A space of projections on the Bergman space

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September 2, 2008

Abstract

We define a set of projections on the Bergman space A^2 parameterized by an affine closed space of a Banach space. This family is defined from an affine space of a Banach space of holomorphic functions in the disk and includes the classical Forelli-Rudin projections.

2000 Mathematics Subject Classification 46E20.

1 Introduction

Recall that the Bergman projection of $L^2(\mathbb{D})$ onto the holomorphic Bergman space $A^2 = L^2(\mathbb{D}) \cap \mathcal{H}(\mathbb{D})$, where $\mathcal{H}(\mathbb{D})$ denotes the space of holomorphic functions in the unit disk, is given by

$$P\varphi(z) = \int_{\mathbb{D}} \frac{\varphi(w)}{(1 - z\overline{w})^2} dA(w),$$

where dA is the normalized Lebesgue measure in the disk. Recall also the family of Forelli-Rudin projections parameterized by $\alpha > -1$

$$P_\alpha\varphi(z) = \int_{\mathbb{D}} (\alpha + 1) \left(\frac{1 - |w|^2}{1 - z\overline{w}} \right)^\alpha \frac{\varphi(w)}{(1 - z\overline{w})^2} dA(w)$$

*Partially supported by the spanish grant MTM2008-04594/MTM

†Partially supported by Conacyt-DAIC U48633-F.

which are the orthogonal projection of the weighted $L^2(\mathbb{D},(1-|w|)^\alpha dA(w))$ onto $\mathcal{H}(\mathbb{D})\cap L^2(\mathbb{D},(1-|w|)^\alpha dA(w))$. It is well known (see [6, Th. 7.1.4]) that P_α is a continuous projection of $L^2(\mathbb{D})$ onto A^2, for each $\alpha > -1/2$.

Since

$$\left\{\frac{1-|w|^2}{1-z\overline{w}}, z, w \in \mathbb{D}\right\} \subset \mathbb{D}_1$$

where $\mathbb{D}_1 = \{z : |z-1| < 1\}$, we may replace the function $g_\alpha(\zeta) = (\alpha+1)\zeta^\alpha$ in the definition of P_α by any holomorphic function g on $\mathbb{D}_1$ to obtain an operator T_g mapping the space $C_c(\mathbb{D})$ of compactly supported continuous functions defined on $\mathbb{D}$ into A^2. An equivalent formulation of the operators defined this way was given by Bonet, Engliš and Taskinen in [1] to construct continuous projections in weighted L^∞ spaces of $\mathbb{D}$ into $\mathcal{H}(\mathbb{D})$. The purpose of this paper is to study the space $\mathcal{P}$ of all holomorphic functions $g \in \mathbb{D}_1$, for which the corresponding operator T_g can be extended continuously to $L^2(\mathbb{D})$. In particular we study the set $\mathcal{P}_0$ of those functions $g \in \mathcal{P}$ that define continuous projections on A^2. For convenience in the notation we will translate the functions in $\mathcal{P}$ to the unit disk $\mathbb{D}$.

We will prove that $\mathcal{P}$ is a Banach space when we define the norm of $g \in \mathcal{P}$ as the operator norm of the operator T_g and that $\Phi(g) = \int_0^1 g(r)dr$ defines a bounded linear functional in $\mathcal{P}^*$. We give an analytic description of the elements of $\mathcal{P}$ and show that if $g \in \mathcal{P}$ then either T_g is identically zero on A^2 or it is a multiple of a continuous projection onto A^2, implying that $\mathcal{P}_0 = \Phi^{-1}(\{1\})$ is a closed affine subspace of $\mathcal{P}$.

As usual, for each $z \in \mathbb{D}$, ϕ_z will denote by ϕ_z the Möbius transform $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$ which satisfies $(\phi_z)^{-1} = \phi_z$ and $\phi_z'(w) = -\frac{1-|z|^2}{(1-\bar{z}w)^2}$. Throughout this paper we will write

$$\psi_z(w) = \frac{1-|w|^2}{1-z\bar{w}}$$

and

$$\mathbb{H} =\{z \in \mathbb{C} : \mathrm{Re}(z) > 1/2\}.$$

Clearly the mapping $z \to \frac{1}{1-z}$ is a bijection of $\mathbb{D}$ onto $\mathbb{H}$, and

$$\psi_z(w) = 1 - \bar{w}\phi_w(z). \tag{1}$$

2 A space of projections on A^2

Let us start by presenting our new definitions and spaces of projections.

Definition 1 *Let g be holomorphic in $\mathbb{D}$. We define*

$$T_g\varphi(z) = \int_{\mathbb{D}} g(\bar{w}\phi_w(z))\varphi(w)\frac{dA(w)}{(1-z\overline{w})^2},$$

for any $\varphi \in C_c(\mathbb{D})$.

We denote by $\mathcal{P}$ (respect. $\mathcal{P}_0$) the space of holomorphic functions $g \in \mathcal{H}(\mathbb{D})$ such that T_g extends continuously to $L^2(\mathbb{D})$ (respect. T_g is a projection on the Bergman space A^2).

