar measure of the group. We denote by $L^0(G)$ the space of Borel measurable functions defined on G and $L^p(G)$ the space of functions in $L^0(G)$ such that $\int_G |f|^p dm_G < \infty$.

Given $1 < p \leq \infty$, a non-negative measure λ on $\mathcal{B}(G)$, a Banach space X and a vector measure $\nu : \mathcal{B}(G) \to X$ we denote by

$$\|\nu\|(A) = \sup_{\|x'\|=1} |\langle \nu, x' \rangle|(A)$$

the semivariation on a Borel set A and write $\|\nu\|$ for $\|\nu\|(G)$, by

$$|\nu|(A) = \sup\{\sum_{E\in\pi} \|\nu(E)\| : \pi \text{ finite partition of } A\}$$

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the variation of ν, by $\|\nu\|_{p,\lambda}$ the p-semivariation of ν with respect to λ (see [6, Page 246]) defined, for $1 < p < \infty$, by

$$\|\nu\|_{p,\lambda} = \sup\left\{\left\|\sum_{A\in\pi}\alpha_A\nu(A)\right\|_X : \pi \text{ partition }, \|\sum_{A\in\pi}\alpha_A\chi_A\|_{L^{p'}(\lambda)} \leq 1\right\}, \quad (1)$$

and

$$\|\nu\|_{\infty,\lambda} = \sup_{\lambda(A)>0}\frac{\|\nu(A)\|}{\lambda(A)}.$$

We shall use the notation $\mathcal{M}(G,X)$ for the space of regular vector measures, $\mathcal{M}_{ac}(G,X)$ for those which are absolutely continuous with respect m_G, i.e. $\nu << m_G$,$\mathcal{M}_p(G,X)$ for those with bounded p-semivariation with respect to m_G, i.e. $\|\nu\|_{p,m_G} < \infty$, $M(G,X)$ for those with bounded variation, and finally we write $M_{ac}(G,X) = \mathcal{M}_{ac}(G,X) \cap M(G,X)$.

As usual, for a given vector measure ν, we write $L^1_w(\nu)$ for the space of functions in $L^0(G)$ such that $\int_G |f|d|\langle\nu,x'\rangle| < \infty$ for any $x' \in X'$ and we write $L^1(\nu)$ for the subspace of $L^1_w(\nu)$ satisfying that for any $A \in \mathcal{B}(G)$ there exists $x_A \in X$ for which $\langle x', x_A\rangle = \int_A fd\langle\nu,x'\rangle$.

For each $f \in L^1(\nu)$ we denote

$$\nu_f(A) = x_A = \int_A fd\nu$$

given as above. Then ν_f is a vector measure and $\|\nu_f\| = \|f\|_{L^1(\nu)}$. We denote I_ν the integration operator, i.e. $I_\nu : L^1(\nu) \to X$ is defined by

$$I_\nu(f) = \nu_f(G) = \int_G fd\nu$$

and satisfies that $\|I_\nu\| \leq \|\nu\|$. In the case $f \in L^1_w(\nu)$ we can look of ν_f as X''-valued measure, using $\langle\nu_f(A), x'\rangle = \int_A fd\langle\nu,x'\rangle$ for each $A \in \mathcal{B}(G)$. As usual, for $1 < p < \infty$ we denote $L^p(\nu) = \{f \in L^0(G) : |f|^p \in L^1(\nu)\}$ and $\|f\|_{L^p(\nu)} = \||f|^p\|^{1/p}_{L^1(\nu)}$.

The Fourier transform of functions in $L^1(\nu)$ was introduced in [3] as the X-valued function defined on the dual group of Γ by

$$\hat{f}^\nu(\gamma) = \int_G f(t)\overline{\gamma(t)}d\nu(t), \gamma \in \Gamma \quad (2)$$

where $f \in L^1(\nu)$ and ν is any vector measure. The validity of the Riemann-Lebesgue lemma in this setting was considered and it was shown, under the assumption $\nu << m_G$, that the fact $\hat{f}^\nu \in c_0(\Gamma, X)$ for any $f \in L^1(\nu)$ reduces to consider the case $f = \chi_G$ (see [3, Teo 2.5]). The following problems were left open:

(a) Does it hold that $\hat{f}^\nu \in c_0(\Gamma, X)$ whenever $\nu << m_G$ and $f \in L^1(\nu)$, for any Banach space X?

(b) Are there natural subclasses of vector measures for which this version of the Riemann-Lebesgue lemma holds?

(c) Are there classes of operators that transform vector measures in vector measures satisfying this formulation of the Riemann-Lebesgue lemma?

They also introduce the Fourier transform $\mathcal{F}_\nu(f)$ of functions $f \in L^1_w(\nu)$ in the case $\nu << m_G$ as bounded operators in $\mathcal{L}(X', \ell^\infty(\Gamma))$ given by

$$\mathcal{F}_\nu(f)(x') = \widehat{fh_{x'}} \tag{3}$$

where $d\langle \nu, x' \rangle = h_{x'} dm_G$ with $h_{x'} \in L^1(G)$.

We shall understand these cases as particular ones of the Fourier transform of a vector measure ν defined by

$$\hat{\nu}(\gamma) = I_\nu(\bar{\gamma}) \tag{4}$$

when dealing with ν_f. The paper is organized into six sections. In Section 2 we give some preliminaries on vector measures to be used in the sequel. We shall study some versions of Riemann-Lebesgue lemma in our context and give answers to the above problems in section 3. In Section 4 we introduce the convolution of a vector measure ν and another complex-valued regular measure $\mu \in M(G)$ by means of the formula

$$\mu * \nu(A) = \int_G \mu(A - t) d\nu(t), A \in \mathcal{B}(G)$$

where the map $t \to \mu(A - t)$ is shown first to be measurable and bounded (and hence in $L^1(\nu)$). This notion is seen to coincide with the symmetric formulation

$$\nu * \mu(A) = \int_G \nu(A - t) d\mu(t), A \in \mathcal{B}(G)$$

for regular measures $\nu \in \mathcal{M}(G; X)$ and $\mu \in M(G)$.

This concept is when restricted to measures $d\mu_f = f dm_G$ becomes

$$f * \nu(A) = I_\nu(\tilde{f} * \chi_A), A \in \mathcal{B}(G) \tag{5}$$

where $\tilde{f}(t) = f(-t)$. Our point of view actually extends the two different convolution maps considered in [3, Def 4.1, Def 4.5]: If ν be a vector measure such that $\nu << m_G$, $f \in L^1(G)$ and $g \in L^1_w(\nu)$ the authors introduced $f *_\nu g : X' \to L^1(G)$ as

$$f *_\nu g(x') = f * (g h_{x'}), x' \in X' \tag{6}$$

where $h_{x'} = \frac{d\langle \nu, x' \rangle}{dm_G}$.

In the case that $g \in L^1(\nu)$ and $f(t - \cdot)g \in L^1(\nu)$ for m_G-almost all $t \in G$ they also defined

$$f *^\nu g(t) = \int_G f(t-s)g(s)d\nu(s). \tag{7}$$

Using the fact that $\nu_g \in \mathcal{M}(G, X'')$ we actually have

$$d\langle f * \nu_g, x' \rangle = f *_\nu g(x') dm_G, x' \in X'$$

and also, in the case $g \in L^1(\nu)$ and $f \in C(G)$, we obtain $\nu_g \in \mathcal{M}(G, X)$ and $f * \nu_g(t) = f *^\nu g(t), \quad t \in G$.

Different formulations of Young's convolution theorems will be provided which will extend several results in [3] when restricted to measures ν_f. In particular we show that for $1 \leq p, q \leq \infty$ and $1/p + 1/q \geq 1$ if $\nu \in \mathcal{M}_p(G, X)$ and $f \in L^q(G)$ then $\nu * f \in P_r(G, X)$ for $1/p + 1/q - 1 = 1/r$ (see definition in Section 2).

Section 5 is devoted to analyze the cases where ν is a translation invariant-type measure. In the paper [7] the notion of "norm integral translation invariant" vector measure was introduced, by the condition

$$\|I_\nu(\tau_a \phi)\| = \|I_\nu(\phi)\|, \phi \in \mathcal{S}(G), a \in G \tag{8}$$

where $\tau_a(\phi)(s) = \phi(s - a)$. For norm integral translation invariant measures ν such that $\nu << m_G$ they showed that $L^1_w(\nu) \subset L^1(G)$ and therefore the convolution and the Fourier transform of functions in $L^1_w(\nu)$ are well defined. One of their main theorems establishes that if $f \in L^1(G)$ and $g \in L^p(\nu)$ then $f * g \in L^p(\nu)$ for $1 \leq p < \infty$. Later in [3] this notion was generalized and used for more general homeomorphisms $H : G \to G$, and the particular case of reflection invariant ones (i.e. $H(s) = -s$) played an important role when considering convolution of functions in $L^1(\nu)$. In this paper we shall introduce a weaker but still useful notion to be denoted "semivariation H-invariant" by

$$\|\nu_f\| = \|(\nu_H)_f\|, \quad f \in \mathcal{S}(G) \tag{9}$$

where $\nu_H(A) = \nu(H(A))$ for $A \in \mathcal{B}(G)$. This definition will be shown to be different to the "norm integral H-invariant". However for semivariation translation invariant measures ν we will still have $L^1(\nu) \subset L^1(G)$. Hence similar results as those in [7] for such a weaker notion will remain valid. We finally include in Section 6 several applications of our general theory for vector measures to the study of convolution and Fourier transform of functions in $L^1(\nu)$.

2 Preliminaries on vector measures

Let us start by recalling that a vector measure ν defined on the Borel σ-algebra $\mathcal{B}(G)$ is called regular if for any $\varepsilon > 0$ and $A \in \mathcal{B}(G)$ there exists a compact set K and an open set O such that $K \subset A \subset O$ and $\|\nu\|(O \setminus K) < \varepsilon$. It is clear that if $\nu << \lambda$ for some finite regular Borel measure λ then $\nu \in \mathcal{M}(G, X)$. In particular $\nu \in \mathcal{M}(G, X)$ if and only if any Rybakov control measure $|\langle \nu, x_0' \rangle|$ is regular.

