Bilinear multipliers and transference.

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Abstract

Let $m(\xi,\eta)$ be a regulated function in $\mathbb{R}\times\mathbb{R}$, $1\le p_1,p_2,p_3\le\infty$ and $1/p_1+1/p_2=1/p_3$. It is shown that m defines a bilinear bounded (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$, if and only if there exists a constant K so that $|\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})|\le K\|\hat{\mu}\|_{B_{p_1}}\|\hat{\nu}\|_{B_{p_2}}\|\hat{\lambda}\|_{B_{p_3'}}$ for all measures μ,ν,λ supported on a finite number of points, where $\|\hat{\mu}\|_{B_p}=\lim_{T\to\infty}(\frac{1}{2T}\int_T^T|\hat{\mu}(\xi)|^p d\xi)^{1/p}$.

1 Introduction.

Let (p_1,p_2,p_3) such that $1\le p_1,p_2,p_3\le\infty$, $1/p_1+1/p_2=1/p_3$ and let $m(\xi,\eta)$ be a bounded measurable function in $\mathbb{R}^2$. It is said to be a bilinear (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if

$$\mathcal{C}_m(f,g)(x)=\int_{\mathbb{R}}\int_{\mathbb{R}}\hat{f}(\xi)\hat{g}(\eta)m(\xi,\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta$$

(defined for functions f,g in the Schwartz class $\mathcal{S}$) extends to a bounded bilinear operator from $L^{p_1}(\mathbb{R})\times L^{p_2}(\mathbb{R})$ into $L^{p_3}(\mathbb{R})$.

The theory of these multipliers has been tremendously developped after the results proved by M. Lacey and R. Thiele ([20, 21, 22]) which establish that $m(\xi,\nu)=sign(\xi+\alpha\nu)$ are (p_1,p_2)-multipliers for each triple (p_1,p_2) such that $1<p_1,p_2\le\infty$, $p_3>2/3$ and each $\alpha\in\mathbb{R}\setminus\{0,1\}$.

The study of such multipliers was started by R. Coifman and Y. Meyer (see [3, 5, 6]) for smooth symbols and new results for non-smooth symbols, extending the ones given by the bilinear Hilbert transform, have been achieved

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by J.E. Gilbert and A.R. Nahmod (see [10, 11, 12]) and also by J. Muscalu, T. Tao and C. Thiele (see [19]).

We refer the reader also to [18, 17, 9, 13] for new results on bilinear multipliers and related topics.

In a recent paper (see [9]) D. Fan and S. Sato have shown certain DeLeeuw type theorems for transferring multilinear operators on Lebesgue and Hardy spaces from $\mathbb{R}^n$ to $\mathbb{T}^n$. Here we will consider bilinear multipliers on Lebesgue spaces $L^p(\mathbb{R})$ and get a characterization which allows us to transfer not only to the bilinear multipliers on $\mathbb{T}$ but also on $\mathbb{Z}$. Our approach will follow closely the ideas in the original paper by DeLeeuw (see [8]) and will provide an alternative proof to some results in [9].

Let us start by setting up natural analogue versions of bilinear multipliers in the periodic and discrete cases. Let $(m_{k,k'})$ be a bounded sequence and $\tilde{m}$ be a periodic function defined on $\mathbb{T} \times \mathbb{T}$. Define

$$\mathcal{P}_m(f,g)(\theta) = \sum_{k\in\mathbb{Z}} \sum_{k'\in\mathbb{Z}} \hat{f}(k)\hat{g}(k')m_{k,k'}e^{2\pi i\theta(k+k')}$$

for periodic functions f, g defined on $\mathbb{T}$ and

$$\mathcal{D}_{\tilde{m}}(a,b)(k) = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2} P(t)Q(s)\tilde{m}(t,s)e^{2\pi ix(t+s)}dtds$$

for sequences $(a(n))_{n\in\mathbb{Z}}$ and $(b(n))_{n\in\mathbb{Z}}$ where $P(t) = \sum_{n\in\mathbb{Z}} a(n)e^{2\pi int}$ and $Q(t) = \sum_{n\in\mathbb{Z}} b(n)e^{2\pi int}$.

