FOURIER ANALYSIS WITH RESPECT TO BILINEAR MAPS

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Abstract. Several results about convolution and Fourier coefficients for X-valued functions defined on the torus satisfying $\sup_{\|y\|=1} \int_{-\pi}^{\pi} \|B(f(e^{i\theta}), y)\| \frac{d\theta}{2\pi} < \infty$ for a bounded bilinear map $B : X \times Y \to Z$ are presented and some applications are given.

1. Introduction and notation

Let $(\mathbb{T}, m)$ be the Lebesgue measure space over $\mathbb{T} = \{|z| = 1\}$, let X be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). An X-valued function $f : \mathbb{T} \to X$ is said to be strongly measurable if there exits a sequence of simple functions, $(s_n) \in \mathcal{S}(\mathbb{T}, X)$, which converges to f a.e. It is called weakly measurable if $\langle f, x^*\rangle$ is measurable for any $x^* \in X^*$. We denote by $L^0(\mathbb{T}, X)$ and $L^0_{weak}(\mathbb{T}, X)$ the spaces of strongly and weakly measurable functions. As usual we denote by $P^p(\mathbb{T}, X)$ the Pettis p-integrable functions and by $L^p(\mathbb{T}, X)$ the Bochner p-integrable functions for $1 \leq p < \infty$.

Convolutions with respect to bilinear maps were introduced and studied in [4, 5] in the setting of Bochner integrable functions:

Let Y and Z be a Banach spaces and let $B : X \times Y \to Z$ be a bounded bilinear map. If $f \in L^1(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}, Y)$ then the map $e^{i\theta} \to B(f(e^{i(t-\theta)}), g(e^{i\theta}))$ is strongly measurable for each t and the fact

$$\|B(f(e^{i(t-\theta)}), g(e^{i\theta})\| \leq \|B\| \|f(e^{i(t-\theta)})\| \|g(e^{i\theta})\|$$

allows to define

$$f *_B g(e^{it}) = \int_{-\pi}^{\pi} B(f(e^{i(t-\theta)}, g(e^{i\theta})) \frac{d\theta}{2\pi} \in L^1(\mathbb{T}, Z)$$

and $\|f *_B g\|_{L^1(\mathbb{T},Z)} \leq \|f\|_{L^1(\mathbb{T},X)} \|g\|_{L^1(\mathbb{T},Y)}$.

Also it is clear that $\hat{f}(n) = \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$ is well defined (as Bochner integral) for $n \in \mathbb{Z}$ and $f \in L^1(\mathbb{T}, X)$.

Actually the following formula holds (see [4, 5]) for $f \in L^1(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}, Y)$,

$$(f *_B g)\hat{}(n) = B(\hat{f}(n), \hat{g}(n)).$$

In this paper we shall try to developp the theory for a wider class of functions integrable with respect the bilinear map that has been recently considered by the authors in [7], and which allows to extend the results in [4, 5].

Given a bounded bilinear map $B : X \times Y \to Z$, we shall be denoting by $B_x \in \mathcal{L}(Y, Z)$ and $B^y \in \mathcal{L}(X, Z)$ the corresponding linear operators $B_x(y) = B(x, y)$ and

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$B^y(x) = B(x,y)$. The following notions were introduced in [7]: A triple (Y,Z,B) is admissible for X if Y and Z are Banach spaces and $B : X \times Y \to Z$ is a bounded bilinear map such that $x \to B_x$ is injective from $X \to \mathcal{L}(Y,Z)$, i.e. $B(x,y) = 0$ for all $y \in Y$ implies $x = 0$. X is said to be a (Y,Z,B)-normed space if there exists $C > 0$ such that $\|x\| \leq C\|B_x\|$ for all $x \in X$, that is X can be understood as a subspace of $\mathcal{L}(Y,Z)$ with some equivalent norm.

Also we define the "adjoints" $B^* : X \times Z^* \to Y^*$ and $B_* : Y \times Z^* \to X^*$ by the formulas

$$\langle B^*(x,z^*), y\rangle = \langle B(x,y), z^*\rangle, \tag{1}$$

$$\langle B_*(y,z^*), x\rangle = \langle B(x,y), z^*\rangle. \tag{2}$$

Hence $(B^*)_x = (B_x)^*$ and $(B_*)_y = (B^y)^*$.

Clearly (Y,Z,B) is admissible for X if and only if (Z^*,Y^*,B^*) is. Observe that X is (Y,Z,B) normed if and only if there exists $C_1, C_2 > 0$ such that

$$C_1 \leq \sup_{\|x\|=\|y\|=\|z^*\|=1} |\langle B(x,y), z^*\rangle| \leq C_2.$$

Therefore X is (Y,Z,B) normed only if X is (Z^*,Y^*,B^*) normed if and only if Y is (Z^*,X^*,B_*) normed.

Throughout the paper we always assume that X is (Y,Z,B) normed. Our aim is to show that some of the results from vector-valued Fourier Analysis can be extended to more general functions and bilinear maps.

As in [7] we say that $f : \mathbb{T} \to X$ is (Y,Z,B)-measurable if $B(f,y) \in L^0(\mathbb{T},Z)$ for any $y \in Y$ and denote the class of such functions by $L^0_B(\mathbb{T},X)$.

