Discretization versus transference for bilinear operators

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Abstract. A very general transference method for bilinear operators is presented and used to show that discretization techniques can also be obtained from transference methods. It is applied to show the boundedness of the discrete version of the bilinear fractional operator and the bisublinear Hardy-Littlewood maximal operator. Also a method for bilinear vector-valued transference is presented.

1. Introduction.

In 1977, a very general and abstract method of transference was introduced by R. Coifman and G. Weiss in [**8**] . Their procedure showed that if a "convolution type" operator defined on a group is bounded on $L^p(G)$ and the group G is represented in the space of bounded linear operators $\mathcal{B}(L^p(\mu))$ for some measure μ then a transferred operator, defined by means of the representation, is also bounded on the corresponding $L^p(\mu)$ spaces. Their method relies on the following result: Let G be an amenable group with left Haar measure m, $1 \le p < \infty$ and let $u \to R_u$ be strongly continuous uniformly bounded representations of the group in $\mathcal{B}(L^p(\mu))$. If $K \in L^1(G)$ is compactly supported and the operator

$$T(g)(v) = \int_G g(u^{-1}v)K(u)dm(u),$$

has norm $N_p(K)$ in $\mathcal{B}(L^p(G))$ then the transferred operator

$$\tilde{T}(f)(x) = \int_G (R_u f)(x)K(u)dm(u),$$

defined for f in some dense subset of $L^p(\mu)$, extends to bounded operator on $L^p(\mu)$ with norm bounded by $CN_p(K)$.

Since then, this method has been developed and extended by many other people (see for example [**3**] or [**4**]) and has shown to be an extremely useful tool to prove the

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boundedness of many operators defined in the setting of measure spaces assuming that we know the boundedness of appropriately related convolution operators in the context of amenable groups.

In 1996, L. Grafakos and G. Weiss (see [**13**]) proved a first result concerning a transference method for multilinear operators. They considered a multilinear operator T defined on an amenable group G by

$$T(g_1, ..., g_k)(v) = \int_{G^k} K(u_1, ..., u_k) g_1(u_1^{-1}v)...g_k(u_k^{-1}v)dm(u_1)...dm(u_k),$$

with g_j in some dense subset of $L^{p_j}(G)$ and where K is a kernel on G^k which may not be integrable, and they were able to transfer the boundedness of $T: L^{p_1}(G)\times...\times L^{p_k}(G) \to L^{p_0}(G)$ whenever $1/p_0 = 1/p_1+...+1/p_k$ to the boundedness of operator $\tilde{T} : L^{p_1}(\mu) \times ...L^{p_k}(\mu) \to L^{p_0}(\mu)$ where (M, μ) is a measure space and

$$\tilde{T}(f_1, ..., f_k)(x) = \int_{G^k} K(u_1, ..., u_k)(R^1_{u_1} f_1)(x)...(R^k_{u_k} f_k)(x)dm(u_1)...dm(u_k),$$

where f_j is in some dense subset of $L^{p_j}(\mu)$, and $R^j : G \to \mathcal{B}(L^{p_j}(\mu))$ $(j = 0, 1, ..., k)$ are representations which are connected through $R^0_v R^j_u = R^j_{uv}$ for all $u, v \in G$ and $1 \le j \le k$, and satisfy certain boundedness conditions.

Recently, pursuing the transference to other groups and measure spaces of the results obtained for the bilinear Hilbert transform and other bilinear multipliers some methods have been developed. In particular, the reader is referred to [**5, 7, 10, 11**] for some different approaches, using DeLeeuw type methods, which also allow to transfer the boundedness of bilinear multipliers from one group to another one.

A technique extending Coiffman-Weiss transference method was introduced in [**6**] for the bilinear situation. Namely, if G is a locally compact abelian group with Haar measure m, $K \in L^1(G)$ is a kernel with compact support, $0 < p_1, p_2, p_3 < \infty$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and the bilinear map

$$B_K(\phi, \psi)(v) = \int_G \phi(u^{-1}v)\psi(uv)K(u)du,$$

is bounded from $L^{p_1}(G) \times L^{p_2}(G) \to L^{p_3}(G)$ then one can define the transference bilinear map $T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ by

$$T_K(f, g)(x) = \int_G (R^1_{u^{-1}} f)(x)(R^2_u g)(x)K(u)du,$$

where $R^j : G \to \mathcal{B}(L^{p_j}(\mu))$ are strongly continuous and uniformly bounded representations for $j = 1, 2$ and (M, μ) is a σ-finite measure space.

Observe that if $p_3 \ge 1$, then $T_K(f, g)(x)$ is well defined at almost every x, but this is not the case if $p_3 < 1$ since it is not true, in general, that if $f \in L^{p_1}(\mu)$, $g \in L^{p_2}(\mu)$ and $K \in L^1(G)$, then $T_K(f, g) \in L^{p_3}(\mu)$ or even that it is well defined. Hence, in this case, we may have to assume something else in the mappings R^j such as, for example, that our operators R^j act also continuously in $L^2(\mu)$. Moreover, whenever $p_3 < 1$, we have to consider $f \in L^{p_1}(\mu) \cap L^2(\mu)$ and $g \in L^{p_2}(\mu) \cap L^2(\mu)$ in order to have that the transferred operator is well defined.

The following result was shown and applied to obtain some new results acting on other groups or measure spaces.

Theorem 1.1. *([**6**, Theorem 2.1]) Under the above conditions, if, for $j = 1, 2$ and every $v \in G$, there exist $A_j > 0$ such that*

$$\|R^j_v f\|_{L^{p_j}} \leq A_j \|f\|_{L^{p_j}} \tag{1.1}$$

and there exists a strongly continuous mapping $R^3 : G \to \mathcal{B}(L^{p_3}(\mu))$ satisfying that, for every $u, v \in G$ and every $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$,

$$R^3_v(R^1_{u^{-1}} f R^2_u g) = R^1_{vu^{-1}} f R^2_{vu} g, \tag{1.2}$$

and such that, for every $v \in G$, there exists $B > 0$ satisfying

$$\|f\|_{L^{p_3}(\mu)} \leq B\|R^3_v f\|_{L^{p_3}(\mu)}. \tag{1.3}$$

then, the bilinear operator $T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{p_3}(\mu)$ is bounded and it has norm bounded by $N_{p_1,p_2}(K)A_1A_2B$ where $N_{p_1,p_2}(K)$ stands for the norm of the bilinear map B_K in the corresponding spaces.

One of the basic aims of the transference methods is to provide machinery for translating estimates in harmonic analysis into discretized counterparts for ergodic operator theory. For the bilinear setting the procedure can, in principle, take the form of direct discretization of bilinear operators initially defined for the real line, and then application of abstract results such as Theorem 1.1. to transfer individual discrete bilinear operators, along with their bounds, to the ergodic theory setting. In [**6**] direct discretization techniques were initiated for the bilinear Hilbert transform, and this approach was advanced in [**1**], where general discretization and transference of bilinear maximal estimates were developed. In particular, the discretization techniques in [**1**] were used to obtain the following counterpart for the integers of the bilinear Hilbert transform for the real line [**19**].

Theorem 1.2. *(see [**6**, Proposition 2] and [**1**, Theorem 1.6]) Let $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$. Then for $a \equiv \{a_j\}_{j=-\infty}^{\infty} \in \ell^{p_1}(\mathbb{Z})$, $b \equiv \{b_j\}_{j=-\infty}^{\infty} \in \ell^{p_2}(\mathbb{Z})$, the series*

$$(H_{\mathbb{Z}}(a, b))(k) \equiv \sum_{j=-\infty, j\neq 0}^{\infty} \frac{a_{k+j} b_{k-j}}{j} \tag{1.4}$$

converges absolutely for each $k \in \mathbb{Z}$, and the bilinear operator $H_{\mathbb{Z}}$ defined on $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ satisfies

$$\|H_{\mathbb{Z}}(a, b)\|_{\ell^{p_3}(\mathbb{Z})} \leq \mathfrak{A}_{p_1,p_2} \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b\|_{\ell^{p_2}(\mathbb{Z})},$$

for all $a \in \ell^{p_1}(\mathbb{Z}), b \in \ell^{p_2}(\mathbb{Z})$, where $\mathfrak{A}_{p_1,p_2}$ is a constant depending only on p_1 and p_2.