As usual we denote $\mathcal{S}(G, X)$ the space of X-valued simple functions and, as usual, we keep the notation $L^p(G, X)$ for the completion of $\mathcal{S}(G, X)$ under the norm

$$\|\mathbf{s}\|_{L^p(G,X)} = (\int_G \|\mathbf{s}\|^p dm_G)^{1/p}$$

in the case $1 \leq p < \infty$ and write $L_0^\infty(G, X)$ the closure of $\mathcal{S}(G, X)$ in $L^\infty(G, X)$. It is well known that $L^1(G, X) \subset M_{ac}(G, X)$. Actually, for each $\mathbf{f} \in L^1(G, X)$ we define

$$\nu_{\mathbf{f}}(A) = \int_A \mathbf{f} dm_G.$$

One has that $\nu_{\mathbf{f}} \in M_{ac}(G, X)$ since $\nu << m_G$ and $|\nu_{\mathbf{f}}|(A) = \int_A \|\mathbf{f}\| dm_G$ (see [4, Page 46]).

We also have that $\mathcal{M}(G, X)$ endowed with the norm given by the semivariation becomes a Banach space and that $\mathcal{M}_{ac}(G, X)$ is a closed subspace of $\mathcal{M}(G, X)$. It is well known (see [4, Page 159]) that $\mathcal{M}(G, X)$ is isometric to the space to weakly compact linear operators. To each $\nu \in \mathcal{M}(G,X)$ corresponds a weakly compact operator $T_\nu : C(G) \to X$ such that $\|T_\nu\| = \|\nu\|$ and we shall write $\int_G \phi d\nu = T_\nu(\phi)$ for each $\phi \in C(G)$. We also recall that $M(G,X)$ is isometric to the space of absolutely summing operators $\Pi_1(C(G),X)$(see [4, Page 162]). For $1 < p \leq \infty$, let us also mention that $\mathcal{M}_p(G,X)$ can be identified with $\mathcal{L}(L^{p'}(G),X)$ (see [6, Page 259]). In other words, if $\nu \in \mathcal{M}_p(G,X)$ then T_ν extends to a bounded operator in $\mathcal{L}(L^{p'}(G),X)$ with $\|\nu\|_{p,m_G} = \|T_\nu\|_{\mathcal{L}(L^{p'}(G),X)}$, and, conversely, for each $T : L^{p'}(G) \to X$ we associate the vector measure $\nu_{_T} : \mathcal{B}(G) \to X$ given by $\nu_{_T}(A) = T(\chi_A)$ satisfying that $\nu_{_T} \in \mathcal{M}_p(G,X)$ and $\|\nu_{_T}\|_{p,m_G} = \|T\|$. This allows to produce easy examples in $\mathcal{M}_p(G,X)$. For instance, for $X = L^p(G)$ the $L^p(G)$-valued measure

$$m_p(A) = \chi_A, \quad A \in \mathcal{B}(G)$$

belongs to $\mathcal{M}_{p'}(G, L^p(G))$. Another important example is produced using Pettis integrable functions, namely if $\mathbf{f} : G \to X$ is Pettis integrable and $\langle \mathbf{f}, x' \rangle \in L^p(G)$ for each $x' \in X'$ then the vector measure

$$m_{\mathbf{f}}(A) = (P) \int_A \mathbf{f} d\mu, \quad \in \mathcal{B}(G),$$

(where the integral denotes the Pettis integral of $\mathbf{f}$ over the set A) belongs to $\mathcal{M}_p(G,X)$ and $\|m_{\mathbf{f}}\|_{p,m_G} = \sup_{\|x'\|=1} \|\langle \mathbf{f}, x' \rangle\|_{L^p(G)}$.

For each $1 \leq p \leq \infty$ and $\mathbf{s} \in \mathcal{S}(G,X)$ and denote

$$\|\mathbf{s}\|_{P_p(G,X)} = \|\nu_{\mathbf{s}}\|_{p,m_G} = \sup_{\|x'\|=1} \|\langle \mathbf{s}, x' \rangle\|_{L^p(G)}. \tag{10}$$

We define $P_p(G,X)$ the closure of $\mathcal{S}(G,X)$ in $\mathcal{M}_p(G,X)$ for $1 \leq p \leq \infty$ where we understand $\mathcal{M}_1(G,X) = \mathcal{M}(G,X)$. Since $C(G)$ is dense in $L^p(G)$ for $1 \leq p < \infty$ and closed for $p = \infty$ we easily see that $C(G,X)$ is dense in $P_p(G,X)$ for $1 \leq p < \infty$ and $C(G,X)$ is closed in $P_\infty(G,X)$.

It is elementary to see that $L^p(G,X) \subseteq P_p(G,X), 1 \leq p < \infty$, $L^\infty_0(G,X) \subseteq P_\infty(G,X)$ and $P_{p_2}(G,X) \subseteq P_{p_1}(G,X), p_1 \leq p_2$. Using that for each $\mathbf{s} \in \mathcal{S}(G,X)$ the measure $\nu_{\mathbf{s}}$ defines a finite rank operator on $L^1(G)$ into X one sees that $P_\infty(G,X) \subseteq L^\infty(G,X)$ (see[4, Page 68]).

Let us finish this preliminary section by showing, for the sake of completeness, that $C(G)$ is dense in $L^1(\nu)$ for regular measures ν.

Lemma 2.1 *Let $\nu \in \mathcal{M}(G, X)$ and $1 \leq p < \infty$. Then $C(G)$ is dense in $L^p(\nu)$. Moreover*

$$I_\nu(f) = \lim_n T_\nu(f_n)$$

for any $(f_n) \in C(G)$ with $\lim f_n = f$ in $L^1(\nu)$.

Proof. Assume ν is regular and let us prove that $C(G)$ is dense in $L^p(\nu)$. Since simple functions are dense in $L^p(\nu)$ it suffices to see that for any $\varepsilon > 0$ and $A \in \mathcal{B}(G)$ there exists $\phi \in C(G)$ such that $\|\chi_A - \phi\|_{L^p(\nu)} < \varepsilon$. Using the regularity of ν we first select a compact set K and an open set O such that $K \subset A \subset O$ with $\|\nu\|(O \setminus K) < \varepsilon^p$. Then use Uryshon's lemma to find $\phi \in C(G)$ such that $0 \leq \phi \leq 1$ and $\phi(t) = 1$ for $t \in K$ and $\phi(t) = 0$ for $t \notin O$. Finally observe that

$$\begin{aligned} \|\chi_A - \phi\|_{L^p(\nu)} &= \sup_{\|x'\|=1} (\int_G |\chi_A - \phi|^p d|\langle \nu, x'\rangle|)^{1/p} \\ &= \sup_{\|x'\|=1} (\int_{O\setminus K} |\chi_A - \phi|^p d|\langle \nu, x'\rangle|)^{1/p} \leq \Big(\|\nu\|(O \setminus K)\Big)^{1/p} < \varepsilon. \end{aligned}$$

Let (f_n) be any sequence of continuous functions converging to f in $L^1(\nu)$. Since $I_\nu(f_n) = T_\nu(f_n)$ we have

$$\|I_\nu(f) - T_\nu(f_n)\| \leq \|f - f_n\|_{L^1(\nu)}$$

and the proof is finished. ■

Corollary 2.2 *Let $\nu \in \mathcal{M}(G, X)$ and $A \in \mathcal{B}(G)$. Then the map*

$$\mathbf{f}(t) = \nu(A - t)$$

is (strongly)-measurable and bounded.

Proof. It is obviously bounded by $\|\nu\|$. We shall show that $\mathbf{f}(t) = \lim_n \mathbf{f}_n(t)$ for some sequence $\mathbf{f}_n \in C(G, X)$. For each $t \in G$ we write ν_t for the regular measure such that

$$\int_G \phi(s + t) d\nu(s) = \int_G \phi(s) d\nu_t(s), \phi \in C(G).$$

From Lemma 2.1 select $\phi_n \in C(G)$ such that $\lim_n \|\chi_A - \phi_n\|_{L^1(\nu_t)} = 0$. Define $\mathbf{f}_n(t) = \int_G \phi_n(s + t) d\nu(s)$ and observe that

$$\|\mathbf{f}_n(t) - \mathbf{f}_n(t')\| \leq \sup_{s\in G} |\phi_n(s + t) - \phi_n(s + t')| \|\nu\|.$$

Hence $\mathbf{f}_n \in C(G, X)$ and we have

$$\mathbf{f}(t) = \int_G \chi_A(s) d\nu_t(s) = \lim_n \int_G \phi_n(s) d\nu_t(s) = \lim_n \mathbf{f}_n(t).$$

■

3 Fourier transform and the Riemann-Lebesgue lemma

Definition 3.1 *Let ν be a vector measure. We define the Fourier transform by*

$$\hat{\nu}(\gamma) = \int_G \bar{\gamma} d\nu = I_\nu(\bar{\gamma}), \quad \gamma \in \Gamma.$$

In the case that $\nu \in \mathcal{M}(G, X)$ and $T_\nu : C(G) \to X$ is the corresponding weakly compact operator representing the measure we have $\hat{\nu}(\gamma) = T_\nu(\bar{\gamma})$.

Of course $\hat{f}^\nu(\gamma) = \widehat{\nu_f}(\gamma)$ whenever $f \in L^1(\nu)$ and, in the case $f \in L^1_w(\nu)$ and $\nu << m_G$, we can consider $\nu_f(A) \in X''$ given by

$$\langle \nu_f(A), x' \rangle = \int_A f d\langle \nu, x' \rangle$$

as a X''-valued vector measure and then $\mathcal{F}_\nu(f)(x')(\gamma) = \langle \widehat{\nu_f}(\gamma), x' \rangle, \quad \gamma \in \Gamma, x \in X'$.

It is straightforward to see that $\hat{\nu} \in \ell^\infty(\Gamma, X)$ with $\sup_{\gamma \in \Gamma} \|\hat{\nu}(\gamma)\| \leq \|\nu\|$. Due to the Radon-Nikodym theorem in the case $X = \mathbb{C}$ (or even for finite dimensional spaces X) we can say that the Riemman-Lebesgue lemma establishes that $\hat{\nu} \in c_0(\Gamma, X)$ whenever $\nu << m_G$.