Now we say that $(m_{k,k'})$ (respect. $\tilde{m}$) is a a bilinear (p_1, p_2)-multiplier on $\mathbb{Z} \times \mathbb{Z}$ (respect. $\mathbb{T} \times \mathbb{T}$) if $\mathcal{P}_m$ (respect. $\mathcal{D}_{\tilde{m}}$) defines a bounded bilinear operator from $L^{p_1}(\mathbb{T}) \times \mathrm{L}^{p_2}(\mathbb{T})$ into $L^{p_3}(\mathbb{T})$ (respect. $\ell^{p_1}(\mathbb{Z}) \times \ell^{p_2}(\mathbb{Z})$ into $\ell^{p_3}(\mathbb{Z})$).

Of course we can see these three cases as instances of the general bilinear multiplier acting on different groups. Let G be a locally compact abelian group and $\hat{G}$ its dual. Let $1 \le p_1, p_2 \le \infty$ and m be a bounded measurable function defined on $\hat{G} \times \hat{G}$. We say that m is a (p_1, p_2)-multiplier on $\hat{G} \times \hat{G}$ if the operator

$$T_m(f,g)(x) = \int_{\hat{G}}\int_{\hat{G}} \mathcal{F}f(\gamma_1)\mathcal{F}g(\gamma_2)m(\gamma_1,\gamma_2)\gamma_1(-x)\gamma_2(-x)dm(\gamma_1)dm(\gamma_2)$$

(defined for simple functions f and g) extends to a bounded bilinear operator from $L^{p_1}(G) \times L^{p_2}(G)$ to $L^{p_3}(G)$ where $1/p_1 + 1/p_2 = 1/p_3$.

The first transference results on linear multiplier were given by K. Deleeuw (see [8]). He showed, among other things, that if m is regulated (all its points are Lebesgue points) and m is a p-multiplier on $\mathbb{R}$ then $(m(\varepsilon k))_k$ are uniformly bounded p-multipliers for all $\varepsilon > 0$ on $\mathbb{Z}$. See [25] page 264 for the converse of this result for continuous multipliers.

In [9] the multilinear version this result was shown, namely that for continuous functions $m(\xi, \eta)$ one has that m is a (p_1, p_2)-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $m(\varepsilon k, \varepsilon k')_{k,k'}$ are uniformly bounded multipliers on $\mathbb{Z} \times \mathbb{Z}$. An extension of the result to Lorentz spaces is achieved in [2].

We shall first characterize the boundedness of bilinear multipliers on $\mathbb{R} \times \mathbb{R}$ by the existence of a constant K such that

$$|\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})| \leq K\|\hat{\mu}\|_{B_{p_1}}\|\hat{\nu}\|_{B_{p_2}}\|\hat{\lambda}\|_{B_{p_3'}}$$

for all measures μ, ν, λ of finite supports.

This allows us to present an alternative proof of the result in [9].

We also obtain the transference from the continuous case $\mathcal{C}_m$ to the periodic case $\mathcal{P}_m$. Our main result establishes that m is (p_1, p_2)-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if $D_\varepsilon m = m_{\varepsilon.,\varepsilon.}\chi_{[-1/2,1/2]\times[-1/2,1/2]}$ are uniformly bounded (p_1, p_2)-multipliers on $\mathbb{T} \times \mathbb{T}$.

Throughout the paper $1 \leq p_1, p_2, p_3 \leq \infty$ and $1/p_3 = 1/p_1 + 1/p_2$. For a given finite Borel measure on $\mathbb{R}$ we write $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi t}d\mu(t)$ and, for an almost periodic function g, we denote $\|g\|_{B_p} = \lim_{T\to\infty}(\frac{1}{2T}\int_{-T}^{T}|g(t)|^p dt)^{1/p}$. We shall use the notations $D_\varepsilon m(x,y) = m(\varepsilon x, \varepsilon y)$ and $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(\frac{x}{\varepsilon})$.