We provide the space $\mathcal{P}$ with the norm $\|g\|_{\mathcal{P}} = \|T_g\|_{L^2(\mathbb{D})\to L^2(\mathbb{D})}$.

Remark 2 *In [1] it was introduced, for each F holomorphic in $\mathbb{H}$ the operator*

$$S_F\varphi(z) = \int_D F\left(\frac{1-z\overline{w}}{1-|w|^2}\right)\varphi(w)\frac{dA(w)}{(1-|w|^2)^2}.$$

We have that $T_g = S_F$, with $F(\eta) = \frac{1}{\eta^2}g(1-\frac{1}{\eta})$. We will say that such $F \in \mathcal{P}$ (respect. $\mathcal{P}_0$) if $g \in \mathcal{P}$ (respect. $\mathcal{P}_0$).

Example 3 *Let $g_\alpha(z) = (\alpha+1)(1-z)^\alpha$ for every $\alpha > -1$. Then $g_\alpha \in \mathcal{P}_0$ for $\alpha > -1/2$. In fact by (1) we have that $T_{g_\alpha} = P_\alpha$, which is a bounded projection from $L^2(\mathbb{D})$ into A^2 if and only if $\alpha > -1/2$.*

Example 4 *If $P(z) = \sum_{k=0}^N a_k z^k$ is a polynomial then $P \in \mathcal{P}$.*

Moreover $P \in \mathcal{P}_0$ if and only if $\sum_{k=0}^N \frac{a_k}{(k+1)} = \int_0^1 P(r)dr = 1$.

Proof. Write $P(z) = \sum_{k=0}^N b_k(1-z)^k$ where $b_k = (-1)^k\frac{P^{(k)}(1)}{k!}$. Hence

$$T_P = \sum_{k=0}^N \frac{b_k}{(k+1)}P_k.$$

This shows that $T_P \in \mathcal{P}$ and $\|P\|_{\mathcal{P}} \le \sum_{k=0}^N \frac{|b_k|}{(k+1)}\|P_k\|$. On the other hand $T_P \in \mathcal{P}_0$ if and only if $\sum_{k=0}^N \frac{b_k}{(k+1)} = 1$. Notice now that $\sum_{k=0}^N \frac{b_k}{(k+1)} = \int_0^1 P(r)dr$ to conclude the proof. ■

Example 5 *If $g \in \mathcal{H}(\mathbb{D})$ is such that $(1-z)^\alpha g(z)$ is bounded for some $\alpha > -1/2$ then $g \in \mathcal{P}$ and $\|g\|_{\mathcal{P}} \le C\sup_{|z|<1}|(1-z)^\alpha g(z)|$. In particular the space of bounded holomorphic functions $H^\infty(\mathbb{D})$ is contained in $\mathcal{P}$ and $\|f\|_{\mathcal{P}} \le C\|f\|_\infty$.*

Proof. Use that $P_\alpha^*\varphi(z) = \int_D \frac{(1-|w|^2)^\alpha}{|1-\bar{w}z|^{2+\alpha}}\varphi(w)dA(w)$ also defines a bounded operator on $L^2(\mathbb{D})$ (see [5, Theorem 1.9]). ∎

Proposition 6 *Let $g : \{z : |z-1| < 2\} \to \mathbb{C}$ be holomorphic such that $g(z) = \sum_{n=1}^\infty a_n(1-z)^n$ for $|z-1| < 2$.*

If $\sum_{n=0}^\infty \frac{2^n|a_n|}{(n+1)^{5/4}} < \infty$ then $g \in \mathcal{P}$ and

$$\|g\|_{\mathcal{P}} \leq C\sum_{n=0}^\infty \frac{2^n\,|a_n|}{(n+1)^{5/4}}.$$

Moreover, $g \in \mathcal{P}_0$ if and only if $\sum_{n=0}^\infty \frac{a_n}{n+1} = 1$.

Proof. Indeed, the norm $\|P_n\| = \frac{\sqrt{(2n)!}}{n!}$ (see [2, 3]). Then for $\varphi \in C_c(\mathbb{D})$

$$T_g\varphi(z) = \sum_{n=1}^\infty \frac{a_n}{(n+1)}P_n\varphi(z),$$

and

$$\|g\|_{\mathcal{P}} \leq \sum_{n=0}^\infty \frac{|a_n|\sqrt{(2n)!}}{(n+1)n!}.$$

Finally observe that, from Stirling's formula, $\frac{\sqrt{(2n)!}}{(n+1)n!} \sim \frac{2^n}{(n+1)^{1/4}}$.