We would like to study the validity of the Riemann-Lebesgue lemma for measures in $\mathcal{M}_{ac}(G, X)$. In other words, if we denote

$$\mathcal{M}_0(G, X) = \{\nu \in \mathcal{M}(G, X) : \hat{\nu} \in c_0(\Gamma, X)\}$$

we ask ourselves whether or not $\mathcal{M}_{ac}(G, X) \subset \mathcal{M}_0(G, X)$.

As expected the answer is negative in general as the following easy example shows: Let $G = \mathbb{T}$, $X = \ell^2(\mathbb{Z})$ and $\nu(A) = (\hat{\chi}_A(n))_{n \in \mathbb{Z}}$. Clearly $T_\nu : C(\mathbb{T}) \to \ell^2(\mathbb{Z})$ corresponds $T(f) = (\hat{f}(n))_{n \in \mathbb{Z}}$. Hence $\hat{\nu}(n) = e_n$ where (e_n) is the canonical basis and $\|\hat{\nu}(n)\| = 1$ for each $n \in \mathbb{Z}$.

However, from the classical Riemann-Lebesgue lemma we have the following weak version in the vector-valued setting.

$$\nu \in \mathcal{M}_{ac}(G,X) \Longrightarrow \langle \hat{\nu}, x' \rangle \in c_o(\Gamma), \quad x' \in X'. \tag{11}$$

Answering question (a) we show now that $\mathcal{M}_0(G,X) \subset \mathcal{M}_{ac}(G,X)$ if and only if X is finite dimensional.

Proposition 3.2 *Let X be an infinite dimensional Banach space and $G = \mathbb{T}$. There exists a regular vector measure $\nu : \mathcal{B}(\mathbb{T}) \to X$ such that $\nu << m_{\mathbb{T}}$ and $\hat{\nu} \notin c_0(\mathbb{Z}, X)$.*

Proof. Let us first take a sequence $x_n \in X$ such that $1/2 \leq \|x_n\| \leq 1$ satisfying

$$\|\sum_{n=1}^{\infty} \alpha_n x_n\| \leq C(\sum_{n=1}^{\infty} |\alpha_n|^2)^{1/2} \tag{12}$$

(see [5, Lemma 1.3]). Define $\nu : \mathcal{B}(\mathbb{T}) \to X$ by

$$\nu(A) = \sum_{n=1}^{\infty} \hat{\chi}_A(n) x_n.$$

Since $\sum_{n=1}^{\infty} |\hat{\chi}_A(n)|^2 \leq \|\chi_A\|^2_{L^2(\mathbb{T})} = m_{\mathbb{T}}(A)$ we have that ν is well defined. Actually we have $\nu(A) = T(m_2(A))$ where $T : L^2(\mathbb{T}) \to X$ is given by $T(\phi) = \sum_{n=1}^{\infty} \hat{\phi}(n) x_n$ and $m_2 : \mathcal{B}(\mathbb{T}) \to L^2(\mathbb{T})$ is given by $m_2(A) = \chi_A$ for $A \in \mathcal{B}(\mathbb{T})$. Hence $\nu \in \mathcal{M}_{ac}(\mathbb{T}, X)$, but $\hat{\nu}(n) = x_n$ does not belong to $c_0(\mathbb{Z}, X)$ since $\|x_n\| \geq 1/2$. ■

Next question is to find some natural classes of measures in $\mathcal{M}_0(G,X)$.

Proposition 3.3 *If $\nu \in \mathcal{M}_{ac}(G,X)$ and ν has relatively compact range then $\nu \in \mathcal{M}_0(G,X)$.*

Proof. Using that $\nu << m_G$ we conclude that $T_\nu^* : X' \to L^1(G)$ is given by $T_\nu^*(x') = h_{x'}$ where $d\langle \nu, x' \rangle = h_{x'} dm_G$ for each $x' \in X'$. Using now that the unit ball of $L^\infty(G)$ is the closed absolutely convex hull of $\{\chi_A : A \in \mathcal{B}(G)\}$ and $\nu(A) = T^{**}(\chi_A)$ we obtain that $T_\nu^{**} : L^\infty(G) \to X''$ is compact (and hence so it is T_ν). This implies that $\{\hat{\nu}(\gamma) = T_\nu(\bar{\gamma}) : \gamma \in \Gamma\}$ is relatively compact and, according to (11) also weakly null. Therefore $\hat{\nu}(\gamma) \in c_0(\Gamma, X)$. ■

Corollary 3.4 *$P_1(G,X) \subset \mathcal{M}_0(G,X)$.*

Let us now study the question of finding classes of bounded operators $T : X \to Y$ that transform measures in $\mathcal{M}_{ac}(G,X)$ into measures in $\mathcal{M}_0(G,Y)$. Recall that an operator $T : X \to Y$ is said to be completely continuous, or Dunford-Pettis, if it maps weakly convergent sequences in X into norm convergent sequences in Y. Hence a simple consequence of (11) and the above definition gives the following result.

Proposition 3.5 *Let $T : X \to Y$ be a completely continuous operator and $\nu \in \mathcal{M}_{ac}(G,X)$. Then $T(\nu) \in \mathcal{M}_0(G,Y)$.*

Let us restrict ourselves to study the version of Riemann-Lebesgue lemma for measures of bounded variation. In general $M_{ac}(G,X)$ is not contained in $\mathcal{M}_0(G,X)$ as it can be seen in the following example: Let $G = \mathbb{T}$, $X = L^1(\mathbb{T})$ and $\nu(A) = \chi_A$. Clearly $T_\nu : C(\mathbb{T}) \to L^1(\mathbb{T})$ corresponds to the inclusion map then $\hat{\nu}(n) = \phi_n$ where $\phi_n(t) = e^{int}$ and $\|\hat{\nu}(n)\| = 1$ for each $n \in \mathbb{Z}$.

However there are conditions which allow to have such a version of the Riemann-Lebesgue lemma. For instance, if X has the Radon Nikodym property then $M_{ac}(G,X) \subset \mathcal{M}_0(G,X)$. Under the RNP we have that $\nu \in M_{ac}(G,X)$ gives $d\nu = \mathbf{f} dm_G$ for some $\mathbf{f} \in L^1(G,X)$ and $\hat{\nu}(n) = \hat{\mathbf{f}}(n) = \int_{\mathbb{T}} \mathbf{f}(e^{it})e^{-int}dt$ for $n \in \mathbb{Z}$, which belongs to $c_0(\mathbb{Z},X)$.

Definition 3.6 *We say that a Banach space satisfies the Riemann-Lebesgue property for measures on G (in short, $X \in (RLP)_G$) if any vector measure ν satisfying $\nu << m_G$ and $|\nu|(G) < \infty$ satisfies that $\hat{\nu} \in c_0(\Gamma, X)$, i.e. $M_{ac}(G,X) \subset \mathcal{M}_0(G,X)$.*

We would like to show that this notion in the case $G = \mathbb{T}$ implies the Riemann-Lebesgue property introduced and considered by S. Bu and R. Chill in [1] for the case $G = \mathbb{T}$. They worked in the spaces

$$L_1^{max}(\mathbb{T}, X'') = \{\mathbf{f} : \mathbb{T} \to X'' weak^* - \text{meas.} : \sup_{\|x'\|=1} |\langle \mathbf{f}, x'\rangle| = \mathbf{f}^{max} \in L^1(\mathbb{T})\}$$

and

$$L_{1,X}^{max}(\mathbb{T}, X'') = \{\mathbf{f} \in L_1^{max}(\mathbb{T}, X'') : \hat{\mathbf{f}}(n) \in X\}$$

where, for a given $weak^*$-measurable function $\mathbf{f} : \mathbb{T} \to X''$ such that $\langle \mathbf{f}, x'\rangle \in L^1(\mathbb{T})$ for any $x' \in X'$, the Fourier coefficient $\hat{\mathbf{f}}(n) \in X''$ is given by

$$\langle \hat{\mathbf{f}}(n), x'\rangle = \int_0^{2\pi} e^{-int}\langle \mathbf{f}(e^{it}), x'\rangle \frac{dt}{2\pi}.$$

The Riemann-Lebesgue property of a complex Banach space X was introduced in [1] by the condition $(\hat{\mathbf{f}}(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}, X)$ for any $\mathbf{f} \in L^{max}_{1,X}(\mathbb{T}, X'')$.

Proposition 3.7 *Let X be a Banach space. If $X \in (RLP)_{\mathbb{T}}$ then X has the Riemann-Lebesgue property.*

Proof. Let $\mathbf{f} \in L^{max}_{1,X}(\mathbb{T}, X'')$. We first observe that or any trigonometric polynomial ψ

$$\begin{aligned}
\|\sum_{n=-N}^{M} \hat{\psi}(n)\hat{\mathbf{f}}(n)\| &= \sup_{\|x'\|=1} |\sum_{n=-N}^{M} \hat{\psi}(n)\langle\hat{\mathbf{f}}(n), x'\rangle| \\
&= \sup_{\|x'\|=1} |\int_0^{2\pi} \langle\hat{\mathbf{f}}(e^{it}), x'\rangle\phi(e^{-it})\frac{dt}{2\pi}| \\
&\leq \int_0^{2\pi} \sup_{\|x'\|=1} |\langle\hat{\mathbf{f}}(e^{it}), x'\rangle||\phi(e^{-it})|\frac{dt}{2\pi} \\
&\leq \|\phi\|_\infty \|\mathbf{f}^{max}\|_1.
\end{aligned}$$

Let us define $T_{\mathbf{f}}(\psi) = \sum_{n=-N}^{M} \hat{\psi}(n)\hat{\mathbf{f}}(n) \in X$ for any trigonometric polynomial ψ. We can use the density of the trigonometric polynomials in $C(\mathbb{T})$, to extend $T_{\mathbf{f}} : C(\mathbb{T}) \to X$ as a bounded operator. The assumption that $\sup_{\|x'\|=1} |\langle\mathbf{f}, x'\rangle| \in L^1(\mathbb{T})$ guarantees not only that $T_{\mathbf{f}}$ is weakly compact (hence there exists a regular measure ν with $T_\nu = T_{\mathbf{f}}$) but also that $T_{\mathbf{f}}$ is absolutely summing (hence $\nu \in M_{ac}(G, X)$). Finally using that $\nu(n) = \hat{\mathbf{f}}(n)$ for $n \in \mathbb{Z}$ we conclude that $(\hat{\mathbf{f}}(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z})$, from the assumption $X \in (RLP)_{\mathbb{T}}$. ■

Remark 3.1 *It was shown (see [1, Prop.3.4]) that the Riemann-Lebesgue property holds for not only spaces X having RNP but also for spaces satisfying the weak RNP (see [9]) o even the "complete continuity property" (see [10, 2]). The reader is referred to [1] for further results.*

4 Convolution for vector measures

From Corollary 2.2 we have that $t \to \mu(A - t)$ is measurable and bounded for each $\mu \in M(G)$ (and hence in $L^1(\nu)$) for any vector measure ν. This allows us to give the following definition.

