# A CHARACTERIZATION OF HILBERT SPACES IN TERMS OF MULTIPLIERS BETWEEN SPACES OF VECTOR VALUED ANALYTIC FUNCTIONS.

Oscar Blasco

## §0.-Introduction.

Given a complex Banach space $(X, ||.||)$ we shall denote by $H^1(X)$ the space of $X$-valued Bochner integrable function on the circle $\mathbb{T} = \{|z| = 1\}$ whose negative Fourier coefficients vanish, that is

$$H^1(X) = \{f \in L^1(\mathbb{T}, X) : \hat{f}(n) = 0 \text{ for } n < 0\}.$$

We write $||f||_{1,X} = \int_0^{2\pi} ||f(e^{it})|| \frac{dt}{2\pi}$ for the norm in $H^1(X)$.

We shall also denote by $BMOA(X)$ the space of vector valued BMO functions on the circle with analytic extension to the unit disc $D$, that is $f \in L^1(\mathbb{T}, X)$ with $\hat{f}(n) = 0$ for $n < 0$ such that

$$||f||_{*,X} = \sup_I \left( \frac{1}{|I|} \int_I ||f(e^{it}) - f_I||^2 \frac{dt}{2\pi} \right)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all intervals $I \in \mathbb{T}$ and $|I|$ stands for the normalized Lebesgue measure of $I$ and $f_I = \frac{1}{|I|} \int_I f(e^{it}) \frac{dt}{2\pi}$.

The norm in the space is given by

$$||f||_{BMO(X)} = || \int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi} || + ||f||_{*,X}.$$

Finally we shall use the notation $Bloch(X)$ for the space of $X$-valued analytic functions on $D$, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, such that $\sup_{|z|<1} (1 - |z|) ||f'(z)|| < \infty$. To avoid constant functions having zero norm we consider

$$||f||_{Bloch(X)} = ||f(0)|| + \sup_{|z|<1} (1 - |z|) ||f'(z)||.$$

Now given two complex Banach spaces $X, Y$ and denoting by $B(X, Y)$ the space of bounded operatos from $X$ into $Y$, simply $B(X)$ when $X = Y$, we can formulate the following definition which is the natural analogue of the scalar-valued notion of convolution multiplier.

Given $F \in Bloch\,(B(X,Y))$, say $F(z) = \sum_{n=0}^{\infty} T_n z^n$ and $f \in H^1(X)$, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$ we shall define

$$F * f(z) = \sum_{n=0}^{\infty} T_n(x_n) z^n = \int_0^{2\pi} F(ze^{it})\left(f(e^{-it})\right)\frac{dt}{2\pi}.$$

Let us write $\left(H^1(X), BMOA(Y)\right)$ for the space of functions $F : D \to B(X,Y)$ such that $F * f \in BMOA(Y)$ for any $f \in H^1(X)$.

The norm on it is induced by the norm as subspace of $B\left(H^1(X), BMOA(Y)\right)$.

It was proved in [6] that the space of multipliers from $H^1$ into $BMOA$ can be identified with the space of Bloch functions, i.e.

$$\left(H^1, BMOA\right) = Bloch. \tag{0.1}$$

It is not hard to see that the vector valued formulation does not hold for general Banach spaces. The aim of this note is to show that the vector-valued extension for $X = Y$ holds only for Hilbert spaces. We shall prove the following theorem.

**Theorem.** *Let $X$ be a complex Banach space.*
*$\left(H^1(X), BMOA(X)\right) = Bloch\,(B(X))$ if and only if $X$ is isomorphic to a Hilbert space.*

Throughtout the paper all Banach spaces appearing are assumed to be vector spaces on the complex field and $C$ will stand for a constant which may vary form line to line.

## §1.-Preliminary results

Let us recall some known facts on vector valued analytic functions that we shall need for the proof.