2 Bilinear multipliers on $\mathbb{R} \times \mathbb{R}$

Let us start by reformulating the condition of (p_1, p_2)-multiplier on $\mathbb{R} \times \mathbb{R}$ using duality. The proof is straightforward and left to the reader.

Lemma 2.1 *Let $m(\xi, \eta)$ be a bounded measurable function on $\mathbb{R} \times \mathbb{R}$.*

m is a (p_1, p_2)-multiplier on $\mathbb{R} \times \mathbb{R}$ if and only if there exists a constant K so that

$$|\int_{\mathbb{R}}\int_{\mathbb{R}} \phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\xi,\eta)d\xi d\eta| \leq K\|\hat{\phi}\|_{p_1}\|\hat{\psi}\|_{p_2}\|\hat{\nu}\|_{p_3'} \qquad (1)$$

for all $\phi, \psi, \nu \in \mathcal{S}$.

Now we present some behavior of multipliers on $\mathbb{R}\times\mathbb{R}$ with respect to convolution and dilation operators to be used later on.

Lemma 2.2 *Let $m(\xi,\eta)$ be a bounded measurable function on $\mathbb{R}\times\mathbb{R}$. If $\Phi\in L^1(\mathbb{R}^2)$ and m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ then $m*\Phi$ a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ and $\|C_{\Phi*m}\|\leq\|\Phi\|_1\|\mathcal{C}_m\|$.*

Proof. Let $\phi,\psi,\nu\in\mathcal{S}$ and $\|\hat\phi\|_{p_1}=\|\hat\psi\|_{p_2}=\|\hat\nu\|_{p_3'}=1$. Applying Lemma 2.1 to ϕ_s,ψ_t,ν_{t+s} where $f_s(x)=f(x+s)$, we have

$$|\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi+s)\psi(\eta+t)\nu(\xi+\eta+t+s)m(\xi,\eta)d\xi d\eta|\leq K$$

for all $(s,t)\in\mathbb{R}^2$.

Therefore

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m*\Phi(\xi,\eta)d\xi d\eta\\ =\;&\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi)\psi(\eta)\nu(\xi+\eta)(\int_{\mathbb{R}^2}m(\xi-s,\eta-t)\Phi(s,t)dsdt)d\xi d\eta\\ =\;&\int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi+s)\psi(\eta+t)\nu(\xi+\eta+s+t)m(\xi,\eta)\Phi(s,t)d\xi d\eta dsdt.\end{aligned}$$

This gives the result applying Lemma 2.1 again. ■

Lemma 2.3 *Let $\varepsilon>0$ and $m(\xi,\eta)$ be a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$. Then $m(\varepsilon\xi,\varepsilon\eta)$ is also a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ and $\|C_{m(\varepsilon.,\varepsilon.)}\|\leq\|\mathcal{C}_m\|$.*

Proof. For $\phi,\psi,\nu\in\mathcal{S}$ and $\|\hat\phi\|_{p_1}=\|\hat\psi\|_{p_2}=\|\hat\nu\|_{p_3'}=1$ we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\varepsilon\xi,\ e\eta)d\xi d\eta\\ =\;&\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\varepsilon^{1/p_1}}\phi(\frac{\xi}{\varepsilon})\frac{1}{\varepsilon^{1/p_2}}\psi(\frac{\eta}{\varepsilon})\frac{1}{\varepsilon^{1/p_3'}}\nu(\frac{\xi+\eta}{\varepsilon})m(\xi,\eta)d\xi d\eta\end{aligned}$$

where the functions now apperaring in the integral are also norm 1 for each ε. Use Lemma 2.1 again to finish the proof. ■

Theorem 2.4 *Let $m(\xi,\eta)$ be a bounded continuous function on $\mathbb{R}\times\mathbb{R}$. The following are equivalent:*

(i) m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$.