For $a \equiv \{a_j\}_{j=-\infty}^{\infty} \in \ell^{p_1}(\mathbb{Z})$, $b \equiv \{b_j\}_{j=-\infty}^{\infty} \in \ell^{p_2}(\mathbb{Z})$, and $N \in \mathbb{N}$, let

$$H_{N,\mathbb{Z}}(a, b)(k) = \sum_{0<|j|\leq N} \frac{a_{k+j} b_{k-j}}{j}, \tag{1.5}$$

for all $k \in \mathbb{Z}$. Then $\{H_{N,\mathbb{Z}}(a, b)\}_{N=1}^{\infty}$ converges to $H_{\mathbb{Z}}(a, b)$ in the metric topology of $\ell^{p_3}(\mathbb{Z})$.

In [**1**] the discretization of the bisublinear maximal operators of [**16**] furnishes the following extension of Theorem 1.2.

THEOREM 1.3. *(See* [**1**, Theorems 1.4 and 1.5]*) Let* $1 < p_1, p_2 < \infty$ *and* $1/p_1 + 1/p_2 = 1/p_3 < 3/2$. *For* $a \equiv \{a_j\}_{j=-\infty}^{\infty} \in \ell^{p_1}(\mathbb{Z})$, $b \equiv \{b_j\}_{j=-\infty}^{\infty} \in \ell^{p_2}(\mathbb{Z})$, *let*

$$\mathcal{H}_{\mathbb{Z}}(a,b)(k) = \sup_{N\in\mathbb{N}} \left| \sum_{0<|j|\le N} \frac{a_{k+j}b_{k-j}}{j} \right|, \text{ for all } k \in \mathbb{Z}. \tag{1.6}$$

$$\mathcal{M}_{\mathbb{Z}}(a,b)(k) = \sup_{N\ge 1} \frac{1}{2N+1} \sum_{n=-N}^{N} |a_{m-n}||b_{m+n}|, \text{ for all } k \in \mathbb{Z}. \tag{1.7}$$

Then there are constants $\mathfrak{B}_{p_1,p_2}$ *and* $\mathfrak{C}_{p_1,p_2}$, *depending only on* p_1 *and* p_2, *such that*

$$\|\mathcal{H}_{\mathbb{Z}}(a,b)\|_{\ell^{p_3}(\mathbb{Z})} \le \mathfrak{B}_{p_1,p_2} \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b\|_{\ell^{p_2}(\mathbb{Z})} \tag{1.8}$$

$$\|\mathcal{M}_{\mathbb{Z}}(a,b)\|_{\ell^{p_3}(\mathbb{Z})} \le \mathfrak{C}_{p_1,p_2} \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b\|_{\ell^{p_2}(\mathbb{Z})} \tag{1.9}$$

for all $a \in \ell^{p_1}(\mathbb{Z})$, $b \in \ell^{p_2}(\mathbb{Z})$.

REMARK 1.4. The boundedness result for the discrete bisublinear Hardy-Littlewood maximal operator in (1.9) is included in Proposition 14.1 of [**9**], which is an article devoted to the treatment of generalized multisublinear Hardy-Littlewood maximal operators, and their transference by measure-preserving transformations to discrete dynamical systems.

The main goal of this paper is to show that the boundedness of some of the discrete versions previously mentioned can be also seen as particular cases of the general method of transference from $G = \mathbb{R}$ to operators acting on $\ell^p(\mathbb{Z})$ when replacing the use of representations by general measurable functions.

To motivate this approach, let us point out that if $G = \mathbb{R}$ and we assume that the bilinear map

$$B_K(\phi,\psi)(x) = \int_{\mathbb{R}} \phi(x-y)\psi(x+y)K(y)dy,$$

given by an integrable kernel K is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ for some values of $0 < p_1, p_2, p_3 < \infty$ then we can consider the map $R : \mathbb{R} \to \mathcal{B}(\ell^p(\mathbb{Z}))$ given by $R_u(x_n) = x_{n+[u+1/2]}$ for which the corresponding transferred bilinear map T_K becomes

$$T_K((\alpha_n),(\beta_n))(m) = \sum_n \alpha_{m-n}\beta_{m+n}K_n$$

for finite sequences (α_n) and (β_n) where $K_n = \int_{[n-1/2,n+1/2)} K(u)du$.

However neither $u \to R_u$ is a representation of $\mathbb{R}$ in the space of operators on $\ell^p(\mathbb{Z})$, nor the map $u \to R_u((\alpha_n))$ is continuous from $\mathbb{R} \to \ell^p(\mathbb{Z})$ for a given $(\alpha_n) \in \ell^p(\mathbb{Z})$. Nevertheless it is still measurable in the strong operator topology.

In this paper we shall present a generalization of Theorem 1.1 where the assumptions are relaxed to obtain the discretization actually as special cases of the general transference principle. A careful look to the proof in Theorem 1.1 allows to see that there are three aspects of the result that can be exploited better and are relevant for applications. First of all the continuity in the strong operator topology of the map $u \to R_u$ is not really needed and actually the fact that R_u are representations is not important once condition (1.2) is assumed. Second of all the conditions (1.1) and (1.3) can be weakened up to new conditions which can be seen as $u \to R_u$ belonging to certain vector-valued spaces. Although only (1.1) and

(1.3) will be needed for applications in this paper we believe that the new and more general conditions can be used in further applications. Finally one observes that the setting where it has been used, that is transferring bounded bilinear operators acting from $L^{p_1}(G) \times L^{p_2}(G) \to L^{q_3}(G)$ where $q_3 = p_3$ and $1/p_3 = 1/p_1 + 1/p_2$ to the case of operators $\mathcal{B}(L^p(\mu))$ can be easily extended, under the assumption $q_3 \geq p_3$, to the case of operators $\mathcal{B}(\ell^p(\mathbb{Z}))$ or, under the assumption $q_3 \leq p_3$, to the case of operators $\mathcal{B}(L^p(\mu))$ whenever $\mu(M) < \infty$.

One of the new applications of our result is the discrete version of boundedness of the bilinear fractional integration obtained by C. Kenig and E. Stein ([**15**]) (see also the work of N. Kalton and L. Grafakos [**12**]).

The paper is organized as follows. We first establish and present the general transference method for bilinear maps and obtain some corollaries in the case of positive kernels. Later we also present similar approach to transfer also maximal bisublinear operators and recover the results about the discrete version of the bisublinear maximal Hardy-Littlewood operator. Finally we present another general transference method in the setting of general bilinear maps acting on Banach spaces, whose application to L^p-spaces allows to recover the result in [**6**].

Throughout the paper G stands for a locally compact abelian topological group, m denotes the Haar measure and we use either $m(A)$ or $|A|$ and $\int_A f(u)dm(u)$ or $\int_A f(u)du$ for the measure of a set and the integral of a function, $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$ and C stands for a constant that may vary from line to line.

2. A general bilinear transference method.

Let us start by giving the following definition, which is related to the amenability condition in the classical theory and that will be convenient for our general framework.

Definition 2.1. A collection $\mathcal{V}$ of measurable sets in G is said to be complete if the following condition holds: for every $\varepsilon > 0$ and every compact set C (that we shall always assume to be symmetric and contain the unit e), there exist $V_0 \in \mathcal{V}$ and $V_1 \in \mathcal{V}$ such that $V_0 C \subset V_1$ and

$$1 \leq \frac{|V_1|}{|V_0|} \leq 1 + \varepsilon.$$

Examples:

1.- If G is a compact group, then every $\mathcal{V}$ containing G is obviously complete.

2.- Let $G = (\mathbf{R^n}, +)$ and let $\mathcal{V}_M = \{(-R, R)^n, R > M\}$. Then $\mathcal{V}_M$ is complete, for every $M > 0$.

3.- Let $G = (\mathbf{R}^+, .)$ and let $\mathcal{V} = \{(\frac{1}{R}, R), R > 1\}$.

4.- Let $G = (\mathbb{Z}, +)$ and let $\mathcal{V} = \{[-N, N] \cap \mathbb{Z}, N \geq 1\}$.