For $1 \leq p < \infty$ and a simple function $s = \sum_{k=1}^n x_k\chi_{A_k}$ one has that

$$\begin{aligned}
\|s\|_{L^p_B(X)} &= \sup_{\|y\|=1} \|B^y(s)\|_{L^p(Z)} \\
&= \sup_{\|y\|=1} (\sum_{k=1}^n \|B(x_k,y)\|^p \mu(A_k))^{1/p} \\
&= \sup\{\| \sum_{k=1}^n B^*(x_k,z_k^*)\mu(A_k)\| : (\sum_{k=1}^n \|z_k^*\|^{p'})^{1/p'} = 1\}.
\end{aligned}$$

We define $L^p_B(\mathbb{T},X)$ as the closure of simple functions $\mathcal{S}(\mathbb{T},X)$ under this norm. Of course $L^p(\mathbb{T},X) \subset L^p_B(\mathbb{T},X)$ and $\|f\|_{L^p_B(X)} \leq \|f\|_{L^p(X)}$ for any $f \in L^p(\mathbb{T},X)$. In particular $L^p_B(\mathbb{T},X)$ for the cases $\mathcal{D} : X \times X^* \to \mathbb{K}$ given by $\mathcal{D}(x,x^*) = \langle x,x^*\rangle$ and $\mathcal{B} : X \times \mathbb{K} \to X$ given by $\mathcal{B}(x,\lambda) = \lambda x$ correspond to $P^p(\mathbb{T},X)$ and $L^p(\mathbb{T},X)$ respectively.

The reader is referred to [7] for some general facts about the theory on these spaces. It is shown there that, under the assumption of X being a (Y,Z,B)-normed space, one obtains that $L^1_B(\mathbb{T},X) \subseteq P^1(\mathbb{T},X)$ and also the existence of the B-integral over sets E for functions in $L^1_B(\mathbb{T},X)$. There are some general examples to have in mind where the general theory can be applied.

Example 1.1. *Let $X = \mathcal{L}(Y,Z)$ for some Banach spaces Y, Z. Define*

$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y,Z) \times Y \to Z, \qquad \mathcal{O}_{Y,Z}(T,y) = T(y). \tag{3}$$

Clearly one has $(\mathcal{O}_{Y,Z})^*(T,z^*) = \mathcal{O}_{Z^*,Y^*}(T^*,z^*)$.

If $f:\mathbb{T}\to\mathcal{L}(Y,Z)$, defined by $f(e^{it})=T_t$, belongs to $L^1_{\mathcal{O}_{Y,Z}}(\mathbb{T},\mathcal{L}(X,Y))$ then

$$\|f\|_{L^1_{\mathcal{O}_{Y,Z}}(\mathcal{L}(X,Y))}=\sup_{\|y\|=1}\int_{-\pi}^{\pi}\|T_\theta(y)\|\frac{d\theta}{2\pi}$$

and there exists $T\in\mathcal{L}(X,Y)$ such that $T(y)=\int_{-\pi}^{\pi}T_\theta(y)\frac{d\theta}{2\pi}$.

Example 1.2. *(Hölder's bilinear map) Let (Ω,η) be a σ-finite measure space, $1\le p_1,p_2\le\infty$ and $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}$. Consider*

$$\mathcal{H}_{p_1,p_2}:L^{p_1}(\eta)\times L^{p_2}(\eta)\to L^{p_3}(\eta),\qquad (f,g)\to fg.$$

It was shown in [7] that for $\Omega=\mathbb{N}$ with the counting measure then

$$\|f\|_{L^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})}=\|(f_n)\|_{\ell_{p_1}(L^{p_3})}$$

where $f=(f_n)\in L^0(\mathbb{T},\ell^{p_1})$.

Example 1.3. *(Young's bilinear map) Let G be locally compact abelian group and m the Haar measure, $1\le p_1,p_2\le\infty$ with $\frac{1}{p_1}+\frac{1}{p_2}\ge1$ and $\frac{1}{p_3}=\frac{1}{p_1}+\frac{1}{p_2}-1$. Consider*

$$\mathcal{Y}_{p_1,p_2}:L^{p_1}(G)\times L^{p_2}(G)\to L^{p_3}(G),\qquad (f,g)\to f*g.$$

It was shown in [7] that $L^p(\mathbb{R})$ is $(L^1(\mathbb{R}),L^p(\mathbb{R}),\mathcal{Y}_{p,1})$-normed whenever $L^1(G)$ has a bounded approximation of the identity. However $(L^2(\mathbb{R}),L^2(\mathbb{R}),\mathcal{Y}_{1,2})$ is an admissible triple for $L^1(\mathbb{R})$, but $L^1(\mathbb{R})$ is not $(L^2(\mathbb{R}),L^2(\mathbb{R}),\mathcal{Y}_{1,2})$-normed.

Also for $G=\mathbb{R}$ with the Lebesgue measure it is easy to show that

$$\|f\|_{L^p_{\mathcal{Y}_{p_1,1}}(L^{p_1}(\mathbb{R}))}=\|f\|_{L^p(L^{p_1}(\mathbb{R}))}$$

for any $f\in L^0(\mathbb{T},L^{p_1}(\mathbb{R}))$.

2. Fourier Analysis with respect to bilinear maps.

We denote by $\mathcal{P}(\mathbb{T},X)$ the space of X-valued trigonometric polynomials. It is clear that $\mathcal{P}(\mathbb{T},X)$ is dense in $L^p_B(\mathbb{T},X)$.