(ii) There exists a constant K so that

$$|\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})| \leq K\|\hat{\mu}\|_{B_{p_1}}\|\hat{\nu}\|_{B_{p_2}}\|\hat{\lambda}\|_{B_{p'_3}}$$

for all measures μ,ν,λ supported on a finite number of points.

Proof. (i) $\Rightarrow$ (ii) Assume that m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$. Denote by ϕ the gaussian function $\phi(x)=e^{-x^2/2}$ and take $0<\alpha,\beta,\gamma$ such that $\alpha+\beta+\gamma=2$.

Let us consider $\mu=\delta_a$, $\nu=\delta_b$ and $\lambda=\delta_c$ for $a,b,c\in\mathbb{R}$ and let us observe that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\varepsilon^2}\phi^\alpha(\frac{\xi-a}{\varepsilon})\phi^\beta(\frac{\eta-b}{\varepsilon})\phi^\gamma(\frac{\xi+\eta-c}{\varepsilon})m(\xi,\eta)d\xi d\eta=$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta+\frac{a+b-c}{\varepsilon})m(a+\varepsilon\xi,b+\varepsilon\eta)d\xi d\eta=$$

$$=\int_{\mathbb{R}}\int_{\mathbb{R}}\mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta.$$

Since

$$\lim_{\varepsilon\to 0}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta+\frac{a+b-c}{\varepsilon})m(a+\varepsilon\xi,b+\varepsilon\eta)=$$

$$\delta_c(a+b)\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)m(a,b),$$

the convergence Lebesgue theorem implies that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{1}{\varepsilon^2}\phi^\alpha(\frac{\xi-a}{\varepsilon})\phi^\beta(\frac{\eta-b}{\varepsilon})\phi^\gamma(\frac{\xi+\eta-c}{\varepsilon})m(\xi,\eta)d\xi d\eta$$

$$=Cm(a,b)\delta_c(a+b)=Cm(a,b)\mu(\{a\})\nu(\{b\})\lambda(\{a+b\}).$$

where $C=\int_{\mathbb{R}}\int_{\mathbb{R}}\phi^\alpha(\xi)\phi^\beta(\eta)\phi^\gamma(\xi+\eta)d\xi d\eta$.

Therefore we have that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\mu*(\phi_\varepsilon)^\alpha(\xi)\nu*(\phi_\varepsilon)^\beta(\eta)\lambda*(\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta$$

$$= C \sum_{(t,s)\in \mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{(t+s)\})$$

for all measures μ, ν, λ having their supports on finite sets of points.

On the other hand, from the assumption and Lemma 2.1 we have

$$|\int_{\mathbb{R}}\int_{\mathbb{R}} \mu * (\phi_\varepsilon)^\alpha(\xi)\nu * (\phi_\varepsilon)^\beta(\eta)\lambda * (\phi_\varepsilon)^\gamma(\xi+\eta)m(\xi,\eta)d\xi d\eta|$$

$$\leq K\|\hat{\mu}(\widehat{\phi_\varepsilon})^\alpha\|_{p_1}\|\hat{\nu}(\widehat{\phi_\varepsilon})^\beta\|_{p_2}\|\hat{\lambda}(\widehat{\phi_\varepsilon})^\gamma\|_{p_3'}.$$

Let us now choose $\alpha = \frac{1}{p_1'}$, $\beta = \frac{1}{p_2'}$ and $\gamma = \frac{1}{p_3}$. Since $(\phi_\varepsilon)^\alpha = \frac{\varepsilon^{1-\alpha}}{\alpha^{1/2}}\phi_{\varepsilon\alpha^{-1/2}}$, we get $(\widehat{\phi_\varepsilon})^\alpha(\xi) = C_\alpha \varepsilon^{1/p_1}e^{-\frac{\varepsilon^2\xi^2}{2\alpha}}$, $(\widehat{\phi_\varepsilon})^\beta(\xi) = C_\beta\varepsilon^{1/p_2}e^{-\frac{\varepsilon^2\xi^2}{2\beta}}$ and $(\widehat{\phi_\varepsilon})^\gamma(\xi) = C_\gamma\varepsilon^{1/p_3}e^{-\frac{\varepsilon^2\xi^2}{2\gamma}}$.