5.- If the group is amenable the collection of neighborhoods of zero is a complete class.

For our theorems we shall see that we can restrict ourselves to complete families of measurable sets $\mathcal{V}$.

Definition 2.2. Let $0 < p < \infty$, let X be a quasi-Banach space and $\mathcal{V}$ be a complete collection of measurable sets in G. We denote by $A^p(X) = A^p(G, \mathcal{V}; X)$

the space of (strongly) measurable functions $F : G \to X$ such that

$$\sup_{V\in\mathcal{V}} \Big(\frac{1}{|V|}\int_V ||F(u)||_X^p du\Big)^{1/p} = ||F||_{A^p(X)} < \infty.$$

If $X = \mathcal{B}(Y)$ for a quasi-Banach space Y we denote by $A_s^p(Y) = A_s^p(G,\mathcal{V};Y)$ the space of functions $F : G \to \mathcal{B}(Y)$ such that $u \to F(u)(y)$ measurable for all $y \in Y$, and satisfy

$$\sup_{V\in\mathcal{V}} \sup_{\|y\|=1} \Big(\frac{1}{|V|}\int_V ||F(u)(y)||_Y^p du\Big)^{1/p} = ||F||_{A_s^p(Y)} < \infty.$$

Of course these classes satisfy $A^{p_2}(X) \subset A^{p_1}(X)$ (respect. $A_s^{p_2}(Y) \subset A_s^{p_1}(Y)$) for all $0 < p_1 \le p_2 < \infty$.

Also it is clear that for all $0 < p < \infty$ and any $\mathcal{V}$ one has

$$L^\infty(G,\mathcal{B}(Y)) \subset A^p(G,\mathcal{V},\mathcal{B}(Y)) \subset A_s^p(G,\mathcal{V},Y).$$

Examples:

1. Let $G = \mathbb{R}$, $X = \mathbb{C}$ and denote by B^p the space of functions such that

$$\lim_{R\to\infty} \frac{1}{2R}\int_R^R |F(u)|^p du < \infty.$$

Hence for $A^p(\mathbb{R},\mathcal{V},\mathbb{C}) \subset B^p$, for $\mathcal{V} = \{(-R,R); R > 1\}$.

Particular examples of functions in B^p are the almost periodic functions $F(x) = \hat{\mu}(x)$ for a finite Borel measure on $\mathbb{R}$ with finite support, say $F(x) = \sum \alpha_t e^{itx}$. If $F \in A^p(\mathbb{R},\mathbb{C})$ then $\alpha = \sum \alpha_t\chi_t$ belongs to $L^p(\hat{\mathbf{D}})$.

This follows from the fact that $||F||_{B^p} = ||\alpha||_{L^p(\hat{\mathbf{D}})}$(see [**20**]) where $\mathbf{D}$ stands for the group $\mathbb{R}$ with the discrete topology, $\hat{\mathbf{D}}$ stands for the dual group of $\mathbf{D}$, which coincides with the Bohr compactification of $\mathbb{R}$ (see [**21**], 1.8) and $\alpha = \sum \alpha_t\chi_t$ where χ_t stands for the corresponding character in $\mathbf{D}$.

2.- Let $\mathcal{V} = \{\mathbb{Z}\cap(-N,N); N > 1\}$. A sequence (x_n) in X belongs to $A^p(\mathbb{Z},\mathcal{V},X)$ if

$$\sup_{N\in\mathbb{N}} \Big(\frac{1}{2N}\sum_{-N}^{N} ||x_n||_X^p\Big)^{1/p} < \infty.$$

A sequence of operators (T_n) in $\mathcal{B}(X)$ belongs to $A_s^p(\mathbb{Z},\mathcal{V},\mathcal{B}(X))$ if

$$\sup_{N\in\mathbb{N}} \sup_{\|y\|=1} \Big(\frac{1}{2N}\sum_{-N}^{N} ||T_n(y)||_Y^p\Big)^{1/p} < \infty.$$

Definition 2.3. Let (M,μ) be a σ-finite measure space and $0 < p_1, p_2, q_3 < \infty$. Let $K \in L^1(G)$ compactly supported and denote

$$B_K(\phi,\psi)(v) = \int_G \phi(u^{-1}v)\psi(uv)K(u)du,$$

for $\phi \in L^{p_1}(G) \cap L^\infty(G)$ and $\psi \in L^{p_2}(G) \cap L^\infty(G)$.

We assume that B_K extends to a bounded bilinear operator from $L^{p_1}(G) \times L^{p_2}(G)$ to $L^{q_3}(G)$ with norm $N(K)$.

Let $R^i : G \to \mathcal{B}(L^{p_i}(\mu))$ be functions which are measurable in the strong operator topology of $\mathcal{B}(L^{p_i}(\mu))$ for $i = 1, 2$, i.e. $u \to R_u^i f$ is measurable for any $f \in L^{p_i}(\mu)$. Assume also that, for all measurable sets A with $\mu(A) < \infty$, one has that $R_u^i\chi_A \in L^2(\mu)$ for $u \in G$ and $u \to \|R_u^i\chi_A\|_2$ is bounded over compact sets.

Now define the transference operator $T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{q_3}(\mu)$ by

$$T_K(f,g)(x) = \int_G (R^1_{u^{-1}} f)(x)(R^2_u g)(x)K(u)du,$$

for simple functions f and g.

Note that $u \to (R^1_{u^{-1}} f)(R^2_u g)K(u)$ belongs to $L^1(G, L^1(\mu))$ if f and g are simple functions, and then $T_K(f,g) \in L^1(\mu)$ in this case.

Let us now state the main result of the paper.

Theorem 2.4. *Let (M,μ) be a σ-finite space, $0 < p_1, p_2, q_3 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, $K \in L^1(G)$ compactly supported and B_K and T_K are defined as above where*

1. *$q_3 = p_3$ for the general case,*
2. *$q_3 \geq p_3$ in the case $(\mathbb{Z}, \nu)$ for the counting measure ν,*
3. *$q_3 \leq p_3$ in the case $\mu(M) < \infty$.*

Let us denote $X_1 = \mathcal{B}(L^{p_1}(\mu))$,$X_2 = \mathcal{B}(L^{p_2}(\mu))$ and $X_3 = \mathcal{B}(L^{p_3}(\mu))$ and assume that:

- *There exist bounded functions ϕ_i with supp$(\phi_i) = G_i$ such that $\sum_{i=1}^n \phi_i(u) = 1$ for any $u \in G$ and there exists a complete family $\mathcal{V}$ in G and $\gamma > 0$ for which $|V| \leq \gamma |V \cap G_i|$ for all i and for all $V \in \mathcal{V}$.*
- *There exist functions (measurable in the strong operator topology) $R : G \to X_3$, $S^i : G \to X_1$ and $T^i : G \to X_2$ satisfying that, for every $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$,*

$$R_v(R^1_{u^{-1}} f R^2_u g) = S^i_{vu^{-1}} f T^i_{vu} g, \qquad u, v \in G_i \tag{2.1}$$

where R_u are invertible operators for all $u \in G$ and $R^{-1} \in A^p(G, \mathcal{V}, X_3)$ for some $0 < p \leq 1$, where $R^{-1}(u) = R_u^{-1}$, $S^i \in A^{p_1}_s(G, \mathcal{V}, L^{p_1})$ and $T^i \in A^{p_2}_s(G, \mathcal{V}, L^{p_2})$ for $i = 1, 2..., n$.