Definition 4.1 *Let ν be a vector valued measure and $\mu \in M(G)$ we define the vector valued set function $\mu * \nu(A)$ given by*

$$\mu * \nu(A) = \int_G \mu(A-t)d\nu(t), A \in \mathcal{B}(G).$$

Let us see that $\mu * \nu$ is always a vector measure.

Proposition 4.2 *If ν is a vector measure and $\mu \in M(G)$ then $\mu * \nu$ is a vector measure. Moreover*

$$\|\mu * \nu\| \leq |\mu|(G)\|\nu\|. \tag{13}$$

Proof. Let (A_n) pairwise disjoint sets in $\mathcal{B}(G)$ with $A = \cup_n A_n$ and $t \in G$. To show that $\mu * \nu(A) = \sum_n \mu * \nu(A_n)$, due to the Orlicz-Pettis theorem (see [4, Page 7]), we simply need to see that $\sum_n \mu * \nu(A_n)$ is weakly unconditionally convergent to $\mu * \nu(A)$. Let $x' \in X'$ and note that for $A \in \mathcal{B}(G)$,

$$\langle \mu * \nu, x' \rangle(A) = \int_G \mu(A-t)d\langle \nu, x' \rangle(t) = \mu * \langle \nu, x' \rangle(A).$$

On the one hand

$$\begin{aligned} \sum_n |\langle \mu * \nu(A_n), x' \rangle| &\leq \sum_n |\mu| * |\langle \nu, x' \rangle|(A_n) \\ &= |\mu| * |\langle \nu, x' \rangle|(A) \\ &\leq |\mu|(G)|\langle \nu, x' \rangle|(G). \end{aligned}$$

On the other hand

$$\begin{aligned} |\sum_{n=1}^m \langle \mu * \nu(A_n), x' \rangle - \langle \mu * \nu(A), x' \rangle| &= |\int_G \mu(\cup_{n=m+1}^\infty A_n - t)d\langle \nu, x' \rangle(t)| \\ &\leq \int_G |\mu|(\cup_{n=m+1}^\infty A_n - t)d|\langle \nu, x' \rangle|(t). \end{aligned}$$

Let $\phi_m(t) = |\mu|(\cup_{n=m+1}^\infty A_n - t)$. We have that $\lim_{m\to\infty} \phi_m(t) = 0$ for each $t \in G$ and $\phi_m(t) \leq |\mu|(G)$ for each $m \in \mathbb{N}$ and $t \in G$. Hence the Lebesgue dominated convergence theorem shows that $\sum_n \langle \mu*\nu(A_n), x' \rangle = \langle \mu*\nu(A), x' \rangle$.

To show (13) we use that

$$|\langle \mu * \nu, x' \rangle|(G) \leq |\mu| * |\langle \nu, x' \rangle|(G) \leq |\mu|(G)|\langle \nu, x' \rangle|(G).$$

Now taking supremum over the unit ball of X' we get the desired estimate. ∎

Lemma 4.3 *If $\nu \in \mathcal{M}(G,X)$ and $\mu \in M(G)$ then $\mu * \nu \in \mathcal{M}(G,X)$. Moreover $T_{\mu*\nu}(g) = T_\nu \circ C_\mu$, where $C_\mu(g)(t) = \int_G g(t+s)d\mu(s)$ for $g \in C(G)$ and*

$$\|T_{\mu*\nu}\| \leq \|T_\nu\||\mu|(G).$$

Proof. It is immediate to see that C_μ is continuous from $C(G)$ into itself and the composition $T_\nu \circ C_\mu$ defines a weakly compact operator from $C(G)$ into X whose representing measure is given by

$$\eta(A) = (T_\nu \circ C_\mu)^{**}(\chi_A) = T_\nu^{**} \circ C_\mu^{**}(\chi_A).$$

Let us show that $\eta = \mu * \nu$. Recall that for each $\lambda \in M(G)$ we have that $\mu * \lambda \in M(G)$ which is defined by

$$\int_G g(u)d(\mu*\lambda)(u) = \int_G \int_G g(t+s)d\mu(s)d\lambda(t) = \int_G C_\mu(g)(t)d\lambda(t).$$

Therefore $C_\mu^*(\lambda) = \lambda * \mu$.

We also have that $C_\mu^{**}(\chi_A) \in (M(G))'$ with

$$C_\mu^{**}(\chi_A)(\lambda) = \lambda * \mu(A) = \int_G \mu(A-t)d\lambda(t).$$

We conclude that the element $C_\mu^{**}(\chi_A)$ is represented by the measurable function $t \to \mu(A-t)$, and taking into account that $M(G)' \subset L^1(\nu)$ we obtain $\eta(A) = T_\nu^{**}(\mu(A-\cdot)) = I_\nu(\mu(A-\cdot)) = \mu * \nu(A)$.

Finally using that $\|C_\mu\| \leq |\mu|(G)$ the proof is completed. ∎

Making use again of Corollary 2.2 we can also define the convolution as follows.

Definition 4.4 *Let $\nu \in \mathcal{M}(G,X)$ and $\mu \in M(G)$ we define the vector valued set function $\nu * \mu(A)$ given by*

$$\nu * \mu(A) = \int_G \nu(A-t)d\mu(t), A \in \mathcal{B}(G)$$

where the map $t \to \nu(A-t)$ is (strongly)-measurable and bounded (and hence in $L^1(\mu)$).

Proposition 4.5 *If $\nu \in \mathcal{M}(G,X)$ and $\mu \in M(G)$ then $\nu * \mu = \mu * \nu$.*

Proof. It suffices to show that $\langle \nu*\mu(A), x'\rangle = \langle \mu*\nu(A), x'\rangle$ for any $A \in \mathcal{B}(G)$ and $x' \in X'$. This now follows from the scalar-valued case: Recall that if $\mu_1, \mu_2 \in M(G)$ then, for each $g \in C(G)$,

$$\begin{aligned}\int_G g d\mu_1 * \mu_2 &= \int_G \Big(\int_G g(t+s)d\mu_1(t)\Big)d\mu_2(s) \\ &= \int_G \Big(\int_G g(t+s)d\mu_2(s)\Big)d\mu_1(t) \\ &= \int_G g d\mu_2 * \mu_1.\end{aligned}$$

Now use that

$$\langle \nu * \mu(A), x'\rangle = \int_G \langle \nu(A-t), x'\rangle d\mu(t) = \langle \nu, x'\rangle * \mu(A)$$

and

$$\langle \mu * \nu(A), x'\rangle = \int_G \mu(A-t) d\langle \nu, x'\rangle(t) = \mu * \langle \nu, x'\rangle(A).$$

■

Following the classical argument we obtain the following easy fact.

Proposition 4.6 *Let $\mu \in M(G)$ and $\nu \in \mathcal{M}(G,X)$. Then*

$$\widehat{\mu * \nu}(\gamma) = \hat{\mu}(\gamma)\hat{\nu}(\gamma), \quad \gamma \in \Gamma. \tag{14}$$

Proof. Let $\gamma \in \Gamma$. Then

$$\begin{aligned}\widehat{\mu * \nu}(\gamma) &= T_\nu(C_\mu(\bar{\gamma})) = T_\nu(\int_G \bar{\gamma}(\cdot + s)d\mu(s)) \\ &= T_\nu(\bar{\gamma}\int_G \bar{\gamma}(s)d\mu(s)) = \hat{\mu}(\gamma)\hat{\nu}(\gamma).\end{aligned}$$

■

Let us now restrict to some classes of measures in $M(G)$ and $\mathcal{M}(G,X)$.

Remark 4.1 *For $f \in L^1(G)$ and $\mathbf{g} \in L^1(G,X)$ we write $d\mu_f = f dm_G$ and $d\nu_{\mathbf{g}} = \mathbf{g} dm_G$. Then $d(\mu_f * \nu_{\mathbf{g}}) = (f * \tilde{\mathbf{g}})dm_G$ where $f * \tilde{\mathbf{g}} \in L^1(G,X)$. Here we use the notation $\tilde{\mathbf{g}}(u) = \mathbf{g}(-u)$ and*

$$f * \tilde{\mathbf{g}}(s) = \int_G f(s-t)\tilde{\mathbf{g}}(t)dm_G(t), \quad m_G - a.e.$$

Indeed

$$\begin{aligned}
\mu_f * d\nu_{\mathbf{g}}(A) &= \int_G \mu_f(A-t)d\nu_{\mathbf{g}}(t) \\
&= \int_G \Big(\int_{A-t} f(s)dm_G(s)\Big)\mathbf{g}(t)dm_G(t) \\
&= \int_G \Big(\int_A f(s+t)dm_G(s)\Big)\mathbf{g}(t)dm_G(t) \\
&= \int_A \Big(\int_G f(s+t)\mathbf{g}(t)dm_G(t)\Big)dm_G(s) \\
&= \int_A \Big(\int_G f(s-u)\tilde{\mathbf{g}}(u)dm_G(u)\Big)dm_G(s),
\end{aligned}$$

Remark 4.2 *For $f \in L^1(G)$ and a vector measure ν we have that*

$$\mu_f * \nu(A) = I_\nu(\chi_A * \tilde{f}), A \in \mathcal{B}(G).$$

Indeed

$$\begin{aligned}
\mu_f * \nu(A) &= \int_G \Big(\int_{A-t} f(s)dm_G(s)\Big)d\nu(t) \\
&= \int_G \Big(\int_G \chi_A(s+t)f(s)dm_G(s)\Big)d\nu(t) \\
&= \int_G \Big(\int_G \chi_A(t-s)\tilde{f}(s)dm_G(s)\Big)d\nu(t) \\
&= I_\nu(\chi_A * \tilde{f}).
\end{aligned}$$

If ν is a vector measure and $f \in L^1(G)$ we denote $\mu_f * \nu = f * \nu$ and we say that $f * \nu \in C(G, X)$ whenever there exists $\mathbf{f}_\nu \in C(G, X)$ such that $d(f * \nu) = \mathbf{f}_\nu dm_G$.