Now taking into account that $\int_{\mathbb{R}} e^{-\frac{\varepsilon^2 p_1\xi^2}{2\alpha}}d\xi = C_\alpha'\varepsilon^{-1}$ we have that

$$\|\hat{\mu}(\widehat{\phi_\varepsilon})^\alpha\|_{p_1} = C(\frac{1}{A(\varepsilon)}\int_{\mathbb{R}}|\hat{\mu}(\xi)|^{p_1}\varepsilon^{-\frac{p_1\varepsilon^2\xi^2}{2\alpha}}d\xi)^{1/p_1},$$

for $A(\varepsilon) = \int_{\mathbb{R}} e^{-\frac{\varepsilon^2 p_1\xi^2}{2\alpha}}d\xi$. Hence $C\|\hat{\mu}\|_{B_{p_1}} = \lim_{\varepsilon\to 0}\|\hat{\mu}\hat{\phi}_\varepsilon^\alpha\|_{p_1}$.

Applying similar procedure for ν and λ we finish this implication.

(ii) $\Rightarrow$(i) From the assumption we can get that the same holds for all finite measures μ, ν, λ with countable support. Let us take ϕ, ψ and ρ such that $\hat{\phi}, \hat{\psi}$ and $\hat{\rho}$ have compact support contain in $[-N/2, N/2]$ for N big enough. Now consider μ_N, ν_N and λ_N the measures with support in $(1/N)\mathbb{Z}$ whose Fourier transform coincide with the periodic extensions of $\hat{\phi}, \hat{\psi}$ and $\hat{\rho}$. In particular we have

$$\mu_N(\{\frac{n}{N}\}) = \frac{1}{N}\phi(\frac{n}{N}), \nu_N(\{\frac{n}{N}\}) = \frac{1}{N}\psi(\frac{n}{N}) \text{ and } \lambda_N(\{\frac{n}{N}\}) = \frac{1}{N}\rho(\frac{n}{N}).$$

Therefore we have

$$\begin{aligned} &\lim_{N\to\infty} N \sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu_N(\{t\})\nu_N(\{s\})\lambda_N(\{t+s\}) \\ = &\lim_{N\to\infty}\sum_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} m(\frac{n}{N},\frac{m}{N})\phi(\frac{n}{N})\psi(\frac{m}{N})\rho(\frac{n+m}{N})\frac{1}{N^2} \\ = &\int_{\mathbb{R}}\int_{\mathbb{R}} m(\xi,\nu)\phi(\xi)\psi(\eta)\rho(\xi+\eta)d\xi d\eta. \end{aligned}$$

Now observe that $\|\hat{\mu}_N\|_{B_{p_1}} = (\frac{1}{2N}\int_{-N}^{N}|\hat{\phi}(\xi)|^{p_1}d\xi)^{1/p_1} = (\frac{1}{2N})^{1/p_1}\|\hat{\phi}\|_{p_1}$ and the same for the others.

Using that $\|\hat{\mu}_N\|_{B_{p_1}}.\|\hat{\nu}_N\|_{B_{p_2}}\|\hat{\lambda}_N\|_{B_{p_3'}} = \frac{1}{2N}$ and passing to the limit we get the result. ■

Recall that a function m is called regulated if

$$\lim_{\varepsilon\to 0}\frac{1}{4\varepsilon^2}\int_{-\varepsilon}^{\varepsilon}\int_{-\varepsilon}^{\varepsilon} m(x-s,y-t)dsdt = m(x,y)$$

for all $(x,y)\in \mathbb{R}^2$.