First of all let us recall the characterization of BMO functions in terms of Carleson measures (see [ 4,Theorem 3.4 ]) that we shall use later on. This is still valid for functions taking values in Hilbert spaces (since it simply relies on Plancherel's theorem). Given a Hilbert space $X$ and an analytic function $f : \mathbb{D} \to X$ we have

$$||f||_{*,X} \approx \sup_{z\in D}\left(\int_0^1\int_0^{2\pi}\frac{(1-s)(1-|z|^2)||f'(se^{it})||^2}{|1-\bar{z}se^{it}|^2}\frac{dt}{2\pi}ds\right)^{\frac{1}{2}}. \tag{1.1}$$

Another fact to be used is that Kintchine's inequalities hold for BMO functions, actually this can be achieved using Paley's inequality (see [3])and duality . That is

$$\left(\sum_{k=0}^{\infty}|\alpha_k|^2\right)^{\frac{1}{2}} \approx ||\sum_{k=0}^{\infty}\alpha_k z^{2^k}||_{BMOA}. \tag{1.2}$$

Regarding vector valued Bloch functions, let us point out the following remarks.

Given $(T_n) \subset B(X,Y)$ and $F(z) = \sum_{n=0}^{\infty} T_n z^n$. It follows obviously from the definition that $F \in Bloch\,(B(X,Y))$ if and only if for any $x \in X, y^* \in Y^*$ the functions $F_{x,y^*}(z) = \sum_{n=0}^{\infty} < T_n(x), y^* > z^n \in Bloch$ . Moreover

$$||F||_{Bloch(B(X,Y))} = \sup_{||x||\leq 1, ||y^*||\leq 1} ||F_{x,y^*}||_{Bloch}. \tag{1.3}$$

According to this it follows from the scalar valued case (see [1,2]) that

$$F(z) = \sum_{n=0}^{\infty} T_n z^{2^n} \text{ if and only if } \sup_{n\in\mathbb{N}} ||T_n|| < \infty. \tag{1.4}$$

Let us now recall a basic inequality, due to Hardy and Littlewood (see [5,Lemma HL1]), which played an important role in the proof of (0.1) and whose vector valued extension we are going to use.

There exists a constant $C > 0$ such that for any $f \in H^1$ one has

$$\left(\int_0^1 (1-r) M_1^2(f', r)dr\right)^{\frac{1}{2}} \leq C||f||_1. \tag{1.5}$$

where $M_1(f', r) = \int_0^{2\pi} |f'(re^{it})| \frac{dt}{2\pi}$.

Using the notation $M_{1,X}(f', r) = \int_0^{2\pi} ||f'(re^{it})|| \frac{dt}{2\pi}$ when dealing with functions in $H^1(X)$ we have the following vector valued extension.

**Lemma 1.1.** *Let $X$ be a Hilbert space. Then there exists a constant $C > 0$ such that*

$$\left(\int_0^1 (1-r) M_{1,X}^2(f', r)dr\right)^{\frac{1}{2}} \leq C||f||_{1,X}$$

*for any $f \in H^1(X)$.*

*Proof.* Let us assume that $X = l^2$ (for general Hilbert spaces it would follow from the previous case and the fact that $X$ is finitely representable in $l^2$).

Given $f \in H^1(l^2)$ we can write $f = (f_n)$ where $f_n \in H^1$ and $\left(\sum_{n=1}^{\infty} |f_n(e^{i\theta})|^2\right)^{\frac{1}{2}} \in L^1(\mathbb{T})$. Denoting by $r_n$ the Rademacher functions in $[0,1]$ we define

$$F(z) = \sum_{n=1}^{\infty} f_n(z) r_n, \qquad F_t(z) = \sum_{n=1}^{\infty} f_n(z) r_n(t).$$

It follows from Fubini's theorem and Kintchine's inequalities that

$$||F||_{1,L^1} \approx ||f||_{1,l^2}, \qquad M_{1,L^1}(F', r) \approx M_{1,l^2}(f', r).$$

Therefore, setting $\alpha_k = 1 - 2^{-k}$,

$$\begin{aligned}\int_0^1 (1-r)M^2_{1,l^2}(f',r)dr &\approx \int_0^1 (1-r)M^2_{1,L^1}(F',r)dr\\ &= \sum_{k=0}^\infty \int_{\alpha_{k+1}}^{\alpha_k} (1-r)M^2_{1,L^1}(F',r)dr\\ &\le \sum_{k=0}^\infty 2^{-2k}M^2_{1,L^1}(F',\alpha_k)\\ &\le \sum_{k=0}^\infty ||2^{-k}M_1(F'_t,\alpha_k)||^2_{L^1([0,1])}.\end{aligned}$$