Proposition 4.7 *Let ν be a vector measure.*
(a) If $f \in C(G)$ then $f * \nu \in C(G, X)$ and

$$\|f * \nu\|_{C(G,X)} \leq \|f\|_{C(G)}\|\nu\|. \tag{15}$$

(b) If $f \in L^1(G)$ then $f * \nu \in P_1(G, X)$ and

$$\|f * \nu\|_{P_1(G,X)} \leq \|f\|_{L^1(G)}\|\nu\|. \tag{16}$$

Proof. We define $\mathbf{f}_\nu(t) \in X$ by

$$\mathbf{f}_\nu(t) = \int_G f(t+s)d\nu(s) = I_\nu(\tau_{-t}f), t \in G. \tag{17}$$

Let us see first that $\mathbf{f}_\nu \in C(G,X)$. For each $t,t' \in G$ we have

$$\|\mathbf{f}_\nu(t) - \mathbf{f}_\nu(t')\| = \|I_\nu(\tau_{-t}f - \tau_{-t'}f)\| \leq \|\nu\| \|\tau_{-t}f - \tau_{-t'}f\|_{C(G)}.$$

Now the result follows using that the map $G \to C(G)$ given by $t \to \tau_{-t}f$ is uniformly continuous.

Let us now show that $d(f * \nu) = \mathbf{f}_\nu dm_G$. Let $A \in \mathcal{B}(G)$ and $x' \in X'$ and note that

$$\begin{aligned}
\langle \int_A \mathbf{f}_\nu dm_G, x' \rangle &= \langle \int_A \Big(\int_G f(t+s)d\nu(s) \Big) dm_G(t), x' \rangle \\
&= \int_A \Big(\int_G f(t+s)d\langle \nu, x' \rangle(s) \Big) dm_G(t) \\
&= \int_G \Big(\int_A f(t+s)dm_G(t) \Big) d\langle \nu, x' \rangle(s) \\
&= \langle \int_G \Big(\int_{A-s} f(t)dm_G(t) \Big) d\langle \nu, x' \rangle(s) \\
&= \langle f * \nu(A), x' \rangle
\end{aligned}$$

Finally (15) follows trivially since

$$\sup_{t \in G} \|\mathbf{f}_\nu(t)\| = \sup_{t \in G, \|x'\|=1} |\int_G f(t+s)d\langle \nu, x' \rangle(s)| \leq \|f\|_{C(G)} \|\nu\|.$$

(b) Assume now that $f \in L^1(G)$. We first find $f_n \in C(G)$ such that $\|f - f_n\|_{L^1(G)} \to 0$. Using the previous case, the estimate (13) and the fact $|\mu_f|(G) = \|f\|_{L^1(G)}$ we conclude that

$$\|\mu_{f_n} * \nu - \mu_f * \nu\| \leq \|f_n - f\|_{L^1(G)} \|\nu\|.$$

Then $f * \nu \in P_1(G,X)$ and $\|f * \nu\|_{P_1(G,X)} = \|\mu_f * \nu\| \leq \|f\|_{L^1(G)} \|\nu\|$. ■

Let us now look for some Young's convolution result when assuming that either $\nu \in \mathcal{M}_p(G,X)$ or $f \in L^p(G)$. Let us mention first that any measure ν with bounded p-semivariation with respect to m_G for some $1 < p \leq \infty$ necessarily belongs to $\mathcal{M}_{ac}(G,X)$. This is due to the fact that it satisfies

$\|\nu(A)\| \leq m_G(A)^{1/p'}$ for each $A \in \mathcal{B}(G)$ which implies $\nu << m_G$ and, in particular ν is automatically regular.

To work with measures in $\mathcal{M}_p(G,X)$ we shall use some lemmas whose proofs we include for the sake of completeness.

Lemma 4.8 *Let $1 < p \leq \infty$ and let ν be a vector measure. Then $\nu \in \mathcal{M}_p(G,X)$ if and only if $\nu \in \mathcal{M}_{ac}(G;X)$ and $T_\nu^*(X') \subset L^p(G)$.*

Proof. Assume $\nu \in \mathcal{M}_p(G,X)$. Now (1) gives that T_ν extends to a bounded operator from $L^{p'}(G)$ into X. Hence T_ν^* is bounded from X' into $L^p(G)$ and therefore $T_\nu^*(X') \subset L^p(G)$.

Assume now that $\nu \in \mathcal{M}_{ac}(G,X)$ and $T_\nu^*(X') \subset L^p(G)$. Use now that $\langle \nu, x'\rangle = T_\nu^*(x')$ and then $d|\langle \nu, x'\rangle| = |h_{x'}|dm_G$ for some $h_{x'} \in L^p(G)$. This implies that for $\|\sum_{A\in\pi} \alpha_A \chi_A\|_{L^{p'}(m_G)} \leq 1$ we have

$$\left\|\sum_{A\in\pi} \alpha_A \nu(A)\right\|_X \leq \sup_{\|x'\|=1} \int_G (\sum_{A\in\pi} |\alpha_A|\chi_A)|h_{x'}|dm_G = \sup_{\|x'\|=1} \|h_{x'}\|_{L^p(G)}.$$

Hence we have ν is of bounded p-semivariation. ■

Next result extends [3, Thm 3.9] to the case $1 < p < \infty$.

Proposition 4.9 *Let $1 < p \leq \infty$ and let $\nu \in \mathcal{M}_{ac}(G,X)$. The following statements are equivalent:*

(i) $\nu \in \mathcal{M}_p(G,X)$.

(ii) $L^{p'}(G) \subset L^1(\nu)$.

(iii) $L^{p'}(G) \subset L^1_w(\nu)$.

Moreover $\|\nu\|_{p.m_G} = \|Id\|_{L^{p'}(G)\to L^1(\nu)}$.

Proof. (i) $\Longrightarrow$ (ii) Assume that ν has bounded p-semivariation with respect to m_G. Hence

$$\|\int_G \phi d\nu\| \leq \|\nu\|_{p,m_G}\|\phi\|_{L^{p'}(G)}, \phi \text{ simple }.$$

This gives, due to the density of simple functions in $L^{p'}(G)$ and $L^1(\nu)$, that $L^{p'}(G) \subset L^1(\nu)$.

(ii) $\Longrightarrow$ (iii) It is obvious.

(iii) $\Longrightarrow$ (i) Assume $L^{p'}(G) \subset L^1_w(\nu)$. Since $\nu << m_G$, using Radon-Nikodym theorem we have, for each $x' \in X'$ the existence of $h_{x'} \in L^1(G)$ for which

$$d\langle \nu, x'\rangle = h_{x'}dm_G. \tag{18}$$

Therefore for each $f \in L^{p'}(G)$ we have

$$\|f\|_{L^1_w(\nu)} = \sup_{\|x'\|=1} \int_G |f||h_{x'}|dm_G \leq K\|f\|_{L^{p'}(G)}.$$

This implies that $h_{x'} \in L^p(G)$ for all $x' \in X'$ and

$$\sup_{\|x'\|=1} \|h_{x'}\|_{L^p(G)} \leq K. \tag{19}$$

This gives $\|\nu\|_{p,m_G} \leq K$. ■

Theorem 4.10 *Let $1 < p < \infty$ and let ν be a vector measure.*

(a) If $\nu \in \mathcal{M}_p(G,X)$ and $f \in L^{p'}(G)$ then $f * \nu \in C(G,X)$ and

$$\|f * \nu\|_{C(G,X)} \leq \|f\|_{L^{p'}(G)} \|\nu\|_{p,m_G}. \tag{20}$$

(b) If $\nu \in \mathcal{M}_{ac}(G,X)$ and $f \in L^\infty(G)$ then $f * \nu \in C(G,X)$ and

$$\|f * \nu\|_{C(G,X)} \leq \|f\|_{L^\infty(G)} \|\nu\|. \tag{21}$$

(c) If $f \in L^p(G)$ then $f * \nu \in P_p(G,X)$. Moreover

$$\|f * \nu\|_{P_p(G,X)} \leq \|f\|_{L^p(G)} \|\nu\|. \tag{22}$$

(d) If $f \in L^q(G)$ and $\nu \in \mathcal{M}_p(G,X)$ with $q' > p$ then $f * \nu \in P_r(G,X)$ for $1/r = 1/p - 1/q'$. Moreover

$$\|f * \nu\|_{P_r(G,X)} \leq \|f\|_{L^q(G)} \|\nu\|_{p,m_G}. \tag{23}$$

Proof. (a) Using Proposition 4.9 we have $L^{p'}(G) \subset L^1(\nu)$. Hence that I_ν is well defined on $L^{p'}(G)$ and, denoting $f_t(s) = f(t+s) = \tau_{-t}f(s)$, we observe that $\mathbf{f}_\nu(t) = I_\nu(f_t)$ makes sense for each value of $t \in G$. Repeating the argument in Proposition 4.7 part (a) and using now that $G \to L^{p'}(G)$ given by $t \to f_t$ is uniformly continuous we obtain that $\mathbf{f}_\nu$ is continuous. And also we have

$$\|f * \nu(t)\| \leq \|I_\nu\|_{L^{p'} \to X} \|f_t\|_{L^{p'}(G)} = \|\nu\|_{p,m_G} \|f\|_{L^{p'}(G)}.$$

(b) If $f \in L^\infty(G)$ then value $I_\nu(f_t)$ makes sense for any $t \in G$. Observe that if $\nu \in \mathcal{M}_{ac}(G,X)$ and $f = \chi_A$ we have

$$\int_G \chi_A(t+s)d\nu(s) = \nu(A-t)$$

is continuous. Hence $f*\nu \in C(G,X)$ for any simple function f and it satisfies $\|I_\nu(f)\| \leq \|f\|_{L^\infty(G)}\|\nu\|$. Finally using the density of simple functions in $L^\infty(G)$ we have the desired result.