Then, the bilinear operator T_K can be extended to a bounded operator

$$T_K : L^{p_1}(\mu) \times L^{p_2}(\mu) \to L^{q_3}(\mu)$$

with norm bounded by $C(n,\gamma) A_1 A_2 A_3 \sup_{1 \leq i \leq n} N(K\phi_i)$ where

$$A_1 = \|R^{-1}\|_{A^p(X_3)},$$

$$A_2 = \sup_{1 \leq i \leq n} \|S^i\|_{A^{p_1}_s(L^{p_1})}$$

and

$$A_3 = \sup_{1 \leq i \leq n} \|T^i\|_{A^{p_2}_s(L^{p_2})}.$$

Proof. Let f, g be simple functions and let $V \in \mathcal{V}$ and denote $k_i = K\phi_i$, $V_i = V \cap G_i$ and $C_i = C \cap G_i$, with $C = \text{supp} K$. Now, for fixed $(v_1, ..., v_n) \in V_1 \times ... \times V_n$, one has

$$
\begin{aligned}
T_K(f,g) &= \sum_{i=1}^{n} \int_{G_i} R^1_{u^{-1}} f R^2_u g k_i(u) du \\
&= \sum_{i=1}^{n} R^{-1}_{v_i} \Big(\int_{G_i} R_{v_i} (R^1_{u^{-1}} f R^2_u g) k_i(u) du \Big) \\
&= \sum_{i=1}^{n} R^{-1}_{v_i} \Big(\int_{G_i} S^i_{v_i u^{-1}} f T^i_{v_i u} g k_i(u) du \Big) \\
&= \sum_{i=1}^{n} R^{-1}_{v_i} \Big(\int_{G} S^i_{v_i u^{-1}} f \chi_{V_i C_i^{-1}} (v_i u^{-1}) T^i_{v_i u} g \chi_{V_i C_i} (v_i u) k_i(u) du \Big)
\end{aligned}
$$

Hence, for every $(v_1, ..., v_n) \in V_1 \times ... \times V_n$, we have that

$$T_K(f,g) = \sum_{i=1}^{n} R^{-1}_{v_i} B_{k_i}((S^i_u f)\chi_{V_i C_i}, (T^i_u g)\chi_{V_i C_i})(v_i).$$

Therefore, if $q_3 \geq 1$,

$$\|T_K(f,g)\|_{q_3} \leq \sum_{i=1}^{n} \|R^{-1}_{v_i}\|_{X_3} \|B_{k_i}((S^i_u f)\chi_{V_i C_i^{-1}}, (T^i_u g)\chi_{V_i C_i})(v_i)\|_{L^{q_3}(\mu)}$$

and, for $0 < q_3 < 1$,

$$\|T_K(f,g)\|^{q_3}_{q_3} \leq \sum_{i=1}^{n} \|R^{-1}_{v_i}\|^{q_3}_{X_3} \|B_{k_i}((S^i_u f \chi_{V_i C_i^{-1}}), (T^i_u g)\chi_{V_i C_i})(v_i)\|^{q_3}_{L^{q_3}(\mu)}.$$

In particular, for any $0 < \alpha \leq \min\{1, q_3\}$,

$$\|T_K(f,g)\|^{\alpha}_{q_3} \leq \sum_{i=1}^{n} \|R^{-1}_{v_i}\|^{\alpha}_{X_3} \|B_{k_i}((S^i_u f \chi_{V_i C_i^{-1}}), (T^i_u g)\chi_{V_i C_i})(v_i)\|^{\alpha}_{L^{q_3}(\mu)}.$$

Let $q = 1 + \frac{p}{q_3}$ and $\alpha = \frac{pq_3}{p+q_3}$. Clearly $0 < \alpha \leq \min\{1, q_3\}$, $q > 1$, $1/q + \alpha/q_3 = 1$ and $q\alpha = p$.

Now integrate over $V_1 \times ... \times V_n$ and denote $\beta = \prod_{j=1}^n |V_j|$ and $\beta_i = \prod_{j\neq i} |V_j|$. Hence, we can write

$$
\begin{aligned}
&\|T_K(f,g)\|_{q_3}^{\alpha} \leq \\
&\leq \frac{1}{\beta}\sum_{i=1}^{n}\beta_i \int_{V_i} \|R_v^{-1}\|_{X_3}^{\alpha} \|B_{k_i}((S_u^i f\chi_{V_iC_i^{-1}}),(T_u^i g)\chi_{V_iC_i})(v)\|_{L^{q_3}(\mu)}^{\alpha} dv \\
= \;& n\sum_{i=1}^{n}\frac{\beta_i|V_i|}{n\beta}\frac{1}{|V_i|}\int_{V_i} \|R_v^{-1}\|_{X_3}^{\alpha} \|B_{k_i}((S_u^i f\chi_{V_iC_i^{-1}}),(T_u^i g)\chi_{V_iC_i})(v)\|_{L^{q_3}(\mu)}^{\alpha} dv \\
\leq \;& n \sup_{1\leq i\leq n}\frac{1}{|V_i|}\Big(\int_{V_i}\|R_v^{-1}\|_{X_3}^{\alpha q} dv\Big)^{1/q} \\
&\Big(\int_{V_i}\|B_{k_i}((S_u^i f\chi_{V_iC_i^{-1}}),(T_u^i g)\chi_{V_iC_i})(v)\|_{L^{q_3}(\mu)}^{q_3} dv\Big)^{\alpha/q_3} \\
\leq \;& n\|R^{-1}\|_{A^p(X_3)}^{\alpha}\sup_{1\leq i\leq n}\Big(\frac{1}{|V_i|}\int_G \|B_{k_i}((S_u^i f\chi_{V_iC_i^{-1}}),(T_u^i g)\chi_{V_iC_i})(v)\|_{L^{q_3}(\mu)}^{q_3} dv\Big)^{\alpha/q_3} \\
= \;& n\|R^{-1}\|_{A^p(X_3)}^{\alpha}\sup_{1\leq i\leq n}\Big(\int_M\frac{1}{|V_i|}\int_G |B_{k_i}((S_u^i f\chi_{V_iC_i^{-1}})(x),(T_u^i g)(x)\chi_{V_iC_i})(v)|^{q_3} dv d\mu(x)\Big)^{\alpha/q_3} \\
\leq \;& n\|R^{-1}\|_{A^p(X_3)}^{\alpha}\sup_{1\leq i\leq n} N^{\alpha}(k_i)\Big(\int_M\Big(\frac{1}{|V_i|}\int_{V_iC_i^{-1}}|S_u^i f(x)|^{p_1}du\Big)^{q_3/p_1} \\
. \;& \Big(\frac{1}{|V_i|}\int_{V_iC_i}|T_u^i g(x)|^{p_2}du\Big)^{q_3/p_2} d\mu(x)\Big)^{\alpha/q_3}.
\end{aligned}
$$

Now, for each $1 \leq i \leq n$, we denote

$$
I_i = \int_M \Big(\frac{1}{|V_i|}\int_{V_iC_i^{-1}}|S_u^i f(x)|^{p_1}du\Big)^{q_3/p_1}\Big(\frac{1}{|V_i|}\int_{V_iC_i}|T_u^i g(x)|^{p_2}du\Big)^{q_3/p_2} d\mu(x).
$$

Hence

$$
\|T_k(f,g)\|_{q_3} \leq n^{1/\alpha}\|R^{-1}\|_{A^p(X_3)}\sup_{1\leq i\leq n} N(k_i) I_i^{1/q_3} \tag{2.2}
$$

Now the proof splits depending the cases for q_3.

Assume first that $q_3 = p_3$. Using now that $1/p_3 = 1/p_1 + 1/p_2$ and Hölder inequality one gets

$$
I_i \;\leq\; \Big(\frac{1}{|V_i|}\int_{V_iC_i^{-1}}\|S_u^i f\|_{p_1}^{p_1}du\Big)^{p_3/p_1}\Big(\frac{1}{|V_i|}\int_{V_iC_i}\|T_u^i g\|_{p_2}^{p_2}du\Big)^{p_3/p_2}.
$$

Assume now that $q_3 \geq p_3$ and $(M,\mu) = (\mathbb{Z},\nu)$. Write $q_3 = \delta p_3$ for some $\delta \geq 1$. Hence

$$
I_i \leq \Big(\int_M \Big(\frac{1}{|V_i|}\int_{V_iC_i^{-1}}|S_u^i f(x)|^{p_1}du\Big)^{p_3/p_1}\Big(\frac{1}{|V_i|}\int_{V_iC_i}|T_u^i g(x)|^{p_2}du\Big)^{p_3/p_2} d\mu(x)\Big)^{\delta}.
$$