We start by pointing out a result which will be used in the sequel.

Proposition 2.1. *(see [7]) If $f\in L^1_B(\mathbb{T},X)$ and $E\in\Sigma$ there exists a unique $x_E\in X$ such that for any $y\in Y$*

$$B(x_E,y)=\int_E B(f,y)d\mu.$$

The value $x_E=(B)\int_E fd\mu$ is called the B-integral of f over E.

Of course $(B)\int_E fd\mu$ coincides always with the Pettis integral, and in the case of Bochner integrable functions then $(B)\int_E fd\mu=\int_E fd\mu$ is the Bochner integral.

It is clear that if $f\in L^1_B(\mathbb{T},X)$ and $\varphi\in L^\infty(\mathbb{T})$ then $f\varphi\in L^1_B(\mathbb{T},X)$. Hence Proposition 2.1 allows to give the following definitions.

Definition 2.2. *Let $n\in\mathbb{Z}$ and $f\in L^1_B(\mathbb{T},X)$. Define the n-Fourier coefficient with respect to B as*

$$\hat{f}^B(n)=(B)\int_{-\pi}^{\pi}f(e^{i\theta})e^{-in\theta}\frac{d\theta}{2\pi}.$$

Hence

$$B(\hat{f}^B(n),y)=\int_{-\pi}^{\pi}B(f(e^{i\theta}),y)e^{-in\theta}\frac{d\theta}{2\pi}=(B^y(f))\hat{}(n).$$

Of course if $f \in L^1(\mathbb{T}, X)$ then $\hat{f}^B(n) = \hat{f}(n)$ for all $n \in \mathbb{Z}$. In particular if $f \in \mathcal{P}(\mathbb{T}, X)$ with $f(e^{i\theta}) = \sum_{k=-N}^{M} x_k e^{ik\theta}$ then $\hat{f}^B(n) = x_n$ for $n \in [-N, M]$ and $\hat{f}^B(n) = 0$ otherwise.

Proposition 2.3. *If $f \in L^1_B(\mathbb{T}, X)$ then $(\hat{f}^B(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}, X)$. Moreover*

$$\|\hat{f}^B(n)\| \leq C\|f\|_{L^1_B(\mathbb{T},X)}.$$

Proof. Using that X is (Y, Z, B)-normed one has

$$\|\hat{f}^B(n)\| \leq C \sup_{\|y\|=1} \|B(\hat{f}^B(n), y))\| \leq C\|f\|_{L^1_B(\mathbb{T},X)}.$$

The standard approximation for polynomials show that $(\hat{f}^B(n))_{n\in\mathbb{Z}} \in c_0(\mathbb{Z}, X)$. □

Let us denote $f_t(e^{i\theta}) = f(e^{i(t-\theta)})$ for $f \in L^p_B(\mathbb{T}, X)$. It is obvious that $f_t \in L^p_B(\mathbb{T}, X)$ and $\|f_t\|_{L^p_B(\mathbb{T},X)} = \|f\|_{L^p_B(\mathbb{T},X)}$.

Definition 2.4. *Let $f \in L^1_B(\mathbb{T}, X)$ and $\varphi \in L^\infty(\mathbb{T})$. Define the convolution with respect to B by*

$$f *^B \varphi(e^{it}) = (B)\int_{-\pi}^{\pi} f_t(e^{i\theta})\varphi(e^{i\theta})\frac{d\theta}{2\pi}, \quad e^{it} \in \mathbb{T}.$$

Hence

$$B(f *^B \varphi(e^{it}), y) = \int_{-\pi}^{\pi} B(f(e^{i(t-\theta)}), y)\varphi(e^{i\theta})\frac{d\theta}{2\pi} = B^y(f) * \varphi(e^{it})$$

for any trigonometric polynomial φ.

Proposition 2.5. *If $f \in L^1_B(\mathbb{T}, X)$ and $\varphi \in L^\infty(\mathbb{T})$ then*

$$\|f *^B \varphi\|_{L^1_B(\mathbb{T},X)} \leq \|f\|_{L^1_B(\mathbb{T},X)}.\|\varphi\|_{L^1(\mathbb{T})}.$$

Proof.

$$\begin{aligned}\int_{-\pi}^{\pi} \|B(f *^B \varphi(e^{it}), y)\|\|\frac{dt}{2\pi} &\leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \|B(f_t(e^{i\theta})\varphi(e^{i\theta}), y)\|\frac{d\theta}{2\pi}\frac{dt}{2\pi} \\ &\leq \int_{-\pi}^{\pi}(\int_{-\pi}^{\pi} \|B(f_t(e^{i\theta}), y)\|\frac{d\theta}{2\pi})|\varphi(e^{it})|\frac{dt}{2\pi} \\ &\leq \|f\|_{L^1_B(\mathbb{T},X)}\|\varphi\|_{L^1(\mathbb{T})}.\end{aligned}$$

□

This allows to give the following definition.

Definition 2.6. *If $f \in L^1_B(\mathbb{T}, X)$ and $\varphi \in L^1(\mathbb{T})$ we define the convolution*

$$f *^B \varphi = \lim_n f *^B \varphi_n$$

for any sequence of polynomials φ_n converging to $\varphi \in L^1(\mathbb{T})$.