(c) From Proposition 4.7 and $f \in C(G)$ we know that $d(f*\nu) = \mathbf{f}_\nu dm_G$ with $\mathbf{f}_\nu \in C(G,X)$. Moreover, for each $x' \in X'$,

$$\begin{aligned}
&\int_G |\langle \mathbf{f}_\nu(t), x'\rangle|^p dm_G(t) \leq \int_G \Big(\int_G |f(t+s)| d|\langle \nu, x'\rangle|(s)\Big)^p dm_G(t) \\
\leq\ & \int_G (|\langle \nu, x'\rangle|(G))^{p-1} \Big(\int_G |f(t+s)|^p d|\langle \nu, x'\rangle|(s)\Big) dm_G(t) \\
=\ & (|\langle \nu, x'\rangle|(G))^{p-1} \int_G \Big(\int_G |f(t+s)|^p dm_G(t)\Big) d|\langle \nu, x'\rangle|(s) \\
=\ & \|f\|^p_{L^p(G)} (|\langle \nu, x'\rangle|(G))^p.
\end{aligned}$$

This shows (22) for continuous functions.

Let $f \in L^p(G)$. We first find $f_n \in C(G)$ such that $\|f - f_n\|_{L^p(G)} \to 0$ and denote $\nu_n = f_n * \nu$. From the previous case conclude that

$$\|\nu_n - \nu_m\|_{p,m_G} = \|\mathbf{f}_{\nu_n} - \mathbf{f}_{\nu_m}\|_{P_p(G,X)} \leq \|f_n - f_m\|_{L^p(G)}\|\nu\|.$$

Therefore ν_n is a Cauchy sequence in $\mathcal{M}_p(G,X)$. Since it converges to $f*\nu$ in $\mathcal{M}(G,X)$ we conclude that $f*\nu \in P_p(G,X)$ and

$$\|f*\nu\|_{P_p(G,X)} = \lim_n \|f_n * \nu\|_{P_p(G,X)} \leq \lim_n \|f_n\|_{L^p(G)}\|\nu\| = \|f\|_{L^p(G)}\|\nu\|.$$

(d) Since $\nu \in \mathcal{M}_p(G,X)$, using Lemma 4.8 we have $d\langle \nu, x'\rangle = h_{x'} dm_G$ with $h_{x'} \in L^p(G)$. Assume again first that $f \in C(G)$ and $\|x'\| = 1$. Hence

$$\langle f*\nu(t), x'\rangle = \int_G f(t+s) h_{x'}(s) dm_G(s) = f * \tilde{h}_{x'}(t).$$

Therefore $|\langle f*\nu(t), x'\rangle| \leq |f| * |\tilde{h}_{x'}|(t)$ and we can apply the classical Young's inequality

$$\begin{aligned}
(\int_G |\langle f*\nu(t), x'\rangle|^r dm_G(t))^{1/r} &\leq \||f| * |\tilde{h}_{x'}|\|_{L^r(G)} \\
&\leq \|f\|_{L^q(G)}\|h_{x'}\|_{L^p(G)} \\
&\leq \|f\|_{L^q(G)}\|\nu\|_{p,m_G}
\end{aligned}$$

This gives (23) for continuous functions. The argument is finished using density as above. ■

5 Invariance under homomorphisms

Throughout this section $H : G \to G$ denotes a homeomorphism and we write $R(s) = -s$ for the reflection and $\tau_a(s) = s + a$ for the translation.

If $f \in L^0(G)$ we shall use the notation $f_H(s) = f(H^{-1}s)$ and, in particular, $\tau_a f(s) = f(s-a)$ and $\tilde{f}(s) = f(-s)$.

If ν is a vector measure we denote $\nu_H(A) = \nu(H(A))$ for $A \in \mathcal{B}(G)$, in particular, $\tau_a\nu(A) = \nu(A+a)$ and $\tilde{\nu}(A) = \nu(-A)$. It is elementary to see that ν_H is also a vector measure with $\|\nu_H\| = \|\nu\|$. If $\nu \in \mathcal{M}(G,X)$ then $\nu_H \in \mathcal{M}(G,X)$ and $T_{\nu_H} = T_\nu \circ \Phi_H$ where $\Phi_H : C(G) \to C(G)$ is the induced operator $g \to g_H = g \circ H^{-1}$.

Let us point out some useful formulae to be used later on. From the fact $(\chi_A)_H = \chi_{H(A)}$ and $I_{\nu_H}(\chi_A) = \nu_H(A) = \nu(H(A)) = I_\nu((\chi_A)_H)$ we conclude

$$I_{\nu_H}(f) = I_\nu(f_H), f \in \mathcal{S}(G). \tag{24}$$

Also we have

$$(\nu_H)_f = (\nu_{f_H})_H \text{ and } (\nu_f)_H = (\nu_H)_{f_{H^{-1}}} \text{ for any } f \in \mathcal{S}(G). \tag{25}$$

This follows by linearity and the obvious case

$$(\nu_H)_f(A) = \int_A f d\nu_H = \int_G (f\chi_A)_H d\nu = \int_{H(A)} f_H d\nu = (\nu_{f_H})_H(A).$$

In particular

$$\widetilde{\nu_f} = \tilde{\nu}_{\tilde{f}}, \quad \tau_a(\nu_f) = (\tau_a\nu)_{\tau_{-a}f}, \quad a \in G. \tag{26}$$

Taking into account (25) we conclude that $f \in L^1(\nu_H)$ if and only if $f_H \in L^1(\nu)$. Moreover with

$$\|f\|_{L^1(\nu_H)} = \|(\nu_H)_f\| = \|\nu_{f_H}\| = \|f_H\|_{L^1(\nu)}, \quad f \in \mathcal{S}(G). \tag{27}$$

Definition 5.1 *Let ν be a vector measure and $\mathcal{H}$ a family of homeomorphisms $H : G \to G$. We say that ν is $\mathcal{H}$- invariant whenever $\nu_H = \nu$ for any $H \in \mathcal{H}$. In particular, we say that ν is translation invariant (respect. reflection invariant) whenever $\tau_a\nu = \nu$ for any $a \in G$ (respect. $\tilde{\nu} = \nu$.)*

Given a vector measure $\nu \in \mathcal{M}(G,X)$ we can define

$$\nu_{inv}(A) = \int_G \tau_t\nu(A) dm_G(t).$$

It is not difficult to show that $\nu_{inv} \in \mathcal{M}(G, X)$. Clearly ν_{inv} is translation invariant. Actually case of translation invariant measures, as in the scalar valued case, reduces to the xm_G for some $x \in X$.

Proposition 5.2 *Let $\nu \in \mathcal{M}(G, X)$ with $\nu(G) \neq 0$. Then ν is translation invariant if and only if $\nu = \nu(G)m_G$.*

Proof. Only the direct implication needs a proof. Assume that ν is translation invariant, that is $\nu = \nu_{inv}$. We shall show that

$$\nu_{inv}(A) = \nu(G)m_G(A), \quad \forall A \in \mathcal{B}(G).$$

It suffices to see that $T_{\nu_{inv}}(g) = (\int_G g dm_G)\nu(G)$ for all $g \in C(G)$. This follows by noticing that

$$\begin{aligned} T_{\nu_{inv}}(g) &= \int_G T_{\tau_t \nu}(g) dm_G(t) = \int_G T_\nu(\tau_t g) dm_G(t) \\ &= T_\nu(\int_G (\tau_t g) dm_G(t)) = T_\nu((\int_G g dm_G)\chi_G) \\ &= \nu(G)(\int_G g dm_G). \end{aligned}$$

■

Using (24) we easily formulate the H-invariance as follows:

Remark 5.1 *Let ν be a vector measure and $H : G \to G$ an homomorphism. The following statements are equivalent.*
(i) ν is H-invariant.
(ii) $I_{\nu_H}(f) = I_\nu(f)$ for any $f \in \mathcal{S}(G)$.
(iii) $L^1(\nu) = L^1(\nu_H)$ and $I_\nu(f_H) = I_\nu(f)$ for any $f \in L^1(\nu)$.

Definition 5.3 *(see [7, 3]) Given an homeomorphism $H : G \to G$ and $f \in L^0(G)$ a vector measure ν is said to be a "norm integral H-invariant" whenever*

$$\|I_{\nu_H}(f)\| = \|I_\nu(f)\|, f \in \mathcal{S}(G). \tag{28}$$

Given a family of homeomorphisms on G, say $\mathcal{H}$, we shall say that ν is "norm integral $\mathcal{H}$-invariant" whenever it is norm integral H-invariant for any $H \in \mathcal{H}$. We say "norm integral reflection invariant" and "norm integral translation invariant" in the cases of $\mathcal{H} = \{R\}$ and $\mathcal{H} = \{\tau_a : a \in G\}$ respectively.

Proposition 5.4 *Let $H : G \to G$ be an homeomorphism and $\nu \in \mathcal{M}(G,X)$. The following are equivalent.*

(i) ν is norm integral H-invariant.

(ii) $\|I_\nu(f_H)\| = \|I_\nu(f)\|, \forall f \in L^1(\nu)$.

(iii) $\|T_{\nu_H}(f)\| = \|T_\nu(f)\|, \forall f \in C(G)$.

(iv) $L^1(\nu) = L^1(\nu_H)$ and $\|I_{\nu_H}(f)\| = \|I_\nu(f)\|$ for any $f \in L^1(\nu)$.