This shows that

$$
I_i^{1/\delta} \leq \Big(\frac{1}{|V_i|}\int_{V_iC_i^{-1}}\|S_u^i f\|_{p_1}^{p_1}du\Big)^{p_3/p_1}\Big(\frac{1}{|V_i|}\int_{V_iC_i}\|T_u^i g\|_{p_2}^{p_2}du\Big)^{p_3/p_2}.
$$

Assume now that $q_3 \le p_3$ and $\mu(M) < \infty$. Write $q_3 = \rho p_3$ for some $\rho \le 1$. Hence

$$I_i \le \mu(M)^{1-\rho}\Big(\int_M \left(\frac{1}{|V_i|}\int_{V_iC_i^{-1}} |S_u^i f(x)|^{p_1} du\right)^{p_3/p_1} \left(\frac{1}{|V_i|}\int_{V_iC_i} |T_u^i g(x)|^{p_2} du\right)^{p_3/p_2} d\mu(x)\Big)^{\rho}.$$

This shows that

$$I_i^{1/\rho} \le \mu(M)^{1/\rho-1}\left(\frac{1}{|V_i|}\int_{V_iC_i^{-1}} \|S_u^i f\|_{p_1}^{p_1} du\right)^{p_3/p_1} \left(\frac{1}{|V_i|}\int_{V_iC_i} \|T_u^i g\|_{p_2}^{p_2} du\right)^{p_3/p_2}.$$

In any case

$$\begin{aligned} I_i^{1/q_3} &\le A\left(\frac{\gamma}{|V|}\int_{VC} \|S_u^i(\frac{f}{\|f\|_{p_1}})\|_{p_1}^{p_1} du\right)^{1/p_1} \\ &\cdot \left(\frac{\gamma}{|V|}\int_{VC} \|T_u^i(\frac{g}{\|g\|_{p_2}})\|_{p_2}^{p_2} du\right)^{1/p_2} \|f\|_{p_1}\|g\|_{p_2}. \end{aligned}$$

Finally for every $\varepsilon > 0$, let $V_0, V_1 \in \mathcal{V}$ such that $V_0C \subset V_1$ and $\frac{|V_1|}{|V_0|} \le 1+\varepsilon$. Therefore, applying the previous estimates for V_0, one gets

$$\begin{aligned} \|T_K(f,g)\|_{q_3} &\le C(n,\gamma)\sup_{1\le i\le n} N(k_i)(1+\varepsilon)^{1/p_3}\|R^{-1}\|_{A^p(X_3)} \\ &.(\sup_{1\le i\le n} \|S^i\|_{A_s^{p_1}(L^{p_1})}\|T^i\|_{A_s^{p_2}(L^{p_2})})\|f\|_{p_1}\|g\|_{p_2}. \end{aligned}$$

Taking limits as ε goes to zero the proof is complete. □

Let us formulate now a corollary from which one can actually get most applications in this paper.

Corollary 2.5. Let $1 < p_1, p_2 < \infty$ and let (M,μ) be either a finite measure space or $\mathbb{Z}$ with the counting measure. Let K be positive, integrable and with compact support defining a bounded bilinear map $B_K : L^{p_1}(G)\times L^{p_2}(G) \to L^{q_3}(G)$ where $q_3 \le p_3$ (in case $\mu(M) < \infty$) or $p_3 \le q_3$ for $M = \mathbb{Z}$. Let us consider a bounded and measurable in the strong operator topology function $R : G \to \mathcal{B}(L^{p_i}(\mu))$ for $i = 1,2,3$ and where R_u are invertible operators for all $u \in G$ such that $R_u^{-1} = R_{u^{-1}}$.

(1) Assume that there exist $G = \cup_{i=1}^n G_i$ for some pairwise disjoint measurable sets and a complete family $\mathcal{V}$ in G and $\gamma > 0$ for which $m(V) \le \gamma m(V \cap G_i)$ for all i and for all $V \in \mathcal{V}$.

(2) Assume that there exist bounded and measurable in the strong operator topology functions $S^i : G \to \mathcal{B}(L^{p_1}(\mu))$ and $T^i : G \to \mathcal{B}(L^{p_2}(\mu))$ satisfying that for every $f \in L^{p_1}(M)$ and $g \in L^{p_2}(M)$,

$$R_v(R_{u^{-1}}fR_ug) = S_{vu^{-1}}^i f T_{vu}^i g, \qquad u, v \in G_i \tag{2.3}$$

Then, the bilinear operator $T_K(f,g) = \int_G R_u f R_{u^{-1}} g K(u) dm(u)$ can be extended to a bounded operator $T_K : L^{p_1}(\mu)\times L^{p_2}(\mu) \to L^{q_3}(\mu)$ with norm bounded by $C(n,\gamma)A_1A_2A_3\sup_{1\le i\le n} N(K)$ where

$$A_1 = \sup_{u\in G, 1\le i\le n} \|S_u^i\|_{\mathcal{B}(L^{p_1}(\mu))},$$

$$A_2 = \sup_{u\in G, 1\le i\le n} \|T_u^i\|_{\mathcal{B}(L^{p_2}(\mu))} \text{ and}$$

$$A_3 = \sup_{u\in G} \|R_u\|_{\mathcal{B}(L^{q_3}(\mu))}.$$

In particular one has the following application:

Corollary 2.6. Let $q_3 \geq p_3$, let K be positive, integrable and with compact support defined in $\mathbb{R}$ such that

$$B_K(\phi,\psi)(v) = \int_{\mathbb{R}} \phi(v-u)\psi(v+u)K(u)dm(u),$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{q_3}(\mathbb{R})$.

Define $K_n = \int_{A_n} K(u)du$ where $A_0 = [-\frac{1}{4}, \frac{1}{4}]$, $A_n = (n - \frac{3}{4}, n + \frac{1}{4}]$ for $n \in \mathbb{N}$ and $A_{-n} = -A_n$.

Then the "discrete bilinear" transform associated to K

$$T_{\mathbb{Z},K}(a,b)(m) = \sum_{n \in \mathbb{Z}} a_{m+n} b_{m-n} K_n$$

is bounded from $\ell_{p_1}(\mathbb{Z}) \times \ell_{p_2}(\mathbb{Z})$ to $\ell_{q_3}(\mathbb{Z})$ and $\|T_{\mathbb{Z},K}\| \leq CN(K)$.

Proof. We shall apply Corollary 2.5 for $G = \mathbb{R}$. Denote $I_k = [k - \frac{1}{4}, k + \frac{1}{4}]$ for $k \in \mathbb{Z}$, $J_k = (k - \frac{3}{4}, k - \frac{1}{4})$ and $J'_k = (-k + \frac{1}{4}, -k + \frac{3}{4})$ for $k \in \mathbb{N}$. Define

$$G_1 = \cup_{k \in \mathbb{Z}} I_k, G_2 = \cup_{k \in \mathbb{N}} J_k \text{ and } G_3 = \cup_{k \in \mathbb{N}} J'_k$$

Consider now $\mathcal{V} = \{(-N, N) : N \in \mathbb{N}\}$.

It is clear that $m((-N,N) \cap G_2) = m((-N,N) \cap G_3) = \frac{1}{2} m((-N,N) \cap G_1) = \frac{1}{4} m((-N,N))$. This gives $\gamma = 2$.

Let us define $R : \mathbb{R} \to \mathcal{B}(\ell_{p_i}(\mathbb{Z}))$ given by

$$R_u = S^{(u)} \chi_{G_1}(u) + S^{(u+\frac{1}{2})} \chi_{G_2}(u) + S^{(u-\frac{1}{2})} \chi_{G_3}(u)$$

where S stands for the Shift operator $S((x_n)) = (x_{n+1})$ and (u) stands for the closest integer to u respectively.

Observe that, for $k \in \mathbb{Z}$, and $u \in I_k$ then $R_u = S^k$. Also, for $k \in \mathbb{N}$, if $u \in J_k$ then $R_u = S^k$ and if $u \in J'_k$ then $R_u = S^{-k}$.

Observe that $R_u(ab) = R_u(a)R_u(b)$ for any sequences a and b.

If $u, v \in G_1$ then $-u \in G_1$ and one has $(v+u) = (v) + (u)$ and $(v-u) = (v) + (-u)$.