Of course $f *^B \varphi \in L^1_B(\mathbb{T}, X)$ and $\|f *^B \varphi\|_{L^1_B(\mathbb{T},X)} \leq \|f\|_{L^1_B(\mathbb{T},X)}\|\varphi\|_{L^1(\mathbb{T})}$.

Remark 2.7. *If $f \in L^1_B(\mathbb{T}, X)$, $\varphi \in L^1(\mathbb{T})$ and $y \in Y$ then*

$$B(f *^B \varphi, y) = B^y(f) * \varphi.$$

We now give the connection between convolution and Fourier coefficients.

Proposition 2.8. *Let $n \in \mathbb{Z}$, $f \in L^1_B(\mathbb{T}, X)$ and $\varphi \in L^1(\mathbb{T})$ then*

$$(f *^B \varphi)\hat{}^B(n) = \hat{f}^B(n)\hat{\varphi}(n).$$

Proof. Assume first that $\varphi(t) = e^{imt}$ for some $m \in \mathbb{Z}$. Then

$$\begin{aligned}(f *^B \varphi)(e^{it}) &= (B)\int_{-\pi}^{\pi} f_t(e^{i\theta})e^{mi\theta}\frac{d\theta}{2\pi} \\ &= (B)\int_{-\pi}^{\pi} f(e^{i(t-\theta)})e^{mi\theta}\frac{d\theta}{2\pi} \\ &= (B)\int_{-\pi}^{\pi} f(e^{i\theta})e^{mi(t-\theta)}\frac{d\theta}{2\pi} \\ &= e^{imt}\hat{f}^B(m).\end{aligned}$$

This shows the result in this particular case. Now by linearity one gets the result for polinomials φ. Finally using Propositions 2.3 and 2.5 one extends to general functions $\varphi \in L^1(\mathbb{T})$.

□

Let us now extend the notion of convolution between two different vector-valued functions.

Definition 2.9. *Let $f \in L^1_B(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}) \otimes Y$, say $g = \sum_{k=0}^{M} y_k\phi_k$ where $y_k \in Y$ and $\phi_k \in L^1(\mathbb{T})$. Define the convolution*

$$f *_B g = \sum_{k=0}^{M} B(f *^B \phi_k, y_k).$$

Remark 2.10. *In particular $B(f *^B \phi, y) = f *_B (\phi \otimes y) = B^y(f) * \phi$ for $\phi \in L^1(\mathbb{T})$ and $y \in Y$.*

Proposition 2.11. *If $f \in L^1_B(\mathbb{T}, X)$ and $g \in \mathcal{P}(\mathbb{T}, Y)$ then*

$$f *_B g(e^{it}) = \int_{-\pi}^{\pi} B(f(e^{i(t-\theta)}), g(e^{i\theta}))\frac{d\theta}{2\pi}.$$

Proof. Take $g = \sum_{k=-N}^{M} \phi_k \otimes y_k$ where $y_k \in Y$ and $\phi_k(e^{it}) = e^{ikt}$. Apply Remark 2.10 to obtain

$$\begin{aligned}f *_B g(e^{it}) &= \sum_{k=-N}^{M} f *_B (\phi_k \otimes y_k)(e^{it}) \\ &= \sum_{k=-N}^{M} \int_{-\pi}^{\pi} B(f(e^{i(t-\theta)}), y_k)\phi_k(e^{i\theta})\frac{d\theta}{2\pi} \\ &= \int_{-\pi}^{\pi} B(f(e^{i(t-\theta)}), g(e^{i\theta}))\frac{d\theta}{2\pi}.\end{aligned}$$

□

Proposition 2.12. *Let $f \in L^1_B(\mathbb{T}, X)$ and $g \in \mathcal{S}(\mathbb{T}, Y)$. Then*

$$\|f *_B g\|_{L^1(Z)} \le \|f\|_{L^1_B(X)}\|g\|_{L^1(Y)}.$$

Proof. Assume $g = \sum_{k=0}^{M} y_k\phi_k$ where $\phi_k = \chi_{I_k}$ for pairwise disjoint intervals. Hence

$$\begin{aligned}\|f *_B g\|_{L^1(Z)} &\leq \sum_{k=0}^{M} \|B(f *^B \phi_k, y_k)\|_{L^1(Z)} \\ &= \sum_{k=0}^{M} \|B^{y_k}(f) * \phi_k\|_{L^1(Z)} \\ &= \sum_{k=0}^{M} \|B^{y_k}(f)\|_{L^1(Z)}\|\phi_k\|_{L^1} \\ &= \sum_{k=0}^{M} \|B^{\frac{y_k}{\|y_k\|}}(f)\|_{L^1(Z)}\|y_k\|\|\phi_k\|_{L^1} \\ &\leq \|f\|_{L^1_B(X)}\|g\|_{L^1(Y)}.\end{aligned}$$

□

This allows us to give the following definition.

Definition 2.13. *If $f \in L^1_B(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}, Y) = L^1(\mathbb{T})\hat{\otimes}Y$ we define the convolution*

$$f *_B g = \lim_n f *_B g_n$$

for any sequence of simple functions $(g_n) \subset \mathcal{S}(\mathbb{T}, Y)$ converging to $g \in L^1(\mathbb{T}, Y)$.

Of course $f *_B g \in L^1(\mathbb{T}, Z)$ and $\|f *_B g\|_{L^1(\mathbb{T},Z)} \leq \|f\|_{L^1_B(\mathbb{T},X)}\|g\|_{L^1(\mathbb{T},Y)}$.