Theorem 2.5 *Let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R}\times\mathbb{R}$. m is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ if and only if here exists a constant K so that*

$$|\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})| \leq K\|\hat{\mu}\|_{B_{p_1}}\|\hat{\nu}\|_{B_{p_2}}\|\hat{\lambda}\|_{B_{p_3'}} \qquad (2)$$

for all measures μ,ν,λ having their supports on finite sets of points.

Proof. Assume that m is (p_1,p_2)-multiplier. Denote $\Phi(s,t) = \frac{1}{4}\chi_{[-1,1]}(s)\chi_{[-1,1]}(t)$ and $\Phi_\varepsilon(\xi,\eta) = \frac{1}{\varepsilon^2}\Phi(\frac{\xi}{\varepsilon},\frac{\eta}{\varepsilon})$ for $\varepsilon > 0$. Now Lemma 2.2, Theorem 2.4 and the fact that $m(x,y) = \lim_{\varepsilon\to 0} m*\Phi_\varepsilon(x,y)$ gives the direct implication.

Conversely, assume (2) for μ,ν,λ having finite supports,

$$\begin{aligned}
&\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m*\Phi_\varepsilon(t,s)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\})\\
= &\int(\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t-u,s-v)\mu(\{t\})\nu(\{s\})\lambda(\{t+s\}))\Phi_\varepsilon(u,v)dudv\\
= &\int(\sum_{(t,s)\in\mathbb{R}\times\mathbb{R}} m(t,s)\mu(\{t+u\})\nu(\{s+v\})\lambda(\{t+s+u+v\}))\Phi_\varepsilon(u,v)dudv.
\end{aligned}$$

This shows that $m*\Phi_\varepsilon$ verifies (2) with uniform constant for all $\varepsilon > 0$. Now apply Theorem 2.4 to get that $m*\Phi_\varepsilon$ are (p_1,p_2)-multipliers with uniform norm.

Finally we have that for $\phi,\psi,\nu\in\mathcal{S}$

$$\begin{aligned}
&|\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m(\xi,\eta)d\xi d\eta|\\
= &|\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}}\phi(\xi)\psi(\eta)\nu(\xi+\eta)m*\Phi_\varepsilon(\xi,\eta)d\xi d\eta|\\
\leq &C||\hat{\phi}||_{p_1}||\hat{\psi}||_{p_2}||\hat{\nu}||_{p_3'}.
\end{aligned}$$

The result follows now from Lemma 2.1. ■

3 Transference theorems

Let us mention the formulations for (p_1, p_2)-multipliers on the groups $\mathbb{T}$ and $\mathbb{Z}$ which follows directly from duality.

Lemma 3.1 *Let $\tilde{m}(t,s)$ be a bounded measurable function on $\mathbb{T}\times\mathbb{T}$.*

m is a (p_1, p_2)-multiplier on $\mathbb{T}\times\mathbb{T}$ if and only if there exists a constant K so that

$$|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} P_a(t)P_b(s)P_c(t+s)\tilde{m}(t,s)dtds| \leq K\|a\|_{p_1}\|b\|_{p_2}\|c\|_{p_3'}$$

for all finite sequences $(a(n))_n, (b(n))_n, (c(n))_n$ where $P_a(t) = \sum_n a(n)e^{2\pi int}$.

Lemma 3.2 *Let $(m_{k,k'})$ be a bounded sequence on $\mathbb{Z}\times\mathbb{Z}$*

m is a (p_1, p_2)-multiplier on $\mathbb{Z}\times\mathbb{Z}$ if and only if there exists a constant K so that

$$|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} m_{k,k'}\hat{P}(k)\hat{Q}(k')\hat{R}(k+k')| \leq K\|P\|_{p_1}\|Q\|_{p_2}\|\|R\|_{p_3'}$$

for all trigonometric polinomials P, Q and R.