Hence $R_v(R_u a R_{-u} b) = S^{(u+v)} a S^{(v-u)} b$ for $u, v \in G_1$. This allows to take $S^1_u = T^1_u = S^{(u)}$.

If $u, v \in G_2$ then $-u \in G_3$ and we have that $-u - \frac{1}{2}, u + \frac{1}{2}, v + \frac{1}{2} \in G_1$. Therefore $(v+u+1) = (v+\frac{1}{2}) + (u+\frac{1}{2})$ and $(v-u) = (v+\frac{1}{2}) + (-u-\frac{1}{2})$.

Hence $R_v(R_u a R_{-u} b) = S^{(u+v+1)} a S^{(v-u)} b$ for $u, v \in G_2$. This allows to take $S^2_u = S^{(u+1)}$ and $T^2_u = S^{(u)}$.

If $u, v \in G_3$ then $-u \in G_2$ and we have that $-u + \frac{1}{2}, u - \frac{1}{2}, v - \frac{1}{2} \in G_1$. Therefore $(v+u-1) = (v-\frac{1}{2}) + (u-\frac{1}{2})$ and $(v-u) = (v-\frac{1}{2}) + (-u+\frac{1}{2})$.

Hence $R_v(R_u a R_{-u} b) = S^{(u+v-1)} a S^{(v-u)} b$ for $u, v \in G_3$. This allows to take $S^3_u = S^{(u-1)}$ and $T^3_u = S^{(u)}$.

Since all operators appearing are norm 1 on $\ell_p(\mathbb{Z})$ for any value of p and for any $u \in \mathbb{R}$, then one gets, using Corollary 2.5, that T_K is bounded from $\ell_{p_1}(\mathbb{Z}) \times \ell_{p_2}(\mathbb{Z})$ to $\ell_{q_3}(\mathbb{Z})$ and $\|T_K\| \leq 2N(K)$.

Let us finally compute T_K in this case

$$\begin{aligned}
& T_K(a,b) = \int_{\mathbb{R}} R_{-u} a R_u b K(u) du \\
= & \sum_{k\in\mathbb{Z}} \int_{I_k} R_{-u} a R_u b K(u) du + \sum_{k\in\mathbb{N}} \int_{J_k \cup J'_k} R_{-u} a R_u b K(u) du \\
= & \; ab \int_{I_0} K(u)du + \sum_{k\in\mathbb{N}} S^{-k} a S^k b \int_{I_k\cup J_k} K(u)du + \sum_{k\in\mathbb{N}} S^k a S^{-k} b \int_{I_{-k}\cup J'_k} K(u)du \\
= & \; ab \int_{I_0} K(u)du + \sum_{k\in\mathbb{N}} S^{-k} a S^k b \int_{(k-\frac{3}{4},k+\frac{1}{4}]} K(u)du \\
+ & \sum_{k\in\mathbb{N}} S^k a S^{-k} b \int_{[-k-\frac{1}{4},-k+\frac{3}{4})} K(u)du \\
= & \sum_{k\in\mathbb{Z}} S^k a S^{-k} b \int_{A_k} K(u)du
\end{aligned}$$

and therefore

$$T_K(a,b)(m) = \sum_{n\in\mathbb{Z}} a_{m+n} b_{m-n} K_n.$$

□

Now one can obtain the following application.

THEOREM 2.7. *Let $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, $0 < \alpha < \min\{1, 1/p_3\}$, and $1/q_3 = 1/p_3 - \alpha$. For $a \equiv \{a_j\}_{j=-\infty}^{\infty} \in \ell^{p_1}(\mathbb{Z})$, $b \equiv \{b_j\}_{j=-\infty}^{\infty} \in \ell^{p_2}(\mathbb{Z})$, let*

$$\mathcal{I}_\alpha(a,b)(m) = \sum_{n\in\mathbb{N}} \frac{a_{m+n} b_{m-n}}{n^{1+\alpha}}$$

Then there is a constant $\mathfrak{D}_{p_1,p_2}$, depending only on p_1 and p_2, such that

$$\|\mathcal{I}_\alpha(a,b)\|_{\ell^{q_3}(\mathbb{Z})} \le \mathfrak{D}_{p_1,p_2} \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b\|_{\ell^{p_2}(\mathbb{Z})}. \tag{2.4}$$

PROOF. Assume first $p_3 \ge \frac{1}{\alpha+1}$, that is $q_3 \ge 1$. This case follows from the vector-valued inequality

$$\begin{aligned}
\|\mathcal{I}_\alpha(a,b)\|_{\ell^{q_3}(\mathbb{Z})} & \le \sum_{n\in\mathbb{N}} \frac{\|a_{\cdot+n} b_{\cdot-n}\|_{\ell^{q_3}(\mathbb{Z})}}{n^{1+\alpha}} \le \sum_{n\in\mathbb{N}} \frac{\|a_{\cdot+n} b_{\cdot-n}\|_{\ell^{p_3}(\mathbb{Z})}}{n^{1+\alpha}} \\
& \le \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b\|_{\ell^{p_2}(\mathbb{Z})} \sum_{n\in\mathbb{N}} \frac{1}{n^{1+\alpha}} \le C \|a\|_{\ell^{p_1}(\mathbb{Z})} \|b_{\cdot-n}\|_{\ell^{p_2}(\mathbb{Z})}.
\end{aligned}$$

In the case $p_3 < \frac{1}{\alpha+1}$ we use transference. It was shown by C. Kenig and E. Stein that for $0 < \alpha < \min\{1, 1/p_3\}$ the operator $I_\alpha : L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{q_3}(\mathbb{R})$ defined by

$$I_\alpha(f,g)(x) = \int_{\mathbb{R}} \frac{f(x-t)g(g+t)}{|t|^{\alpha+1}} dt$$

is bounded for $1/q_3 = 1/p_3 - \alpha$. Apply Corollary 2.6 to the Kernels $K^N(t) = \frac{1}{|t|^{1+\alpha}} \chi_{(-N,N)}(t)$ for $G = \mathbb{R}$ and notice that $K^N_n \approx \frac{1}{|n|^{1+\alpha}}$ for $0 < |n| \le N$. □

3. Transference for maximal operators

In this section we do not give complete proofs since the arguments are quite similar to the previous ones. For a complete treatment of maximal bisublinear discretization and transference without the special assumptions used below, see [**1**].

THEOREM 3.1. *Let us assume the hypotheses in Theorem 2.4 in the case* $q_3 = p_3$ *and that* R_u^{-1} *are positive operators. Let* $\{K_j\}_j$ *be a family of kernels in* $L^1(G)$ *with compact supports* $\{C_j\}_j$ *and assume that, for* $i = 1, ..., n$, *the corresponding bisublinear maximal operator*

$$B_K^*(\phi,\psi)(v) = \sup_{j\in\mathbb{N}, 1\le i\le n} \left|\int_G \phi(vu^{-1})\psi(vu)K_j(u)\phi_i(u)dm(u)\right|, \tag{3.1}$$

is bounded from $L^{p_1}(G)\times L^{p_2}(G)$ *to* $L^{p_3}(G)$ *with norm less than or equal to* $N(\{K_j\}_j)$.

Then we have that the maximal operator

$$T^*(f,g) = \sup_j |T_{K_j}(f,g)| = \sup_j \left|\int_G R_u^1 f R_{u^{-1}}^2 g K_j(u)dm(u)\right|$$

is bounded from $L^{p_1}(\mu)\times L^{p_2}(\mu)$ *to* $L^{p_3}(\mu)$ *and it has norm bounded by* $C(n,p)A_1A_2A_3N(\{K_j\}_j)$ *where* A_i *for* $i = 1, 2, 3$ *are the same constants appearing in Theorem 2.4.*

PROOF. Denote $k_i^j(u) = K_j(u)\phi_i(u)$. As shown in Theorem 2.4, for every $(v_1, ..., v_n) \in V_1 \times ... \times V_n$, and $j \in \mathbb{N}$ we have that

$$T_{K_j}(f,g) = \sum_{i=1}^n R_{v_i}^{-1} B_{k_i^j}((S_u^i f)\chi_{V_iC_i}, (T_u^i g)\chi_{V_iC_i})(v_i).$$

Hence, using the positivity of $R_{v_i}^{-1}$ one has

$$\sup_{1\le j\le N} |T_{K_j}(f,g)| \le \sum_{i=1}^n R_{v_i}^{-1} \sup_{1\le j\le N} |B_{k_i^j}((S_u^i f)\chi_{V_iC_i}, (T_u^i g)\chi_{V_iC_i})(v_i))|.$$

Therefore

$$T^*(f,g) \le \sum_{i=1}^n R_{v_i}^{-1}(B_K^*((S_u^i f)\chi_{V_iC_i}, (T_u^i g)\chi_{V_iC_i})(v_i)).$$

Now repeat the argument in the Theorem 2.4. □

Similarly it is not difficult to show the following maximal version of Corollary 2.6.