Theorem 2.14. *Let $n \in \mathbb{Z}$, $f \in L^1_B(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}, Y)$. Then*

$$(f *_B g)\hat{}(n) = B(\hat{f}^B(n), \hat{g}(n)).$$

Proof. Assume first that $g = \phi \otimes y$ for $\phi \in L^\infty(\mathbb{T})$ and $y \in Y$.
Therefore

$$\begin{aligned}(f *_B g)\hat{}(n) &= B(f *^B \phi, y)\hat{}(n) \\ &= B((f *^B \phi)\hat{}(n), y) \\ &= B(\hat{f}^B(n)\hat{\phi}(n), y) \\ &= B(\hat{f}^B(n), \hat{g}(n)).\end{aligned}$$

This extends to $g \in \mathcal{P}(\mathbb{T}, Y)$ by linearity. Now use the density of $\mathcal{P}(\mathbb{T}, Y)$ in $L^1(\mathbb{T}, Y)$ to obtain

$$\begin{aligned}(f *_B g)\hat{}(n) &= \lim_{k\to\infty} (f *_B g_k)\hat{}(n) \\ &= \lim_{k\to\infty} B(\hat{f}^B(n), \hat{g}_k(n)) \\ &= B(\hat{f}^B(n), \hat{g}(n)).\end{aligned}$$

□

3. Young's Theorem

We shall present here several analogues to Young's theorems about convolutions in our setting.

Note that for any $f \in L^1(\mathbb{T}, X)$ and $\varphi \in L^1(\mathbb{T})$ the following pointwise estimate holds

$$\|f * \varphi\| \leq \|f\| * |\varphi|.$$

Using the scalar-valued Young theorem one clearly obtains that if $f \in L^p(\mathbb{T}, X)$ and $\varphi \in L^q(\mathbb{T})$ then $f * \varphi \in L^r(\mathbb{T}, X)$ with

$$\|f * \varphi\|_{L^r(\mathbb{T},X)} \leq \|f\|_{L^p(\mathbb{T},X)} \|\varphi\|_{L^q(\mathbb{T})}$$

where $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

Using Remark 2.7 and the previous observation we can formulate the following extension.

Proposition 3.1. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} \geq 1$ and let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. If $f \in L^p_B(\mathbb{T}, X)$ and $\varphi \in L^q(\mathbb{T})$ then $f *^B \varphi \in L^r_B(\mathbb{T}, X)$.*

Moreover

$$\|f *^B \varphi\|_{L^r_B(\mathbb{T},X)} \leq \|f\|_{L^p_B(\mathbb{T},X)} \|\varphi\|_{L^q(\mathbb{T})}.$$

Let us establish the dualities to be used in our bilinear setting.

Lemma 3.2. *Let $B : X \times Y \to Z$ bounded bilinear map and $B_* : Y \times Z^* \to X^*$ given by $\langle B(x, y), z^* \rangle = \langle x, B_*(y, z^*) \rangle$.*

If $f \in \mathcal{P}(\mathbb{T}, X)$, $g \in \mathcal{P}(\mathbb{T}, Y)$ and $h \in \mathcal{P}(\mathbb{T}, Z^*)$ then

$$\langle f *_B \bar{g}, h \rangle = \langle f, g *_{B_*} h \rangle \text{ and } \langle \bar{f} *_B g, h \rangle = \langle f *_{B^*} h, g \rangle,$$

where $\bar{g}(e^{i\theta}) = g(e^{-i\theta})$.

Proof. Observe that if F and G are polynomial with values in a Banach space and its dual respectively then

$$\langle F, G \rangle = \int_{-\pi}^{\pi} \langle F(e^{i\theta}), G(e^{i\theta}) \rangle \frac{d\theta}{2\pi} = \sum \langle \hat{F}(n), \hat{G}(-n) \rangle$$

Taking into account that

$$f *_B \bar{g}(e^{it}) = \sum B(\hat{f}^B(n), \hat{g}(-n)) e^{int}$$

one obtains

$$\begin{aligned} \langle f *_B \bar{g}, h \rangle &= \sum \langle B(\hat{f}(n), \hat{g}(-n)), \hat{h}(-n) \rangle \\ &= \sum \langle \hat{f}(n), B_*(\hat{g}(-n), \hat{h}(-n)) \rangle \\ &= \sum \langle \hat{f}(n), (g *_{B_*} h)\hat{}(-n) \rangle \\ &= \langle f, g *_{B_*} h \rangle. \end{aligned}$$

Similarly

$$\begin{aligned} \langle \bar{f} *_B g, h \rangle &= \sum \langle B(\hat{f}(-n), \hat{g}(n)), \hat{h}(-n) \rangle \\ &= \sum \langle B^*(\hat{f}(-n), \hat{h}(-n)), \hat{g}(n) \rangle \\ &= \sum \langle (f *_{B^*} h)\hat{}(-n), \hat{g}(n) \rangle \\ &= \langle f *_{B*} h, g \rangle. \end{aligned}$$

□

Let us now present the version of Young's theorem in our general setting.