To conclude the result note that $\sum_{n=0}^\infty \frac{|a_n|}{n+1} < \infty$ and

$$T_g\varphi(z) = (\sum_{n=1}^\infty \frac{a_n}{(n+1)})\varphi(z),$$

for $\varphi \in A^2$. ∎

Example 7 *Let $h_\beta(z) = A_\beta(1+z)^{-\beta}$ for $\beta > 0$ where $A_\beta = \frac{1-\beta}{2^{-\beta+1}-1}$ if $\beta \neq 1$ and $A_1 = (\log 2)^{-1}$. Then $h_\beta \in \mathcal{P}_0$ for $0 < \beta < 5/4$.*

Proof. Since, for $\beta > 0$, $\frac{1}{(1-w)^\beta} = \sum_{n=0}^\infty \beta_n w^n$ for $|w| < 1$, where $\beta_n \sim (n+1)^{\beta-1}$, we have that

$$h_\beta(z) = \frac{A_\beta}{2^\beta(1-(1-z)/2)^\beta} = \sum_{n=0}^\infty A_\beta 2^{-(n+\beta)}\beta_n(1-z)^n.$$

Now Proposition 6 implies that $h_\beta \in \mathcal{P}$.

Note that

$$1 = \int_1^2 A_\beta s^{-\beta} ds = \int_0^1 h_\beta(r) dr = \sum_{n=0}^{\infty} \frac{A_\beta 2^{-(n+1)}\beta_n}{n+1}.$$

Apply again Proposition 6 to finish the proof. ■

Let us now give some necessary conditions that functions g in $\mathcal{P}$ should satisfy.

Theorem 8 *If $g \in \mathcal{P}$ then*

$$\sup_{z\in\mathbb{D}} \left\{ \int_{\mathbb{D}} |g(\bar{w}\phi_w(z))|^2 \, dA(w) \right\}^{1/2} \leq 2 \|g\|_{\mathcal{P}}, \tag{2}$$

$$\left(\int_0^1 |g(r)|^2 dr \right)^{1/2} \leq 2\|g\|_{\mathcal{P}}, \tag{3}$$

$$\left(\int_0^1 \Big(\int_{\mathbb{D}} \frac{|g(ru))|^2}{|1-ru|^4} dA(u) \Big) (1-r^2)^2 r dr \right)^{1/2} \leq 2 \|g\|_{\mathcal{P}}. \tag{4}$$

Proof. If $g \in \mathcal{P}$ and $\varphi \in C_c(\mathbb{D})$ one has $T_g\varphi \in A^2$. Hence for each $z \in \mathbb{D}$

$$|T_g\varphi(z)| \leq \frac{\|T_g\varphi\|_2}{(1-|z|)} \leq \frac{\|g\|_{\mathcal{P}} \|\varphi\|_2}{(1-|z|)}.$$

Therefore

$$\left| \int_{\mathbb{D}} g(\bar{w}\phi_w(z)) \varphi(w) \frac{dA(w)}{(1-z\overline{w})^2} \right| \leq \frac{\|g\|_{\mathcal{P}} \|\varphi\|_2}{(1-|z|)}.$$

Then by duality,

$$\left\{ \int_{\mathbb{D}} |g(\bar{w}\phi_w(z))|^2 \frac{dA(w)}{|1-z\overline{w}|^4} \right\}^{1/2} \leq \frac{\|g\|_{\mathcal{P}}}{(1-|z|)} \leq 2 \frac{\|g\|_{\mathcal{P}}}{(1-|z|^2)}. \tag{5}$$

Let us show the following formula:

$$\overline{\phi_z(u)} \phi_{\phi_z(u)}(z) = u\overline{\phi_u(z)}. \tag{6}$$

Indeed, since

$$1-|\phi_z(u)|^2=\frac{(1-|z|^2)(1-|u|^2)}{|1-\bar{z}u|^2},$$

then

$$\psi_z(\phi_z(u))=\frac{1-|\phi_z(u)|^2}{1-\overline{\phi_z(u)}z}=\frac{(1-|u|^2)}{(1-\bar{z}u)}=\overline{\psi_z(u)}. \tag{7}$$

Now (6) follows from (1) and (7)

$$\overline{\phi_z(u)}\phi_{\phi_z(u)}(z)=1-\psi_z(\phi_z(u))=u\overline{\phi_u(z)}. \tag{8}$$

Changing the variable $u=\phi_z(w)$ in (5) and using (6) we obtain

$$\left\{\int_{\mathbb{D}}\left|g\left(u\overline{\phi_u(z)}\right)\right|^2 dA(u)\right\}^{1/2}\leq 2\left\|f\right\|_{\mathcal{P}}.$$

Now replacing u and $\bar{z}$ by $\bar{w}$ and z respectively the inequality (2) is achieved.

Part (3) follows selecting $z=0$ in (2).