With this estimate together with the well known fact, due to the cotype 2 condition on $L^1$ (see [6]), that

$$\left(\sum_{k=0}^\infty ||\phi_k||^2_{L^1([0,1])}\right)^{\frac{1}{2}} \le C||\big(\sum_{k=0}^\infty (|\phi_k(t)|)^2\big)^{\frac{1}{2}}||_{L^1([0,1])}$$

and applying the scalar inequality (1.5), we can write

$$\begin{aligned}\left(\int_0^1 (1-r)M^2_{1,l^2}(f',r)dr\right)^{\frac{1}{2}} &\le \int_0^1 \left(\sum_{k=0}^\infty 2^{-2k}M_1^2(F'_t,\alpha_k)\right)^{\frac{1}{2}} dt\\ &\le C\int_0^1 \left(\int_0^{2\pi} (1-r)M_1^2(F'_t,r)dr\right)^{\frac{1}{2}} dt\\ &\le C\int_0^1\int_0^{2\pi} |F_t(e^{i\theta})|\frac{d\theta}{2\pi}dt = C||F||_{1,L^1} \approx ||f||_{1,l^2}. \quad \square\end{aligned}$$

Let us finish this section by recalling the notions of type and cotype (where we replace the Rademacher functions by lacunary sequences). The reader is referred to [8, 10 ] for information on these properties.

A Banach space has cotype 2 (respec. type 2) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and for all $x_1, x_2, ...x_N \in X$ one has

$$\left(\sum_{k=1}^N ||x_k||^2\right)^{\frac{1}{2}} \le C||\sum_{k=1}^N x_k e^{2^k it}||_{1,X}$$

(respect. $$||\sum_{k=1}^N x_k e^{2^k it}||_{1,X} \le C\left(\sum_{k=1}^N ||x_k||^2\right)^{\frac{1}{2}}. \quad )$$

Let us finally recall Kwapien's characterization of Hilbert spaces (see [9]):

$X$ is isomorphic to a Hilbert space if and only if $X$ has type and cotype 2.

## §3.- Proof of the theorem

**Lemma 3.1.** *Let $X, Y$ be two complex Banach spaces. Then*

$$\big(H^1(X), BMOA(Y)\big) \subset Bloch\,(B(X,Y))\,.$$

*Proof.* Given $F \in \big(H^1(X), BMOA(Y)\big)$ and $x \in X, y^* \in Y^*$ then clearly one has $< F(z)(x), y^* > \in \big(H^1, BMOA\big) = Bloch$. Moreover

$$|| < F(z)(x), y^* > ||_{Bloch} \leq ||F||_{(H^1(X),BMOA(Y))}||x||||y^*||.$$

Hence (1.3) shows that $F \in Bloch\,(B(X,Y))$. □

**Proof of the theorem.**

From Kwapien's result we shall show first that $\big(H^1(X), BMOA(X)\big) = Bloch\,(B(X))$ implies cotype 2 and type 2 on $X$.

Let us take $x_1, x_2, ...x_N \in X$. Then choose $x_n^* \in X^*$ so that $< x_n^*, x_n >= ||x_n||$ and $||x_n^*|| = 1$ . Then, using (1.2)

$$\left(\sum_{k=1}^N ||x_k||^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^N | < x_k^*, x_k > |^2\right)^{\frac{1}{2}} \approx || \sum_{k=1}^N < x_k^*, x_k > z^{2^k} ||_{BMOA}.$$

Now let us fix $x \in X$ with $||x|| = 1$ and consider $F(z) = \sum_{n=1}^N T_n z^{2^n}$ where $T_n$ are the operators in $B(X)$ defined by $T_n(y) =< x_n^*, y > x$. From (1.2) we have $F \in Bloch\,(B(X))$ and $||F||_{Bloch(B(X))} = 1$.