Theorem 3.3. *Let $1 < p < \infty$.*

(i) If $f \in L^p_B(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T}, Y)$ then $f *_B g \in L^p(\mathbb{T}, Z)$. Moreover

$$\|f *_B g\|_{L^p(Z)} \leq \|f\|_{L^p_B(X)} \|g\|_{L^1(Y)}.$$

(ii) If $f \in L^p_{B^*}(\mathbb{T},X)$ and $g \in L^{p'}(\mathbb{T},Y)$ then $f *_B g \in L^\infty(\mathbb{T},Z)$. Moreover
$$\|f *_B g\|_{L^\infty(Z)} \le \|f\|_{L^p_{B^*}(X)}\|g\|_{L^{p'}(Y)}.$$

(iii) If $f \in L^{p'}(\mathbb{T},X)$ and $g \in L^p_{B_*}(\mathbb{T},Y)$ then $f *_B g \in L^\infty(\mathbb{T},Z)$. Moreover
$$\|f *_B g\|_{L^\infty(Z)} \le \|f\|_{L^1(X)}\|g\|_{L^p_{B_*}(Y)}.$$

(iv) If $f \in L^p_B(\mathbb{T},X) \cap L^p_{B^*}(\mathbb{T},X)$, and $g \in L^q(\mathbb{T},Y)$ for $1 \le q \le p'$ then $f *_B g \in L^r(\mathbb{T},Z)$ where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Moreover
$$\|f *_B g\|_{L^r(\mathbb{T},Z)} \le \|f\|^{p/r'}_{L^p_B(\mathbb{T},X)}\|f\|^{1-p/r'}_{L^p_{B^*}(\mathbb{T},X)}\|g\|_{L^q(\mathbb{T})}.$$

Proof. (i) Assume $g = \sum_{k=0}^M y_k\phi_k$ where $\phi_k = \chi_{I_k}$ for pairwise disjoint intervals. Hence

$$\begin{aligned}
\|f *_B g\|_{L^p(Z)} &\le \sum_{k=0}^M \|B(f *^B \phi_k, y_k)\|_{L^p(Z)} \\
&= \sum_{k=0}^M \|B^{y_k}(f) * \phi_k\|_{L^p(Z)} \\
&= \sum_{k=0}^M \|B^{\frac{y_k}{\|y_k\|}}(f)\|_{L^p(Z)}\|y_k\|\|\phi_k\|_{L^1} \\
&\le \|f\|_{L^p_B(X)}\|g\|_{L^1(Y)}.
\end{aligned}$$

As usual one extends to general functions $g \in L^1(\mathbb{T},Y)$ using the density of simple functions.

(ii) Using Lemma 3.2 and (i) one gets, for $f \in \mathcal{P}(\mathbb{T},X)$ and $g \in \mathcal{P}(\mathbb{T},Y)$, that

$$\begin{aligned}
\|f *_B g\|_{L^\infty(Z)} &= \sup\{|\langle f *_B g, h\rangle| : h \in \mathcal{P}(\mathbb{T},Z^*), \|h\|_{L^1(Z^*)} = 1\} \\
&= \sup\{|\langle \bar f *_{B^*} h, g\rangle| : h \in \mathcal{P}(\mathbb{T},Z^*), \|h\|_{L^1(Z^*)} = 1\} \\
&\le \sup\{\|g\|_{L^{p'}(Y)}\|\bar f *_{B^*} h\|_{L^p(Y^*)} : h \in \mathcal{P}(\mathbb{T},Z^*), \|h\|_{L^1(Z^*)} = 1\} \\
&\le \|f\|_{L^p_{B^*}(X)}\|g\|_{L^{p'}(Y)}.
\end{aligned}$$

Using the density of polynomials the result is completed.

(iii) is analogous to (ii).

(iv) follows from interpolation using (i) and (ii). □

Remark 3.4. *For $\mathcal{D} : X \times X^* \to \mathbb{K}$ given by $\mathcal{D}(x,x^*) = \langle x, x^*\rangle$ and $\mathcal{B} : X \times \mathbb{K} \to X$ given by $\mathcal{B}(x,\lambda) = \lambda x$ one has that $\mathcal{B}^* = \mathcal{D}$ and $\mathcal{D}^* = \mathcal{B}$. Therefore $L^p_{\mathcal{B}}(\mathbb{T},X) \subset L^p_{\mathcal{B}^*}(\mathbb{T},X)$, and there exists $f \in L^p_{\mathcal{D}}(\mathbb{T},X) \setminus L^p_{\mathcal{D}^*}(\mathbb{T},X)$.*

We shall now observe that Young's theorem (see Theorem 3.3, (iv)) does not hold without the extra assumption $f \in L^p_{B^*}(\mathbb{T},X)$.

Proposition 3.5. *For any infinite dimensional Banach space X there exists $f \in L^1_{\mathcal{D}}(\mathbb{T},X)$ and $g \in L^\infty(\mathbb{T},X^*)$ such that $f *_{\mathcal{D}} g \notin L^\infty(\mathbb{T})$.*

Proof. Assume the result does not hold true. Then if $f \in \mathcal{P}(\mathbb{T},X)$ we have that for any $g \in \mathcal{P}(\mathbb{T},X^*)$
$$|f *_{\mathcal{D}} g(0)| = |\int_{-\pi}^{\pi} \langle f(e^{-i\theta}), g(e^{i\theta})\rangle \frac{d\theta}{2\pi}| \le C\|f\|_{L^1_{\mathcal{B}}(X)}\|g\|_{L^\infty(Y)}.$$

This would imply that $\|f\|_{L^1(\mathbb{T},X)} \leq C\|f\|_{P^1(\mathbb{T},X)}$ for any polynomial, and then X would be finite dimensional. □

Let us point out an application of our convolution which extends the bilinear Marcinkiewicz-Zygmund theorem (see [5], Corollary 2.7).