Theorem 3.3 *(See [9]) Let $m(\xi,\eta)$ be a regulated bounded function on $\mathbb{R}\times\mathbb{R}$. If $m(\xi,\eta)$ is a (p_1, p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$ then $(m(k,k'))_{k,k'}$ is a (p_1, p_2)-multiplier on $\mathbb{Z}\times\mathbb{Z}$.*

Proof. According to Lemma 3.2 we have to show that there exists a constant K so that

$$|\sum_{k\in\mathbb{Z}}\sum_{k'\in\mathbb{Z}} m(k,k')\hat{P}(k)\hat{Q}(k')\hat{R}(k+k')| \leq K\|P\|_{p_1}\|Q\|_{p_2}\|\|R\|_{p_3'}$$

for all trigonometric polinomials P, Q and R.

This follows by selecting in Theorem 2.5 the measures μ, ν, λ such that $\hat{\mu} = P$, $\hat{\nu} = Q$ and $\hat{\lambda} = R$. ■

Theorem 3.4 *Let $m(\xi,\eta)$ be a bounded regulated function on $\mathbb{R}\times\mathbb{R}$. The following are equivalent:*

(i) $m(\xi,\eta)$ is a (p_1,p_2)-multiplier on $\mathbb{R}\times\mathbb{R}$.

(ii) $m(\varepsilon.,\varepsilon.)\chi_{[-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}]}\chi_{[-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}]}$ are uniformly bounded (p_1,p_2)-multipliers on $\mathbb{T}\times\mathbb{T}$.

Proof. (i)$\Rightarrow$(ii). Using Lemma 3.1, it suffices to show that for any finite sequences $(a(n))_n$, $(b(n))_n$ and $(c(n))_n$ with $\|a\|_{p_1}=\|b\|_{p_2}=\|c\|_{p_3'}=1$ there exists a constant $K>0$ such that

$$|\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} m(\xi,\eta)P_a(\xi)P_b(\eta)P_c(\xi+\eta)d\xi d\eta|\leq K$$

where $P_a(\xi)=\sum_n a(n)e^{2\pi i n\xi}$.

Since $P_a(x)\chi_{[-1/2,-1/2]}(x)=\hat{\phi}_a(x)$ where $\phi_a(x)=\sum_n a(n)\frac{\sin(\pi(x-n))}{\pi(x-n)}$ and $P_c(x)\chi_{[-1,-1]}(x)=\hat{\psi}_c(x)$ where $\psi_c(x)=\sum_n c(n)\frac{\sin(2\pi(x-n))}{\pi(x-n)}$ we can write

$$\begin{aligned}&\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} m(\xi,\eta)P_a(\xi)P_b(\eta)P_c(\xi+\eta)d\xi d\eta\\ =\ &\int_{\mathbb{R}}\int_{\mathbb{R}} m(\xi,\eta)\hat{\phi}_a(\xi)\hat{\phi}_b(\eta)\hat{\psi}_c(\xi+\eta)d\xi d\eta\end{aligned}$$

Using now the assumption and the known facts that $\|\phi_a\|_{L^p(\mathbb{R})}\approx\|a\|_{\ell_p}\approx\|\psi_a\|_{L^p(\mathbb{R})}$ for all $1\leq p\leq\infty$ we obtain the desired inequality.

Now we apply Lemma 2.3 to get the result for each ε.

(ii)$\Longrightarrow$(i) Let us take ϕ and ψ such that $supp\phi$ and $supp\psi$ are contained in $[-1/4,1/4]$. For a fixed $u\in[-1/2,1/2]$ consider the periodic extension of the functions $\hat{\phi}(\xi)e^{2\pi iu\xi}$, $\hat{\psi}(\eta)e^{2\pi iu\eta}$ to be denoted $\tilde{P}_u$ and $\tilde{Q}_v$ respectively.