Proof. (i) $\Longrightarrow$ (ii). It was shown in [3, Thm 3.3], due to the fact $I_\nu(f_H) = I_{\nu_H}(f)$ for all $f \in \mathcal{S}(G)$.

(ii) $\Longrightarrow$ (iii) It follows using that $C(G) \subset L^1(\nu)$ and $I_\nu(f) = T_\nu(f)$ for $f \in C(G)$.

(iii) $\Longrightarrow$ (iv) Let us show that $\|f\|_{L^1(\nu)} = \|f\|_{L^1(\nu)}$ for any $f \in C(G)$. Indeed, for each $f \in C(G)$ we can write

$$\begin{aligned}
\|f\|_{L^1(\nu)} &= \|\nu_f\| = \|T_{\nu_f}\| \\
&= \sup\{\|T_{\nu_f}(g)\| : \|g\|_{C(G)} = 1\} = \sup\{\|T_\nu(fg)\| : \|g\|_{C(G)} = 1\} \\
&= \sup\{\|T_{\nu_H}(fg)\| : \|g\|_{C(G)} = 1\} = \sup\{\|T_{(\nu_H)_f}(g)\| : \|g\|_{C(G)} = 1\} \\
&= \|(\nu_H)_f\| = \|f\|_{L^1(\nu_H)}.
\end{aligned}$$

Finally use Lemma 2.1 to extend $\|I_{\nu_H}(f)\| = \|I_\nu(f)\|, \forall f \in C(G)$ to all $f \in L^1(\nu)$.

(iv) $\Longrightarrow$ (i) It is immediate. ∎

Let us also consider some weaker notions still good enough for our purposes.

Definition 5.5 *Let ν be a vector measure and let $\mathcal{H}$ be a family of homeomorphisms $H : G \to G$. Then ν is said to be "semivariation $\mathcal{H}$-invariant" whenever*

$$\|\nu_f\| = \|(\nu_H)_f\|, \quad f \in \mathcal{S}(G), H \in \mathcal{H}. \tag{29}$$

In particular ν is said to be "semivariation translation invariant" and "semivariation reflection invariant" in the cases of $\mathcal{H} = \{\tau_a : a \in G\}$ and $\mathcal{H} = \{R\}$ respectively.

Remark 5.2 *If ν is norm integral translation invariant or semivariation translation invariant then so it is $\tilde{\nu}$.*

Assume first that ν is norm integral translation invariant, $f \in \mathcal{S}(G)$ and $a \in G$. Then

$$\|I_{\tilde{\nu}}(\tau_a f)\| = \|I_\nu(\tau_{-a}\tilde{f})\| = \|I_\nu(\tilde{f})\| = \|I_{\tilde{\nu}}(f)\|.$$

Assume now that ν is semivariation translation invariant, $f \in \mathcal{S}(G)$ and $a \in G$. Then applying (26) we obtain $(\tau_a\tilde{\nu})_f = (\widetilde{\tau_{-a}\nu})_f = \widetilde{(\tau_{-a}\nu)_{\tilde{f}}}$ and therefore

$$\|(\tau_a\tilde{\nu})_f\| = \|(\tau_{-a}\nu)_{\tilde{f}}\| = \|\nu_{\tilde{f}}\| = \|\widetilde{\nu_{\tilde{f}}}\| = \|(\tilde{\nu})_f\|.$$

Proposition 5.6 *Let ν be a vector measure and let $H : G \to G$ be an homeomorphism. The following statements are equivalent:*

(i) ν is semivariation H-invariant.

(ii) $L^1(\nu) = L^1(\nu_H)$ isometrically.

(iii) $\|T_{\nu_H} \circ M_f\| = \|T_\nu \circ M_f\|, \forall f \in C(G)$ where $M_f : C(G) \to C(G)$ stands for the multiplication operator $g \to fg$.

Proof. (i) $\iff$ (ii) follows using that $\|f\|_{L^1(\nu)} = \|\nu_f\| = \|(\nu_H)_f\| = \|f\|_{L^1(\nu_H)}$ and the density of simple functions in $L^1(\nu)$ and $L^1(\nu_H)$.

(i) $\iff$ (iii) follows observing that $\|T_\nu \circ M_f\| = \|\nu_f\|$ whenever $f \in C(G)$ and Lemma 2.1. ■

In Theorem 3.3 [3] (see also Proposition 3.5 [7]) it was shown that if ν is norm integral H-invariant then it is semivariation H-invariant. We shall see now that the converse is not true in general.

Proposition 5.7 *Let $G = \mathbb{T}$ and $X = L^1(\mathbb{T})$. For each $n \in \mathbb{Z} \setminus \{0\}$ denote $\phi_n(t) = t^n$ for $t \in \mathbb{T}$ and define*

$$\nu_{(n)}(A) = \hat{\chi}_A(n)\phi_n, A \in \mathcal{B}(\mathbb{T}).$$

Then $\nu_{(n)}$ is semivariation reflection invariant but not norm integral reflection-invariant.

Proof. To see that $\nu_{(n)}$ is not norm integral reflection-invariant we use an argument similar to that of example 3.6 (c) in [3]. Note that $T_{\nu_{(n)}}(g) = \hat{g}(n)\phi_n$ for any $g \in C(\mathbb{T})$ and $T_{\widetilde{\nu_{(n)}}}(g) = \hat{g}(-n)\phi_n$ for any $g \in C(\mathbb{T})$. Hence $T_{\nu_{(n)}}(\phi_n) = \phi_n$ and $T_{\nu_{(n)}}(\tilde{\phi}_n) = 0$. Hence $\|T_{\nu_{(n)}}(\tilde{\phi}_n)\| \neq \|T_{\nu_{(n)}}(\phi_n)\|$.

We shall see that $\|(\nu_{(n)})_f\| = \|f\|_1$ for any $f \in C(\mathbb{T})$. Due to (26) this gives $\|(\nu_{(n)})_f\| = \|(\widetilde{\nu_{(n)}})_f\|$ for any $f \in C(\mathbb{T})$.

First we notice that if $f, g \in C(\mathbb{T})$ we have

$$T_{(\nu_{(n)})_f}(g) = \widehat{fg}(n)\phi_n.$$

Hence for each $\psi \in X' = L^\infty(\mathbb{T})$ we obtain

$$\langle T_{(\nu_{(n)})_f}(g), \psi\rangle = \widehat{fg}(n)\hat{\psi}(n) = \int_{\mathbb{T}} g(t)\Big(\hat{\psi}(n)f(t)\bar{\phi}_n(t)\Big)dm_{\mathbb{T}}(t).$$

This shows that $d\langle (\nu_{(n)})_f, \psi\rangle = \hat{\psi}(n)f\bar{\phi}_n dm_{\mathbb{T}}$ and

$$\|(\nu_{(n)})_f\| = \sup_{\|\psi\|_{L^\infty(\mathbb{T})}=1} |\langle (\nu_{(n)})_f, \psi\rangle|(\mathbb{T}) = \sup_{\|\psi\|_{L^\infty(\mathbb{T})}=1} \|f\|_1|\hat{\psi}(n)| = \|f\|_1.$$

■

Proposition 5.8 *Let $\nu_{\mathbf{f}}(A) = \int_A \mathbf{f}(s)dm_G(s)$ with $\mathbf{f} \in L^\infty(G, X)$ non constant function satisfying that*

$$\|\mathbf{f}(t)\| = 1, \quad t \in G$$

and there exists $A \in \mathcal{B}(G)$ and $a \in G$ for which

$$\nu_{\mathbf{f}}(A) = 0, \quad \nu_{\mathbf{f}}(A + a) \neq 0.$$

Then $\nu_{\mathbf{f}}$ is semivariation translation invariant but not norm integral translation invariant.

Proof. Note that $\tau_t\nu_{\mathbf{f}} = \nu_{\tau_t\mathbf{f}}$ and $\tau_t\mathbf{f} \in L^\infty(G, X)$ for each $t \in G$. In particular $\tau_t\nu_{\mathbf{f}}$ is of bounded variation and $d|\tau_t\nu_{\mathbf{f}}| = \tau_t\|\mathbf{f}\|dm_G = dm_G$. Hence $L^1(\nu) = L^1(\tau_t\nu) = L^1(m_G)$ for any $t \in G$. Invoking now Proposition 5.6 we obtain that ν is semivariation translation invariant.

On the other hand $I_\nu(g) = \int_G g\mathbf{f}dm_G$ and we have $\|I_\nu(\tau_a\chi_A)\| \neq 0$ while $\|I_\nu(\chi_A)\| = 0$, showing that $\nu_{\mathbf{f}}$ is not norm integral translation invariant. ■

Remark 5.3 *Select $X = \mathbb{C}$, $G = \mathbb{T}$, $\mathbf{f}(s) = \chi_{[0,1/2)}(e^{2\pi is}) - \chi_{[1/2,1)}(e^{2\pi is})$, $A = \{e^{2\pi is} : 1/4 \leq s < 1/2\}$ and $a = e^{i\pi/2}$) to have an example satisfying conditions of Proposition 5.8.*

One of the basic properties of semivariation translation invariant measures is the following fact.