Theorem 3.6. *Let $1 \leq p_i < \infty$ and (Ω_i, μ_i) be σ-finite measure spaces for $i = 1, 2$. If X is $(L^{p_1}(\mu_1), L^{p_2}(\mu_2), B)$-normed then there exists $C > 0$ such that*

$$\|(\sum_{j=1}^{n} |B(x_j, \phi_j)|^2)^{1/2}\|_{L^{p_2}} \leq C \sup_{\|\varphi\|_{p_1}=1} \|(\sum_{j=1}^{n} |B(x_j, \varphi)|^2)^{1/2}\|_{L^{p_2}} \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^{p_1}}$$

for all $x_1, ..., x_n \in X$, $\phi_1, ..., \phi_n \in L^{p_1}(\mu_1)$, $n \in \mathbb{N}$.

Proof. Let $f(e^{it}) = \sum_{j=1}^{n} x_j e^{i2^j t} \in \mathcal{P}(\mathbb{T}, X)$ and $g(e^{it}) = \sum_{j=1}^{n} \phi_j e^{i2^j t} \in \mathcal{P}(\mathbb{T}, L^{p_1}(\mu_1))$. Hence $f *_B g(e^{it}) = \sum_{j=1}^{n} B(x_j, \phi_j) e^{i2^j t}$. Now use Kintchine's inequalities (see [10]), which assert that

$$\|\sum_{j=1}^{n} \varphi_j e^{i2^j t}\|_{L^1(\mathbb{T}, L^p(\mu))} \approx \|(\sum_{j=1}^{n} |\varphi_j|^2)^{1/2}\|_{L^p(\mu)}$$

for any $0 < p < \infty$ and $\varphi_1, ...\varphi_n \in L^p(\mu)$, together with

$$\|f *_B g\|_{L^1(\mathbb{T}, L^{p_2}(\mu_2))} \leq \|f\|_{L^1_B(\mathbb{T},X)} \|g\|_{L^1(\mathbb{T}, L^{p_1}(\mu_2))}$$

to obtain the result. □

Corollary 3.7. *Let (Ω, μ) be σ-finite measure space and $1 \leq p_i < \infty$ for $i = 1, 2, 3$ such that $1/p_3 = 1/p_1 + 1/p_2$. Then there exists $C > 0$ such that*

$$\|(\sum_{j=1}^{n} |\psi_j \phi_j|^2)^{1/2}\|_{L^{p_3}} \leq C \|(\sum_{j=1}^{n} |\psi_j|^2)^{1/2}\|_{L^{p_1}} \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^{p_2}}$$

for all $\psi_1, ..., \psi_n \in L^{p_1}(\mu)$, $\phi_1, ..., \phi_n \in L^{p_2}(\mu)$, $n \in \mathbb{N}$.

Proof. Apply Theorem 3.6 for $B : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ given by $(\phi, \psi) \to \phi\psi$ and use the fact

$$\sup_{\|\varphi\|_{p_2}=1} \|(\sum_{j=1}^{n} |B(\phi_j, \varphi)|^2)^{1/2}\|_{L^{p_3}} = \sup_{\|\varphi\|_{p_2}=1} \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2} |\varphi|\|_{L^{p_3}} = \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^{p_1}}$$

□

Corollary 3.8. *Let $(\mathbb{R}, m)$ be the Lebesgue measure space and $1 \leq p < \infty$. Then there exists $C > 0$ such that*

$$\|(\sum_{j=1}^{n} |\psi_j * \phi_j|^2)^{1/2}\|_{L^p} \leq C \sup_{\|\varphi\|_1=1} \|(\sum_{j=1}^{n} |\psi_j * \varphi|^2)^{1/2}\|_{L^p} \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^p}$$

for all $\psi_1, ..., \psi_n \in L^p(\mathbb{R})$, $\phi_1, ..., \phi_n \in L^1(\mathbb{R})$, $n \in \mathbb{N}$.

Proof. Apply Theorem 3.6 for $B : L^p(\mathbb{R}) \times L^1(\mathbb{R}) \to L^p(\mathbb{R})$ given by $(\phi, \psi) \to \phi * \psi$. □

We now present the following different generalization of the Marcinkiewicz-Zygmund Theorem (see [10]).

Theorem 3.9. *Let $1 \leq p_i < \infty$ and (Ω_i, μ_i) be σ-finite measure spaces for $i = 1, 2$. Then there exists $C > 0$ such that*

$$\|(\sum_{j=1}^{n} |T_j(\phi_j)|^2)^{1/2}\|_{L^{p_2}} \leq C \sup_{\|\varphi\|_{p_1}=1} \|(\sum_{j=1}^{n} |T_j(\varphi)|^2)^{1/2}\|_{L^{p_2}} \|(\sum_{j=1}^{n} |\phi_j|^2)^{1/2}\|_{L^{p_1}}$$

for all $\phi_1, ..., \phi_n \in L^{p_1}(\mu_1)$, $T_j : L^{p_1}(\mu_1) \to L^{p_2}(\mu_2)$ bounded linear operators for $1 \leq j \leq n$, and $n \in \mathbb{N}$.