If $a^u(n)=\int_{-1/2}^{1/2}\tilde{P}_u(\xi)e^{-i2\pi n\xi}d\xi$, $b^u(n)=\int_{-1/2}^{1/2}\tilde{Q}_u(\xi)e^{-i2\pi n\xi}d\xi$ for all $n\in\mathbb{Z}$ we have that if $x=k+u$ for some $k\in\mathbb{Z}$ and $u\in[-1/2,1/2)$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} m(\xi,\eta)\hat{\phi}(\xi)\hat{\psi}(\eta)e^{2\pi ix(\xi+\eta)}d\xi d\eta=$$

$$=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} m(\xi,\eta)\tilde{P}_u(\xi)\tilde{Q}_u(\eta)e^{2\pi ik(\xi+\eta)}d\xi d\eta.$$

Denote $\tilde{m}(\xi,\eta)=m(\xi,\eta)\chi_{[-1/2,1/2]}(\xi)\chi_{[-1/2,1/2]}(\eta)$. Hence for $x=u+k$

$$\mathcal{C}_m(\phi,\psi)(x)=\mathcal{D}_{\tilde{m}}(a^u,b^u)(k).$$

Now

$$
\begin{aligned}
&\int_{\mathbb{R}} |\mathcal{C}_m(\phi,\psi)(x)|^{p_3} dx = \\
= \ & \sum_k \int_{-1/2}^{1/2} |\mathcal{C}_m(\phi,\psi)(k+u)|^{p_3} du \\
= \ & \int_{-1/2}^{1/2} \sum_k |\mathcal{D}_{\tilde{m}}(a^u,b^u)(k)|^{p_3} du \\
\leq \ & \|\mathcal{D}_{\tilde{m}}\|^{p_3} \int_{-1/2}^{1/2} (\sum_k |a^u(k)|^{p_1})^{p_3/p_1} (\sum_k |b^u(k)|^{p_2})^{p_3/p_2} du \\
\leq \ & \|\mathcal{D}_{\tilde{m}}\|^{p_3} (\int_{-1/2}^{1/2} \sum_k |a^u(k)|^{p_1} du)^{p_3/p_1} (\int_{-1/2}^{1/2} \sum_k |b^u(k)|^{p_2} du)^{p_3/p_2} \\
= \ & \|\mathcal{D}_{\tilde{m}}\|^{p_3} (\int_{-1/2}^{1/2} \sum_k |\phi(u+k)|^{p_1} du)^{p_3/p_1} (\int_{-1/2}^{1/2} \sum_k |\psi(u+k)|^{p_2} du)^{p_3/p_2} \\
= \ & \|\mathcal{D}_{\tilde{m}}\|^{p_3} \|\phi\|_{p_1}^{p_3} \|\psi\|_{p_2}^{p_3}
\end{aligned}
$$

In the general case if ϕ, ψ are such that $\hat{\phi}, \hat{\psi}$ have compact support, then there exists $\varepsilon > 0$ so that $\hat{\phi}_\varepsilon, \hat{\psi}_\varepsilon$ have their support in $[-1/4, 1/4]$. Now observe that

$$\mathcal{C}_m(\phi,\psi)(x) = \varepsilon^2 C_{m(\varepsilon.,\varepsilon.)}(\phi_\varepsilon, \psi_\varepsilon)(\varepsilon x).$$

Applying the previous case and the assumption we obtain

$$
\begin{aligned}
\|\mathcal{C}_m(\phi,\psi)\|_{p_3} &= \varepsilon^{2-1/p_3} \|C_{m(\varepsilon.,\varepsilon.)}(\phi_\varepsilon,\psi_\varepsilon)\|_{p_3} \\
&\leq K\varepsilon^{2-1/p_3} \|\phi_\varepsilon\|_{p_1} \|\psi_\varepsilon\|_{p_2} \\
&= K\varepsilon^{2-1/p_3} \|\phi\|_{p_1} \varepsilon^{-1/p_1'} \|\psi\|_{p_1} \varepsilon^{-1/p_2'} \\
&= K \|\phi\|_{p_1} \|\psi\|_{p_1}.
\end{aligned}
$$

■

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