Part (4) follows from (2) replacing the supremum by an integral over $\mathbb{D}$ and changing the variable $u=\phi_w(z)$,

$$\begin{aligned}\int_{\mathbb{D}}\int_{\mathbb{D}}\left|g\left(\bar{w}\phi_w(z)\right)\right|^2 dA(w)dA(z) &= \int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{\left|g\left(\bar{w}u\right)\right)\right|^2}{|1-\bar{w}u|^4}dA(u)\Big)(1-|w|^2)^2dA(w)\\ &= \int_{\mathbb{D}}\Big(\int_{\mathbb{D}}\frac{\left|g\left(|w|u\right)\right)\right|^2}{|1-|w|u|^4}dA(u)\Big)(1-|w|^2)^2dA(w)\\ &= \int_0^1\Big(\int_{\mathbb{D}}\frac{\left|g\left(ru\right)\right)\right|^2}{|1-ru|^4}dA(u)\Big)(1-r^2)^2rdr.\end{aligned}$$

■

Remark 9 *$(\mathcal{P},\|\cdot\|_{\mathcal{P}})$ is a normed space and $\Phi(g)=\int_0^1 g(r)dr\in\mathcal{P}^*$.*

Indeed, the only condition which needs a proof is the fact that $\|g\|_{\mathcal{P}}=0$ implies that $g=0$. It follows from (3) that if $\|g\|_{\mathcal{P}}=0$, then $g(r)=0$ for $0<r<1$. Hence by analytic continuation, $g(z)=0$ for $z\in\mathbb{D}$.

Notice also that (3) implies that $\|\Phi\|\leq 2$.

Remark 10 *The space $\mathcal{P}$ is not invariant under under rotations. Given $\theta \in [0, 2\pi)$ denote $R_\theta(f)(z) = f(e^{i\theta}z)$ for $f \in \mathcal{H}(\mathbb{D})$. Observe that $R_\theta T_g(\varphi) = T_g(R_\theta \varphi)$. However, that T_g is bounded in $L^2(\mathbb{D})$ does not imply that $T_{R_\theta g}$ is bounded in $L^2(\mathbb{D})$. For instance, the function $g(z) = (1+z)^{-1/2}$ belongs to $\mathcal{P}$, but by (3), its reflection $g(z) = (1-z)^{-1/2} \notin \mathcal{P}$.*

Let us now also give some necessary conditions to belong to the class $\mathcal{P}_0$.

Theorem 11 *If $g \in \mathcal{P}_0$ then*

$$\int_{\mathbb{D}} g(u\overline{\phi_u(z)})\psi(u)dA(u) = \psi(0) \tag{9}$$

for all $\psi \in A_2$ and $z \in \mathbb{D}$.

In particular,

(i) If $g \in \mathcal{P}_0$ then $\int_0^1 g(r)dr = 1$.

(ii) Let $S_2 = \{\bar{z}(1-|z|^2)\varphi(\bar{z}) : \varphi \in A^2\}$. If $g \in \mathcal{P}_0$ and $g' \in \mathcal{P}$ then $S_2 \subset Ker(T_{g'})$.

Proof. Assume

$$\int_{\mathbb{D}} g(\bar{w}\phi_w(z))\frac{\varphi(w)}{(1-\bar{w}z)^2}dA(w) = \varphi(z)$$

for all $\varphi \in A^2$.

Given $\psi \in A^2$ and $z \in D$, consider $\varphi(w) = \psi(\phi_z(w))\frac{(1-|z|^2)^2}{(1-\bar{z}w)^2}$. Clearly $\varphi \in A_2$ and $\|\varphi\|_2 = (1-|z|^2)\|\psi\|_2$. From the assumption,

$$\int_{\mathbb{D}} g(\bar{w}\phi_w(z))\psi(\phi_z(w))\frac{(1-|z|^2)^2}{|1-\bar{w}z|^4}dA(w) = \psi(0).$$

for all $\psi \in A^2$ and $z \in \mathbb{D}$.

Now changing the variable $u = \phi_z(w)$, and using (6), one gets

$$\int_{\mathbb{D}} g(u\overline{\phi_u(z)})\psi(u)dA(u) = \psi(0)$$

for all $\psi \in A_2$ and $z \in \mathbb{D}$. Finally changing u by $\bar{w}$ one obtains

$$\int_{\mathbb{D}} g(\bar{w}\phi_w(z))\psi(\bar{w})dA(w) = \psi(0) \tag{10}$$

for all $\psi \in A_2$ and $z \in \mathbb{D}$.

(i) follows selecting $\psi = 1$ and $z = 0$ in (10).

Differentiating in (10) with respect to z one obtains

$$\int_{\mathbb{D}} g'(\bar{w}\phi_w(z)) \frac{-\bar{w}(1-|w|^2)}{(1-\bar{w}z)^2} \psi(\bar{w}) dA(u) = T_{g'}(\psi_1) = 0$$

where $\varphi_1(u) = -\bar{u}(1-|u|^2)\varphi(\bar{u})$. Hence (ii) is finished. ■

Let us now show that $(\mathcal{P}, \|\cdot\|_{\mathcal{P}})$ is complete. For such a purpose, let us define $h_z : \mathbb{D} \to \mathbb{H}$ by

$$h_z(w) = \frac{1}{\psi_z(w)} = \frac{1-z\overline{w}}{1-|w|^2},$$

and let us mention that

$$\mathbb{D}_1 = \{\frac{1-|w|^2}{1-z\overline{w}} : z, w \in \mathbb{D}\} = \{\psi_z(w) : z, w \in \mathbb{D}\}.$$

Lemma 12 *For every $\xi \in \mathbb{H}$, there exist $0 \leq \alpha < 1$ and $w \in \mathbb{D}$ such that $\xi = h_\alpha(w)$ and h_α is an diffeomorfism of a neighborhood U of w onto an open neighborhood of ξ.*

Proof. For $0 \leq r, \alpha < 1$ fixed,

$$h_\alpha(re^{i\theta}) = \frac{1}{1-r^2} - \frac{r\alpha}{1-r^2}e^{-i\theta} \tag{11}$$

describes the circle $C_{r,\alpha}$ centered at the complex number $\frac{1}{1-r^2}$ with radius $\frac{r\alpha}{1-r^2}$.