Proof. Apply Theorem 3.6 for $B : \mathcal{L}(L^{p_1}(\mu_1), L^{p_2}(\mu_2)) \times L^{p_1}(\mu_1) \to L^{p_2}(\mu_2)$ given by $(T, \psi) \to T(\psi)$. □

4. Haussdorff-Young inequality

We recall that for $1 \leq p \leq 2$ a complex Banach space X is said have Fourier type p if there exists $C > 0$ such that

$$(\sum_{k \in \mathbb{Z}} \|\hat{f}(k)\|^{p'})^{1/p'} \leq C\|f\|_{L^p(\mathbb{T},X)}$$

for any $f \in \mathcal{P}(\mathbb{T}, X)$ and p', as usual, verifies $1/p + 1/p' = 1$.

This notion was firstintroduced in [13] and it has been extensively studied by different authors (see [11] for a survey on that). It is well known that X has Fourier type 2 if and only if X is isomorphic to a Hilbert space ([9]) and that X has Fourier type p if and only if X^* has.

Definition 4.1. *Let $1 \leq p \leq 2$. X is said to have B-Fourier type p if there exists $C > 0$ such that*

$$\sup_{\|y\|=1} \|(B(\hat{f}^B(k), y))_{k \in \mathbb{Z}}\|_{\ell_{p'}(Z)} \leq C\|f\|_{L^p_B(\mathbb{T},X)}$$

for any $f \in \mathcal{P}(\mathbb{T}, X)$.

Remark 4.2. *Every Banach space X has B-Fourier type 1.*

Proposition 4.3. *If Z has Fourier type p then X has B-Fourier type p.*
In particular every Banach space X has $\mathcal{D}$-Fourier type 2.

Proof. Let $f \in \mathcal{P}(\mathbb{T}, X)$ and $y \in Y$. From the assumption

$$\begin{aligned} \|(B(\hat{f}^B(k), y))_{k \in \mathbb{Z}}\|_{\ell_{p'}(Z)} &= \|(B^y(f)\hat{}(k))_{k \in \mathbb{Z}}\|_{\ell_{p'}(Z)} \\ &\leq C\|B^y(f)\|_{L^p(Z)} \\ &\leq C\|y\|\|f\|_{L^p(\mathbb{T},X)} \end{aligned}$$

Taking suprema one gets the result. □

It is well known that ℓ_q has Fourier-type $\min\{q, q'\}$.

Proposition 4.4. *Let $2 \leq q \leq \infty$. For each $r \in [q', 2]$ there exists B such that ℓ_q has B-Fourier type r.*

Proof. For $r = 2$ take $B = \mathcal{D}$ and for $r = q'$ take $B = \mathcal{B}$. Assume now $q' < r < 2 < q$.

Consider $B = \ell_q \times \ell_p \to \ell_r$ given by $((\alpha_n), (\beta_n)) \to (\alpha_n \beta_n)$ for $1/p = 1/r - 1/q$. Using Proposition 4.3 and $F(\ell_r) = r$ one obtains the result. □

We now present some applications. theorems.

Theorem 4.5. *Let $1 \leq p < \infty$ and (Ω, μ) be σ-finite measure space. If $T_n : X \to L^p(\mu)$ be a sequence of bounded linear operator then there exists $C > 0$ such that*

$$\sup_{\|x\|=1} (\sum_{j=1}^{n} \|T_j(x)\|_{L^p}^{\max\{p,p'\}})^{1/\max\{p,p'\}} \leq C \sup_{\|x\|=1} \|(\sum_{j=1}^{n} |T_j(x)|^2)^{1/2}\|_{L^p}$$

for all $n \in \mathbb{N}$.

Proof. Since $L^p(\mu)$ has Fourier-type $\min\{p, p'\}$, applying Proposition 4.3 for the bilinear map $B : \mathcal{L}(X, L^p(\mu)) \times X \to L^p(\mu)$ given by $(T, x) \to T(x)$ one has that $\mathcal{L}(X, L^p(\mu))$ has B-Fourier type $\min\{p, p'\}$. Now apply the result to the function $f(e^{it}) = \sum_{j=1}^{n} T_j e^{i2^j t}$ and Kintchine's inequality one more time. □

Remark 4.6. *The previous result is immediate for $p \geq 2$, since*

$$\|(\sum_{j=1}^{n} |T_j(x)|^p)^{1/p}\|_{L^p} \leq \|(\sum_{j=1}^{n} |T_j(x)|^2)^{1/2}\|_{L^p}.$$

Corollary 4.7. *Let $1 < p < 2$ and denote $\Delta_j(f)(e^{i\theta}) = \sum_{n=2^j+1}^{2^{j+1}} \hat{f}(n)e^{in\theta}$. Then there exists $C > 0$ such that*

$$(\sum_{j} \|\Delta_j(f)\|_{L^p(\mathbb{T})}^{p'})^{1/p'} \leq C\|f\|_{L^p(\mathbb{T})}.$$

Proof. Apply Theorem 4.5 for $T_j : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ given by $T_j(f) = \Delta_j(f)$ together with Littlewood-Paley inequalities

$$\|(\sum_{j=1}^{n} |\Delta_j(f)|^2)^{1/2}\|_{L^p} \approx \|f\|_{L^p}.$$

□

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