THEOREM 3.2. *Let* $\bar{K} = \{K_j\}_j$ *be a family of positive and integrable functions defined in* $\mathbb{R}$ *such that*

$$B_{\bar{K}}^*(\phi,\psi)(v) = \sup_j \left|\int_{\mathbb{R}} \phi(v-u)\psi(v+u)K_j(u)du\right|$$

is bounded from $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$, *with norm* $N(B^*)$.

Define $K_n^j = \int_{n-1/4}^{n+1/4} K_j(u)du$. *Then the maximal "discrete bisublinear" transform associated to* $\bar{K}$

$$T_{\bar{K}}^*(a,b)(m) = \sup_j \left|\sum_{n\in\mathbb{Z}} a_{m-n}b_{m+n}K_n^j\right|$$

is bounded from $\ell_{p_1}(\mathbb{Z}) \times \ell_{p_2}(\mathbb{Z})$ *to* $\ell_{p_3}(\mathbb{Z})$ *and* $\|T_{\bar{K}}^*\| \le CN(B^*)$.

Then one can transfer the bisublinear Hardy-Littlewood maximal operator in $\mathbb{R}$. It was shown by M. Lacey (see [**16**]) that

$$M(f,g)(x) = \sup_{R>0} \frac{1}{2R}\int_{R}^{R} |f(x+t)||g(x-t)|dt$$

maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R}) \to L^{p_3}(\mathbb{R})$ for $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$. The reader should be aware that the case $p_3 > 1$ is elementary, and only the case $p_3 \leq 1$ is relevant.

We can now give the following alternative proof of (1.9) whose statement we repeat as the next corollary.

COROLLARY 3.3. Let $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$. Then

$$M(a,b)(m) = \sup_{N\geq 1} \frac{1}{2N} \sum_{n=-N}^{N} |a_{m-n}||b_{m+n}|.$$

is bounded from $\ell_{p_1}(\mathbb{Z}) \times \ell_{p_2}(\mathbb{Z})$ into $\ell_{p_3}(\mathbb{Z})$.

PROOF. Let us consider $K_j = \frac{1}{2j}\chi_{(-j-\frac14, j+\frac14)}$. Clearly

$$\begin{aligned} B^*_K(\phi,\psi)(v) &= \sup_{j\in\mathbb{N}, 1\leq i\leq n} |\int_{G_i} \phi(v-u)\psi(v+u)K_j(u)du| \\ &\leq \sup_{j\in\mathbb{N}} |\frac{1}{2j}\int_{-j-\frac14}^{j+\frac14} \phi(v-u)\psi(v+u)K_j(u)du| \\ &\leq 2M(\phi,\psi)(v). \end{aligned}$$

Notice that

$$K_n^j = \int_{A_n} K_j(u)du = \frac{1}{2j}\chi_{\{|n|\leq j\}}(n)m(A_n)$$

where $A_0 = [-\frac14, \frac14]$, $A_n = (n-\frac34, n+\frac14]$ and $A_{-n} = [-n-\frac14, -n+\frac34)$ for $n \in \mathbb{N}$.

Therefore

$$T_{\mathbb{Z},K_j}(a,b) = \frac{1}{2j}\sum_{|k|\leq j} S^k a S^{-k} b - \frac{1}{4j}ab$$

Then the "maximal discrete bilinear" transform can be estimated, for $a, b \geq 0$, as follows

$$\begin{aligned} M(a,b)(m) &= \sup_{j\geq 1} \frac{1}{2j}\sum_{k=-j}^{j} |a_{m-k}||b_{m+k}| \\ &\leq \sup_{j} |T_{\mathbb{Z},K_j}(a,b)(m)| + a(m)b(m) \end{aligned}$$

And the result follows from Theorem 3.2 □

In turn, Corollary 3.3 can be transferred so as to yield the bisublinear ergodic maximal operator, which we formulate here in the following special case of [**1**, Theorem 4.3].

THEOREM 3.4. *Let (Ω, Σ, μ) be a σ-finite measure space, $\tau : \Omega \to \Omega$ be an invertible measure-preserving transformation and define $T(f) = f \circ \tau$.*

Then the "bilinear ergodic maximal transform"

$$M_\tau(f,g)(x) = \sup_{N\in\mathbb{N}} \frac{1}{2N} \sum_{|n|\le N} |T^n f(x)||T^{-n}g(x)|$$

is bounded from $L^{p_1}(\Omega)\times L^{p_2}(\Omega)$ *into* $L^{p_3}(\Omega)$ *whenever* $p_1, p_2 > 1$ *and* $1/p_1 + 1/p_2 = 1/p_3 < 3/2$.

In particular, let A be a matrix with $|det(A)| = 1$ and consider $Tf(x) = f(Ax)$ for $x \in \mathbb{R}^n$ one obtains the following:

Corollary 3.5. (see [**1**]) The maximal transform

$$M_A(f,g)(x) = \sup_{N\in\mathbb{N}} \frac{1}{2N} \sum_{|n|\le N} |f(A^n x)||g(A^{-n}x)|$$

is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ whenever $p_1, p_2 > 1$ and $1/p_1 + 1/p_2 = 1/p_3 < 3/2$.

4. Bilinear vector-valued transference

Throughout this section X, Y, and Z will be arbitrary Banach spaces, and β will be a bounded bilinear mapping of $X \times Y$ into Z and G will be an arbitrary locally compact abelian group with given Haar measure m (sometimes abbreviated by du) , and K will be an arbitrary m-integrable complex-valued function on G. When (Ω, μ) is a measure space and $1 \le p < \infty$, we shall denote by $L^p_X(\mu)$ the usual Lebesgue space of X-valued μ-measurable functions ψ such that

$$\|\psi\|^p_{L^p_X(\mu)} \equiv \int_\Omega \|\psi\|^p_X \; d\mu < \infty.$$

In the special case when μ is the Haar measure m of G (respectively, in the special case when X is the field of complex numbers $\mathbb{C}$), $L^p_X(\mu)$ will also be symbolized by $L^p_X(G)$ (respectively, by $L^p(\mu)$).

$R^{(1)}_{(\cdot)}$, $R^{(2)}_{(\cdot)}$, and $R^{(3)}_{(\cdot)}$ will designate given functions defined on G which take values in $\mathcal{B}(X)$, $\mathcal{B}(Y)$, and $\mathcal{B}(Z)$, respectively, while satisfying the following hypotheses (a) through (d).

(a) For $j = 1, 2, 3$, $R^{(j)}_{(\cdot)}$ is a strongly continuous function on G.

(b) For $j = 1, 2$,

$$A_j \equiv \sup_{u\in G} \left\| R^{(j)}_u \right\| < \infty. \tag{4.1}$$

(c) There is a positive real constant A_3 such that

$$\|z\| \le A_3 \left\| R^{(3)}_u z \right\|, \text{ for all } z \in Z,\ u \in G. \tag{4.2}$$

(d) For all $u \in G$, $v \in G$, $x \in X$, $y \in Y$,

$$R^{(3)}_v \left(\beta \left(R^{(1)}_{u^{-1}} x, R^{(2)}_u y \right) \right) = \beta \left(R^{(1)}_{vu^{-1}} x, R^{(2)}_{vu} y \right). \tag{4.3}$$

Under the foregoing assumptions and notation, we now use Z-valued Bochner integration to define the bilinear mapping $H_K : X \times Y \to Z$ by putting

$$H_K(x,y) = \int_G \beta \left(R^{(1)}_{u^{-1}} x, R^{(2)}_u y \right) K(u)\, du, \text{ for all } x \in X,\ y \in Y. \tag{4.4}$$

Notice that H_K is a bounded bilinear mapping, with

$$\|H_K\| \leq \|\beta\| A_1 A_2 \|K\|_{L^1(G)} . \tag{4.5}$$

Since L^1-norms of integration kernels tend to have higher orders of magnitude than corresponding integration operators defined by them, it is desirable to replace the factor $\|K\|_{L^1(G)}$ in the majorant of (4.5) with a quantity which has a milder size in principle. This will be accomplished in our main transference result below (Theorem 4.3), where vector-valued transference methods effectively replace $\|K\|_{L^1(G)}$ in (4.5) with the norm of the bilinear mapping $B_{K,\beta}$ defined as follows.