Lemma 5.9 *Let $1 \leq p < \infty$, let $\nu \in \mathcal{M}(G, X)$ be a semivariation translation invariant measure and $f \in L^p(\nu)$. Then $a \to \tau_a f$ is uniformly continuous from G into $L^p(\nu)$.*

Proof. Invoking Lemma 2.1, for each $\varepsilon > 0$ we find $g \in C(G)$ such that $\|f - g\|_{L^p(\nu)} < \varepsilon$. Now use the standard argument

$$\begin{aligned}
\|\tau_a f - \tau_b f\|_{L^p(\nu)} &\leq \|\tau_a(f-g)\|_{L^p(\nu)} + \|\tau_b(f-g)\|_{L^p(\nu)} + \|\tau_a g - \tau_b g\|_{L^p(\nu)} \\
&= \|f-g\|_{L^p(\nu_a)} + \|f-g\|_{L^p(\nu_b)} + \|\tau_a g - \tau_b g\|_{L^p(\nu)}
\end{aligned}$$

Hence, using Proposition 5.6, we conclude

$$\|\tau_a f - \tau_b f\|_{L^p(\nu)} \leq 2\varepsilon + \|\tau_a g - \tau_b g\|_{C(G)} \|\nu\|^{1/p}$$

and the proof finishes using that g is uniformly continuous. ■

Theorem 5.10 *Let $1 \leq p < \infty$ and let $\nu \in \mathcal{M}(G, X)$ be semivariation translation invariant with $\nu(G) \neq 0$. Then $L^p(\nu) \subset L^p(G)$ and*

$$\|f\|_{L^p(G)} \leq \|f\|_{L^p(\nu)} \|\nu(G)\|^{-1/p}.$$

Proof. Using that $\nu_{inv} = \nu(G)m_G$, in particular we know that $L^p(\nu_{inv}) = L^p(G)$ and $\|f\|_{L^p(\nu_{inv})} = \|\nu(G)\|^{1/p}\|f\|_{L^p(G)}$. Therefore it suffices to show that $L^p(\nu) \subset L^p(\nu_{inv})$ and $\|f\|_{L^p(\nu_{inv})} \leq \|f\|_{L^p(\nu)}$. We first point out the following trivial estimate between positive measures

$$|\langle \nu_{inv}, x' \rangle| \leq \int_G |\langle \tau_t \nu, x' \rangle| dm_G(t)$$

for any $x' \in X'$. Hence, for each $f \in L^p(\nu) = L^p(\tau_t \nu)$ for all $t \in G$, we have

$$\begin{aligned}
\|f\|_{L^p(\nu_{inv})} &= \sup_{\|x'\|=1} \Big(\int_G |f|^p d|\langle \nu_{inv}, x' \rangle|\Big)^{1/p} \\
&\leq \sup_{\|x'\|=1} \Big(\int_G \Big(\int_G |f|^p d|\langle \tau_t \nu, x' \rangle|\Big) dm_G(t)\Big)^{1/p} \\
&\leq \Big(\int_G \Big(\sup_{\|x'\|=1} \int_G |f|^p d|\langle \tau_t \nu, x' \rangle|\Big) dm_G(t)\Big)^{1/p} \\
&= \Big(\int_G \|f\|^p_{L^p(\tau_t \nu)} dm_G(t)\Big)^{1/p} \\
&= \|f\|_{L^p(\nu)}.
\end{aligned}$$

■

Proposition 5.11 *Let $\nu \in \mathcal{M}(G, X)$ a semivariation translation invariant measure and $f \in L^1(\nu)$. Then $f * \nu \in C(G, X)$ and*

$$\|f * \nu\|_{C(G,X)} \leq \|f\|_{L^1(\nu)} \|\nu\|. \tag{30}$$

Proof. Recall that

$$\mathbf{f}_\nu(t) = \int_G f(t+s) d\nu(s) = I_\nu(\tau_{-t} f) = I_{\tau_{-t}\nu}(f)$$

which it is well defined for each $t \in G$ using that $L^1(\nu) = L^1(\tau_{-t}\nu)$. Moreover

$$\|f * \nu(t)\| \leq \|\tau_{-t} f\|_{L^1(\nu)} \|\tilde{\nu}\| = \|f\|_{L^1(\nu)} \|\nu\|.$$

Using now Lemma 5.9 we have that $t \to \tau_{-t} f$ is continuous from G into $L^1(\nu)$ which shows that $f * \nu \in C(G, X)$. ∎

Corollary 5.12 *Let $\nu \in \mathcal{M}(G, X)$. Then ν is norm integral translation invariant if and only if for each $f \in L^1(\nu)$ one has that $f * \nu \in C(G, X)$ with*

$$\|f * \nu(t)\| = \|\int_G f d\nu\| \quad \forall t \in G.$$

Proof. Assume that ν is norm integral translation invariant. From Proposition 5.11 we have $f * \nu \in C(G, X)$. Moreover in this case

$$\|f * \nu(t)\| = \|I_\nu(\tau_{-t} f)\| = \|I_\nu(f)\| = \|\int_G f d\nu\| \quad \forall t \in G.$$

The converse is immediate because any simple function f belongs to $L^1(\nu)$ and by assumption

$$\|I_\nu(\tau_t f)\| = \|f * \nu(-t)\| = \|\int_G f d\nu\| = \|I_\nu(f)\|.$$

∎

6 Applications to Fourier Analysis on $L^1(\nu)$

In this section we analyze the properties of ν_f, where $\nu_f(A) = \int_A f d\nu$ for each $A \in \mathcal{B}(G)$, in terms of those of ν and $f \in L^1(\nu)$ and apply them to recover and improve the results in [3] and [7].

Proposition 6.1 *Let ν be vector measure.*

(i) If $\nu \in \mathcal{M}(G, X)$ and $f \in L^1(\nu)$ then $\nu_f \in \mathcal{M}(G, X)$.

(ii) If $1 < p \leq \infty$, $\nu \in \mathcal{M}_p(G, X)$ and $f \in L^q(G)$ for some $p' \leq q \leq \infty$ then $\nu_f \in \mathcal{M}_r(G, X)$ for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

Proof. (i) Note from Lemma 2.1 we have that $T_{\nu_f} = \lim_n T_{\nu_{f_n}}$ where $f_n \in C(G)$ for each n. It suffices to see that $T_{\nu_g} : C(G) \to X$ given by $\varphi \to \int \varphi g d\nu$ is weakly compact for any $g \in C(G)$. On the other hand $T_{\nu_g} = T_\nu \circ M_g$ where M_g stands for the multiplication operator on $C(G)$ given by $M_g(\phi) = g\phi$. Therefore, since ν is regular, hence T_ν is weakly compact and then we obtain that ν_f is regular.

(ii) We use Proposition 4.9 to have that $L^q(G) \subset L^{p'}(G) \subset L^1(\nu)$. Hence ν_f is well defined. Now use the fact that $T^*_{\nu_f} = M_f \circ T^*_\nu$ where $M_f : L^p(G) \to L^r(G)$ is the multiplication operator $M_f(g) = fg$ and the fact that $\nu_f \in \mathcal{M}_r(G, X)$ is equivalent to $T^*_{\nu_f} \in \mathcal{L}(X', L^r(G))$ to finish the proof. ■

Corollary 6.2 *Let $1 \leq p < \infty$ and $\nu \in \mathcal{M}(G, X)$.*

(i) If $f \in L^p(G)$ and $g \in L^1(\nu)$ then $f *^\nu g \in P_p(G, X)$. Moreover

$$\|f *^\nu g\|_{P_p(G,X)} \leq \|f\|_{L^p(G)} \|g\|_{L^1(\nu)} (see[3]).$$

(ii) If $\nu \in \mathcal{M}_{p_1}(G, X)$, $g \in L^{p_2}(G)$ and $f \in L^{p_3}(G)$ with $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ and $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 1$ then $f *^\nu g \in P_r(G, X)$ for $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r'}$. Moreover

$$\|f *^\nu g\|_{P_r(G,X)} \leq \|\nu\|_{p_1, m_G} \|f\|_{L^{p_2}(G)} \|g\|_{L^{p_3}(G)}.$$

Proof. As mentioned in the introduction $f *^\nu g = f * \nu_g$. Now both cases follow combining Proposition 6.1 and Theorem 4.10. ■

From Theorem 5.10 when assuming that $\nu \in \mathcal{M}(G, X)$ is semivariation translation invariant we have $L^p(\nu) \subset L^p(G)$ for any $p \geq 1$. Hence, as pointed out in [7], we can consider the classical convolution

$$f *_G g(t) = \int_G f(t - s) g(s) dm_G(s)$$

between $f \in L^1(G)$ and $g \in L^p(\nu)$ and between $f \in L^p(\nu)$ and $g \in L^q(\nu)$.

Theorem 6.3 *Let $1 \leq p < \infty$ and let $\nu \in \mathcal{M}(G, X)$ semivariation translation invariant.*

(i) If $f \in L^1(G)$ and $g \in L^p(\nu)$ then $f *_G g \in L^p(\nu)$. Moreover

$$\|f *_G g\|_{L^p(\nu)} \leq \|f\|_{L^1(G)} \|g\|_{L^p(\nu)}. \tag{31}$$

(ii) If $f \in L^p(G)$ and $g \in L^1(\nu)$ then $f *_G g \in L^p(\nu)$. Moreover

$$\|f *_G g\|_{L^p(\nu)} \leq \|f\|_{L^p(G)} \|g\|_{L^1(\nu)}. \tag{32}$$

Proof. We first analyze the case $p = 1$. Note that for $f, g \in C(G)$ we have $s \to \tau_s g$ is continuous function with values in $L^1(\nu)$. We write the $L^1(\nu)$-valued Riemann integral

$$f *_G g = \int_G \tau_s g f(s) dm_G(s).$$

Using Minkowsky's inequality and Proposition 5.6 we get

$$\|f *_G g\|_{L^1(\nu)} \leq \int_G \|\tau_s g\|_{L^1(\nu)} |f(s)| dm_G(s) = \|f\|_{L^1(G)} \|g\|_{L^1(\nu)}.$$

To extend to general functions, we use that $C(G)$ is dense in $L^1(\nu)$ and in $L^1(G)$.

Assume now $p > 1$. As above we start with $g \in C(G)$ and use Hölder's inequality together with $L^1(\nu) \subset L^1(G)$ to have

$$|f *_G g(t)|^p \leq \min\{\|f\|^{p-1}_{L^1(G)} (|f| *_G |g|^p)(t), \|g\|^{p-1}_{L^1(\nu)} (|f|^p *_G |g|)(t)\}.$$

Therefore (i) follow from the case $p = 1$. Indeed,

$$\|f *_G g\|^p_{L^p(\nu)} \leq \|f\|^{p-1}_{L^1(G)} \||f| *_G |g|^p\|^p_{L^1(\nu)} \leq \|f\|^p_{L^1(G)} \||g|^p\|_{L^1(\nu)} = \|f\|^p_{L^1(G)} \|g\|^p_{L^p(\nu)}.$$

The case (ii) is analogue. ■

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