Let $\xi \in \mathbb{H}$. To prove that $\xi \in h_\alpha(\mathbb{D})$ it is enough to see that $\xi \in C_{r,\alpha}$ for some $0 \leq r, \alpha < 1$.

Let

$$\beta = \frac{1}{r^2}\left[(1-r^2)^2|\xi|^2 + 1 - 2(1-r^2)\operatorname{Re}\xi\right] = \frac{|(1-r^2)\xi - 1|^2}{r^2}. \tag{12}$$

It is clear that $\beta \geq 0$ and

$$\beta < 1 \Leftrightarrow (1-r^2)|\xi|^2 + 1 < 2\operatorname{Re}\xi.$$

Also, since $\xi \in \mathbb{H}$, we have for some $\varepsilon > 0$ that $2\operatorname{Re}\xi > 1 + \varepsilon$. Hence if $|\xi|^2 < \frac{\varepsilon}{(1-r^2)}$ then $\beta < 1$. We conclude that there exists r_0 such that $0 \leq \beta < 1$ provided $r_0 < r < 1$. Then if $r_0 < r < 1$ and we let $\alpha = \sqrt{\beta}$ we have $0 \leq \alpha < 1$ and

$$\left|\xi - \frac{1}{1-r^2}\right| = \frac{r\alpha}{1-r^2},$$

that is $\xi \in C_{r,\alpha}$. Hence there exists θ_r and $0 \leq \alpha_r < 1$ such that $h_{\alpha_r}(re^{i\theta_r}) = \xi$.

To find θ_r explicitly, we let $\varphi_r = \pi - \theta_r$. From (11) we can write

$$\xi = \frac{1}{1-r^2} + \frac{r\alpha_r}{1-r^2}e^{i\varphi_r}.$$

Hence φ_r is the argument of ξ in polar coordinates centered at the complex number $\frac{1}{1-r^2}$. Then if $\frac{1}{1-r^2} \geq \operatorname{Re}(\xi)$,

$$\sin\theta_r = \sin\varphi_r = \frac{\operatorname{Im}(\xi)}{r\alpha_r}(1-r^2)$$

$$\begin{aligned} \cos\theta_r &= -\cos\varphi_r = \frac{(1-r^2)}{r\alpha_r}\left(\frac{1}{1-r^2} - \operatorname{Re}(\xi)\right) \\ &= \frac{1-(1-r^2)\operatorname{Re}(\xi)}{r\alpha_r}. \end{aligned} \tag{13}$$

Now we will prove that possibly except for a finite number of values of $r \geq r_0$, the jacobian matrix $Dh_{\alpha_r}(re^{i\theta_r})$ is not singular, where α_r and θ_r are chosen so that $h_{\alpha_r}(re^{i\theta_r}) = \xi$ as before. To this end, it is enough to see that the set of values of r such that the vectors

$$\frac{\partial h_{a_r}}{\partial\rho}(\rho e^{i\theta_r})_{|\rho=r} \text{ and } \frac{1}{r}\frac{\partial h_{a_r}}{\partial\theta}(re^{i\theta})_{|\theta=\theta_r} \tag{14}$$

are linearly dependent is finite.

We have

$$\frac{\partial h_a}{\partial\rho}(\rho e^{i\theta}) = \left(\frac{2\rho}{(1-\rho^2)^2} - \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\cos\theta, \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\sin\theta\right),$$

$$\frac{1}{\rho}\frac{\partial h_a}{\partial\theta}(\rho e^{i\theta}) = \left(\frac{\alpha}{(1-\rho^2)}\sin\theta, \frac{\alpha}{(1-\rho^2)}\cos\theta\right),$$

and the jacobian of h_α

$$Jh_\alpha(\rho e^{i\theta}) = \det\left[\frac{\partial h_a}{\partial \rho}(\rho e^{i\theta})|\frac{1}{\rho}\frac{\partial h_a}{\partial \theta}(\rho e^{i\theta})\right]$$

$$= \det\begin{bmatrix} \frac{2\rho}{(1-\rho^2)^2} - \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\cos\theta & \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\sin\theta \\ \frac{\alpha}{(1-\rho^2)}\sin\theta & \frac{\alpha}{(1-\rho^2)}\cos\theta \end{bmatrix}$$

$$= \frac{\alpha}{(1-\rho^2)^3}\left(2\rho\cos\theta - \alpha(1+\rho^2)\right). \tag{15}$$