Definition 4.1. Suppose that $1 < p_1, p_2 < \infty$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 \leq p_3 < \infty$. This notation will be tacitly in effect henceforth. In terms of the preceding notation for X, Y, Z, β, G, and K, we use Z-valued Bochner integration to define the bilinear mapping $B_{K,\beta} : L_X^{p_1}(G) \times L_Y^{p_2}(G) \to L_Z^{p_3}(G)$ by writing

$$B_{K,\beta}(f,g)(v) = \int_G \beta\left(f\left(u^{-1}v\right), g(uv)\right) K(u)\, du, \quad \text{for all } f \in L_X^{p_1}(G),\ g \in L_Y^{p_2}(G). \tag{4.6}$$

Remark 4.2. It is straightforward to see that the integral on the right of (4.6) exists for m-almost all $v \in G$ and defines a Z-valued m-measurable function of v satisfying the crude estimate

$$\|B_{K,\beta}(f,g)\|_{L_Z^{p_3}(G)} \leq \|K\|_{L^1(G)} \|\beta\| \|f\|_{L_X^{p_1}(G)} \|g\|_{L_Y^{p_2}(G)} . \tag{4.7}$$

In the special case where X, Y, and Z coincide with the complex field $\mathbb{C}$, and $\beta(x,y) \equiv xy$, we shall denote the bounded bilinear mapping $B_{K,\beta} : L^{p_1}(G) \times L^{p_2}(G) \to L^{p_3}(G)$ by s_K. (When K has compact support, s_K coincides with the bilinear operator B_K defined in Section 1.)

We are now ready to take up the result of this section, which is stated as follows(compare with Theorem (3.2) of [**2**]).

Theorem 4.3. *Let p_1, p_2, p_3 be as in Definition 4.1. Then in terms of the above hypotheses and notation, we have*

$$\|H_K(x,y)\| \leq A_1 A_2 A_3 \|B_{K,\beta}\| \|x\| \|y\| , \quad \textit{for all } x \in X,\ y \in Y. \tag{4.8}$$

Proof. In view of (4.5), (4.7), together with standard approximations in $L^1(G)$, it suffices to establish (4.8) in the special case wherein K is compactly supported (which we now assume). Let C be a compact subset of G such that K vanishes outside of C. Temporarily fix vectors $x \in X$, $y \in Y$. By (4.2) and (4.3), we see that

$$\|H_K(x,y)\|^{p_3} \leq A_3^{p_3} \left\| \int_G \beta\left(R_{vu^{-1}}^{(1)} x, R_{vu}^{(2)} y\right) K(u)\, du \right\|^{p_3}, \text{ for all } v \in G. \tag{4.9}$$

Now let $\varepsilon > 0$ be arbitrary, and use that G is a l.c.a. group to get an open neighborhood V of the identity in G such that V has compact closure, and

$$\frac{|m\left(V\left(C \cup C^{-1}\right)\right)|}{m(V)} < 1 + \varepsilon. \tag{4.10}$$

Denote by χ the characteristic function, defined on G, of $V\left(C\cup C^{-1}\right)$. Integrating (4.9) over V with respect to dv, we see that
(4.11)

$$\|H_K(x,y)\|^{p_3}\leq\frac{A_3^{p_3}}{m(V)}\int_V\left\|\int_G\beta\left(R_{vu^{-1}}^{(1)}x,R_{vu}^{(2)}y\right)\chi\left(vu^{-1}\right)\chi(vu)\,K(u)\,du\right\|^{p_3}dv.$$

Next, let us define $f\in L_X^{p_1}(G)$ and $g\in L_Y^{p_2}(G)$ by writing, for all $u\in G$,

$$f(u)=\chi(u)\,R_u^{(1)}x;\ g(u)=\chi(u)\,R_u^{(2)}y. \tag{4.12}$$

We can accordingly rewrite (4.11) in the following form:

$$\|H_K(x,y)\|^{p_3}\leq\frac{A_3^{p_3}}{m(V)}\int_V\|B_{K,\beta}(f,g)(v)\|^{p_3}\,dv.$$

Consequently,

$$\begin{aligned}\|H_K(x,y)\| &\leq \frac{A_3}{[m(V)]^{1/p_3}}\|B_{K,\beta}(f,g)\|_{L_Z^{p_3}(G)} \\ &\leq \frac{A_3}{[m(V)]^{1/p_3}}\|B_{K,\beta}\|\,\|f\|_{L_X^{p_1}(G)}\,\|g\|_{L_Y^{p_2}(G)}\,.\end{aligned}\tag{4.13}$$

By (4.12),

$$\begin{aligned}\|f\|_{L_X^{p_1}(G)} &\leq A_1\|x\|\left[m\left(V\left(C\cup C^{-1}\right)\right)\right]^{1/p_1};\\ \|g\|_{L_Y^{p_2}(G)} &\leq A_2\|y\|\left[m\left(V\left(C\cup C^{-1}\right)\right)\right]^{1/p_2}.\end{aligned}$$

Applying these estimates to (4.13), we see directly that

$$\|H_K(x,y)\|\leq\frac{A_1A_2A_3}{[m(V)]^{1/p_3}}\|B_{K,\beta}\|\left[m\left(V\left(C\cup C^{-1}\right)\right)\right]^{1/p_3}\|x\|\,\|y\|\,,$$

and hence by (4.10),

$$\|H_K(x,y)\|\leq(1+\varepsilon)^{1/p_3}A_1A_2A_3\|B_{K,\beta}\|\,\|x\|\,\|y\|\,. \tag{4.14}$$

We can now let $\varepsilon\to0$ in (4.14) to obtain (4.8), and thereby complete the proof of Theorem 4.3. □

We now specialize our last result to the $L^p(\mu)$-spaces. Actually, we show that the estimate in the general transference result for bilinear maps (Theorem 4.3) can be refined when we specialize the general Banach spaces X, Y, and Z to be, respectively, $L^{p_1}(\mu)$, $L^{p_2}(\mu)$, and $L^{p_3}(\mu)$. This refinement is accomplished by the following lemma which can be demonstrated by suitably adapting the reasoning of Lemma (4.2) of [**2**].

Lemma 4.4. *Let p_1, p_2, p_3 be as in Definition 4.1, and let (Ω,μ) be an arbitrary measure space. Specialize the preceding hypotheses and notation surrounding the arbitrary function $K\in L^1(G)$ to the case where $X=L^{p_1}(\mu)$, $Y=L^{p_2}(\mu)$, and $Z=L^{p_3}(\mu)$, and let the bounded bilinear form $\beta:X\times Y\to Z$ be defined in this case as the pointwise product on Ω: $\beta(f,g)=fg$ (in particular, it is automatic that $\|\beta\|\leq1$ here). Then, in terms of the bilinear mapping s_K defined in Remark 4.2 above, we have*

$$\|B_{K,\beta}\|\leq\|s_K\|\,. \tag{4.15}$$

Remark 4.5. When the hypotheses of Theorem 4.3 are specialized in accordance with the statement of Lemma 4.4, the theorem and lemma combine to furnish the following transference estimate in the resultant measure-theoretic context (Theorem 2.1 of [**6**]):

$$\|H_K\| \leq A_1 A_2 A_3 \|s_K\| .$$

This estimate has the pleasant feature that $\|s_{K,p_1,p_2}\|$ is independent of the abstract measure μ (in contrast to $\|B_{K,\beta}\|$).

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