If $2r\cos\theta_r - \alpha_r(1+r^2) = 0$, then multiplying this equation by $\alpha_r r^2$ we obtain

$$2r^2\alpha_r r\cos\theta_r - \alpha_r^2 r^2(1+r^2) = 0. \tag{16}$$

However, from (12) and (13) we see that $2r^2\alpha_r r\cos\theta_r - \alpha_r^2 r^2(1+r^2)$ is a polynomial of degree 6 in the variable r. We conclude that the vectors in (14) are linearly dependent for six values of r at the most and the proof of the lemma is complete. ■

Theorem 13 *$\mathcal{P}$ is a Banach space*

Proof. Let $g \in \mathcal{P}$ we have by Theorem 8 that

$$\sup_{z\in\mathbb{D}}\left\{\int_{\mathbb{D}}|g(\bar{w}\phi_w(z))|^2\,dA(w)\right\}^{1/2} \le 2\,\|g\|_{\mathcal{P}}\,. \tag{17}$$

Fix $\xi \in \mathbb{D}$. Since $\psi_z = 1/h_z$, then the local invertibility statement of Lemma 12 holds for the family of functions $1-\psi_z$ taking $\xi \in \mathbb{D}$, namely, there exist $\alpha \in (0,1)$, $w_\xi \in \mathbb{D}$ and open neighborhoods U and V of ξ and w_ξ respectively, such that $1-\psi_z$ is a diffeomorphism of V into U.

Hence

$$\left\{\int_U |g(u)|^2 \, dA(u)\right\}^{1/2} = \left\{\int_V |g(1-\psi_\alpha(w))|^2 \, |J\psi_\alpha(w)| \, dA(w)\right\}^{1/2}$$

$$\leq C(\xi)\left\{\int_V |g(\bar{w}\phi_w(\alpha))|^2 \, dA(w)\right\}^{1/2}$$

$$\leq C(\xi)\, \|g\|_{\mathcal{P}} \, .$$

It follows that

$$\left\{\int_K |g(u)|^2 \, dA(u)\right\}^{1/2} \leq C_K \, \|g\|_{\mathcal{P}} \, ,$$

for every compact set $K \subset \mathbb{D}$. This implies that

$$\sup_{u \in K} |g(u)| \leq \|g\|_{\mathcal{P}} \, C'_K. \tag{18}$$

If $\{g_n\}$ is a Cauchy sequence in $\mathcal{P}$, we have by (18) that $\{g_n\}$ converges to uniformly on compact sets of $\mathbb{D}$ to a holomorphic function g.

If $\varphi \in C_c(\mathbb{D})$, we have

$$T_{g_n}\varphi(z) \to T_g\varphi(z),$$

uniformly on $\mathbb{D}$ in $L^2(\mathbb{D})$. Since $\|g_n\|_{\mathcal{P}}$ is a bounded sequence then by the Fatou lemma it follows that

$$\|T_g\varphi\|_2 \leq M \, \|g\|_{\mathcal{P}} \, ,$$

and $g \in \mathcal{P}$. Also, from

$$\|T_{g_n}\varphi - T_{g_m}\varphi\|_2 \leq \|g_n - g_m\|_{\mathcal{P}} \, \|\varphi\|_2$$

we conclude that $T_{g_n} \to T_g$, namely $g_n \to g$ in $\mathcal{P}$. ■

3 Main results

Let us now describe the norm in $\mathcal{P}$ in a more explicit way. We shall use the formulation of the space given in [1].

Theorem 14 *Let $g \in \mathcal{H}(\mathbb{D})$ and put $F(\xi) = \frac{1}{\xi^2} g(1 - \frac{1}{\xi})$.*
Then $g \in \mathcal{P}$ if and only

$$\sup_j \frac{1}{j!\sqrt{j+1}} \left(\int_1^\infty [(x-1)x]^j \left| x F^{(j)}(x) \right|^2 dx \right)^{1/2} < \infty.$$

Proof. We use the expression

$$T_g \varphi(z) = \int_{\mathbb{D}} F\left(\frac{1 - z\overline{w}}{1 - |w|^2} \right) \varphi(w) \frac{dA(w)}{(1 - |w|^2)^2}.$$

Consider the space M of functions of the form

$$\varphi = \sum_{finite} \varphi_j(r) e^{ij\theta},$$

with $\varphi_j \in L^2((0,1), rdr)$. Then M is a dense subspace of $L^2(\mathbb{D})$.

For $z \in \mathbb{D}$ and $0 \le r < 1$ fixed, let $f(\zeta) = F\left(\frac{1 - rz\zeta}{1 - r^2}\right)$, which is holomorphic on $\overline{\mathbb{D}}$. We have .

$$f(\zeta) = F\left(\frac{1 - rz\zeta}{1 - r^2} \right) = \sum_{j \ge 0} \frac{1}{j!} \left(\frac{-rz}{1 - r^2} \right)^j F^{(j)}(\frac{1}{1 - r^2}) \zeta^j, |\zeta| \le 1.$$

Then for $g \in M$,

$$\int_0^{2\pi} f(re^{-i\theta}) \varphi(re^{i\theta}) \frac{d\theta}{2\pi} = \sum_{j \ge 0} \varphi_j(r) \frac{(-1)^j}{j!} \left(\frac{r}{1 - r^2} \right)^j F^{(j)}(\frac{1}{1 - r^2}) z^j,$$

Hence

$$T_g(\varphi)(z) = \sum_{j \ge 0} \gamma_j(\varphi_j) \sqrt{j+1} z^j, \tag{19}$$

where γ_j is the functional in $L^2((0,1), rdr)$ defined by

$$\gamma_j(\varphi) = \frac{(-1)^j}{\sqrt{j+1} j!} \int_0^1 \varphi(r) \left(\frac{r}{1 - r^2} \right)^j F^{(j)}(\frac{1}{1 - r^2}) \frac{r}{(1 - r^2)^2} dr.$$

Using the normalized Lebesgue measure dA, the set $\{\sqrt{j+1} z^j\}$ is an orthonormal basis for A^2, so we conclude that T_g is bounded in $L^2(\mathbb{D})$ if and only if

$$\left\|(\gamma_j(\varphi_j))_{j\geq 0}\right\|_{\ell^2} \leq C\|\varphi\|_{L^2(\mathbb{D})}$$

$$= C\left(\sum_j \int |\varphi_j(r)|^2 r dr\right)^{1/2}.$$

Using duality, this will hold if and only if

$$\sup_{j\geq 0}\frac{1}{\sqrt{j+1}j!}\left(\int_0^1\left(\frac{r}{1-r^2}\right)^{2j}\left|F^{(j)}(\frac{1}{1-r^2})\right|^2\frac{rdr}{(1-r^2)^4}\right)^{1/2} < \infty. \quad (20)$$

Letting the change of variables $x = \frac{1}{1-r^2}$, the integrals above equal

$$\frac{1}{2}\int_1^\infty [(x-1)x]^j \left|xF^{(j)}(x)\right|^2 dx$$

and the proof is complete. ■

We can now give an alternative proof of a well know result.

Corollary 15 *P_α is bounded on $L^2(\mathbb{D})$ if and only if $\alpha > -1/2$.*

Proof. Consider $g_\alpha(z) = (1-z)^\alpha$. Assume first that $g_\alpha \in \mathcal{P}$. Then (3) in Theorem 8 implies that $\int_0^1 (1-r)^{2\alpha} dr < \infty$ and therefore $\alpha > -1/2$.

Assume now that $\alpha > -1/2$. Since $F_\alpha(\xi) = \xi^{-m}$ with $m = 2+\alpha$ and $2m-3 > 0$, one has for $j \geq 0$ that

$$F_\alpha^{(j)}(x) = (-1)^j m(m+1)...(m+j-1)x^{-(m+j)} = (-1)^j\frac{\Gamma(m+j)}{\Gamma(m)}x^{-(m+j)}.$$

Therefore

$$\begin{aligned}
\int_1^\infty [(x-1)x]^j \left|xF_\alpha^{(j)}(x)\right|^2 dx &= \int_1^\infty (1-\frac{1}{x})^j (x^{j+1}F_\alpha^{(j)}(x))^2 dx \\
&= (\frac{\Gamma(m+j)}{\Gamma(m)})^2 \int_1^\infty (1-\frac{1}{x})^j x^{-2m+4}\frac{d}{x^2} \\
&= (\frac{\Gamma(m+j)}{\Gamma(m)})^2 \int_0^1 (1-r)^j r^{2m-4} dr \\
&= (\frac{\Gamma(m+j)}{\Gamma(m)})^2 B(2m-3, j+1).
\end{aligned}$$

Using that $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ one concludes that

$$\frac{1}{(j!)^2(j+1)} \int_1^\infty [(x-1)x]^j \left| x F_\alpha^{(j)}(x) \right|^2 dx = \frac{B(2m-3, j+1)}{B(m,j)^2 j^2 (j+1)}.$$

Finally since for p fixed, $B(p,j) \sim j^{-p}$ one obtains that

$$\frac{B(2m-3, j+1)}{B(m,j)^2 j^2 (j+1)} \sim 1.$$

■

Example 16 *In Example 7 it was shown that, for* $0 < \beta < 5/4$, $g(z) = (1+z)^{-\beta} \in \mathcal{P}$ *(which corresponds to* $F(\xi) = \frac{\xi^{\beta-2}}{(2\xi-1)^2}$*).*

Let us show for instance that $g(z) = (1+z)^{-2} \notin \mathcal{P}$.

In this case $F(\xi) = \frac{1}{(2\xi-1)^2}$ *and*

$$F^{(j)}(\xi) = \frac{(-1)^j (j+1)! 2^j}{(2\xi - 1)^{2+j}}.$$

Since $\frac{x}{2} \leq x - 1 \leq x$ *for* $x \geq 2$ *we have*

$$\Big(\int_2^\infty (x(x-1))^j |xF^{(j)}(x)|^2 dx \Big)^{1/2} \sim 2^j (j+1)! \Big(\int_2^\infty \frac{x^{2j+2}}{(2x-1)^{4+2j}} dx \Big)^{1/2} \sim 2^j (j+1)!$$

Hence the condition in Theorem 14 does not hold.

The conditions

$$\sup_{j \geq 0} \frac{1}{j!} \int_1^\infty \left| (x-1)^j F^{(j)}(x) \right| dx < \infty, \tag{21}$$

$$\lim_{x \to \infty} x^{j+1} F^{(j)}(x) = 0 \tag{22}$$

were introduced in [1]. These conditions imply that on the space of all the holomorphic functions φ such that $S_F \varphi$ is well defined, the operator S_F is a constant multiple of the identity . Now we will see that (21) and (22) hold for every $g \in \mathcal{P}$ what allows to show the following result.

Theorem 17 *Let* $g \in \mathcal{P}$ *and* $c_0 = \int_0^1 g(r) dr$. *Then*

$$T_g(\varphi) = c_0 \varphi, \quad \varphi \in A^2.$$

Proof. Let us notice first that $(x-1)^j F^{(j)}(x) \in L^1([1,\infty), dx)$ for $j \geq 0$. Indeed,

$$\begin{aligned} & \int_1^\infty |x-1|^j \left|F^{(j)}(x)\right| dx \\ = & \int_1^\infty |x(x-1)|^j \left|xF^{(j)}(x)\right| \frac{dx}{x^{j+1}} \\ \leq & \Big(\int_1^\infty (x(x-1))^j \left|xF^{(j)}(x)\right|^2 dx\Big)^{1/2} \Big(\int_1^\infty \frac{(x(x-1))^j}{x^{2j+2}} dx\Big)^{1/2} \\ = & \Big(\int_1^\infty (x(x-1))^j \left|xF^{(j)}(x)\right|^2 dx\Big)^{1/2} \Big(\int_0^1 (1-r)^j \, dr\Big)^{1/2} \\ = & \frac{1}{\sqrt{j+1}} \Big(\int_1^\infty |x(x-1)|^j \left|xF^{(j)}(x)\right|^2 dx\Big)^{1/2} \leq Cj! \|g\|_{\mathcal{P}}. \end{aligned}$$

Applying (19) in Theorem 14 to $\varphi(z) = \sum_{j=0}^N a_j z^j$ one obtains

$$T_g \varphi = \sum_{j=0}^N c_j a_j z^j, \tag{23}$$

and

$$c_j = \frac{(-1)^j}{j!} \int_1^\infty (x-1)^j F^{(j)}(x) dx,$$

where c_j is well defined.

As in [1, Th. 1] we have by integration by parts

$$c_j - c_{j+1} = \frac{(-1)^j}{(j+1)!} \lim_{x\to\infty} (1-x)^{j+1} F^{(j)}(x).$$

Let us now show that $\lim_{x\to\infty}(1-x)^{j+1}F^{(j)}(x) = 0$.

Note first that $(x-1)^{j+1}F^{(j)}(x) \in L^2([1,\infty), dx)$ for $j \geq 0$. Indeed

$$\int_1^\infty |(x-1)^{j+1}F^{(j)}(x)|^2 dx \leq \int_1^\infty |x(x-1)|^j |xF^{(j)}(x)|^2 dx \leq C(j+1)(j!)^2. \tag{24}$$

In particular $(x-1)^j F^{(j)}(x) \in L^2([1,\infty), dx)$ for $j \geq 1$. From Cauchy-Schwarz and the previous estimates one has that if $f_j(x) = [(x-1)^{j+1}F^{(j)}(x)]^2$ then $(f_j)' \in L^1([1,\infty))$ for every $j \geq 0$.

Therefore writing

$$[(x-1)^{j+1}F^{(j)}(x)]^2 = \int_1^x (f_j)'(y)dy$$

we see that the $\lim_{x\to\infty}((x-1)^{j+1}F^{(j)}(x))^2$ exists and by (24) it vanishes for all j.

Hence (23) becomes $T_g(\varphi) = c_0\varphi$ where

$$c_0 = \int_1^\infty F(x)dx = \int_1^\infty g(1-\frac{1}{x})\frac{dx}{x^2} = \int_0^1 g(r)dr.$$

■

Corollary 18 *Let $g \in \mathcal{P}$. Then $A^2 \subset KerT_g$ if and only if $\int_0^1 g(r)dr = 0$.*

Corollary 19 *Let $\Phi(g) = \int_0^1 g(r)dr$ for $g \in \mathcal{P}$. Then $\mathcal{P}_0 = \Phi^{-1}(\{1\})$.*

Corollary 20 *Let $g \in \mathcal{P}$. If T_g is not identically zero in A^2 then there exists $\lambda \neq 0$ and $g_0 \in g \in \mathcal{P}_0$ such that $g = \lambda g_0$.*

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