Therefore

$$\left(\sum_{k=1}^N ||x_k||^2\right)^{\frac{1}{2}} \leq C|| \sum_{k=1}^N T_k(x_k) z^{2^k} ||_{BMOA(X)} \leq C|| \sum_{k=1}^N x_k z^{2^k} ||_{1,X}.$$

This shows that $X$ has cotype 2.

Now given $x_1, x_2, ...x_N \in X$ we fix $x \in X$ and $x^* \in X^*$ with $||x|| = 1$ and $< x^*, x >= 1$. Define $F(z) = \sum_{n=1}^N T_n z^{2^n}$ where $T_n$ are the operators in $B(X)$ defined by $T_n(y) =< x^*, y > \frac{x_n}{||x_n||}$. From (1.2) we have $F \in Bloch\,(B(X))$ and $||F||_{Bloch(B(X))} = 1$.

Observe that

$$\sum_{k=1}^N x_k z^{2^k} = \sum_{k=1}^N T_k(||x_k||x) z^{2^k} = F * f$$

where $f(z) = \sum_{k=1}^N ||x_k|| x z^{2^k}$. Then, since $BMOA(X) \subset H^1(X)$, we have

$$\begin{aligned} || \sum_{k=1}^N x_k z^{2^k} ||_{1,X} &\leq || \sum_{k=1}^N x_k z^{2^k} ||_{BMOA(X)} \\ &\leq C|| \sum_{k=1}^N ||x_k|| x z^{2^k} ||_{1,X} \\ &\leq C|| \sum_{k=1}^N ||x_k|| z^{2^k} ||_1 \leq C \left(\sum_{k=1}^N ||x_k||^2\right)^{\frac{1}{2}}. \end{aligned}$$

This shows that $X$ has type 2.

Conversely, let us assume that $X$ is a Hilbert space. From Lemma 3.1 we only have to prove

$$Bloch\,(B(X)) \subset \big(H^1(X), BMOA(X)\big)\,.$$

Let us take $F(z) = \sum_{n=0}^{\infty} T_n z^n \in Bloch\,(B(X))$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Now let us observe that

$$\begin{aligned}
z(F*f)'(z^2) &= \sum_{n=1}^{\infty} nT_n(x_n)z^{2n-1} \\
&= \int_0^{2\pi} F'(ze^{it})\big(f(ze^{-it})\big)e^{it}\frac{dt}{2\pi} \\
&= 2\int_0^1\int_0^{2\pi} (\sum_{n=1}^{\infty} nT_n z^{n-1}r^{n-1}e^{i(n-1)t})(\sum_{n=1}^{\infty} nx_n r^{n-1}e^{-i(n-1)t})\frac{dt}{2\pi}rdr \\
&= 2\int_0^1\int_0^{2\pi} F'(zre^{it})\big(f'(re^{-it})\big)e^{it}\frac{dt}{2\pi}rdr.
\end{aligned}$$

Therefore, since $F \in Bloch\,(B(X))$, we have

$$\begin{aligned}
||z(F*f)'(z^2)|| &\le C\int_0^1 \frac{1}{(1-s|z|)}M_{1,X}(f', s|z|)ds \\
&\le C\left(\int_0^1 \frac{ds}{(1-s|z|)^2}\right)^{\frac12}\left(\int_0^{|z|} M_{1,X}^2(f', s)ds\right)^{\frac12} \\
&\le \frac{C}{(1-|z|)^{\frac12}}\left(\int_0^{|z|} M_{1,X}^2(f', s)ds\right)^{\frac12}.
\end{aligned}$$

Finally, using (1.1), we get

$$\begin{aligned}
||F*f||_{*,X}^2 &\approx \sup_{z\in D}\int_0^1\int_0^{2\pi} \frac{(1-s)(1-|z|^2)||(F*f)'(se^{it})||^2}{|1-\bar{z}se^{it}|^2}\frac{dt}{2\pi}ds \\
&= 2\sup_{z\in D}\int_0^1\int_0^{2\pi} \frac{(1-r^2)(1-|z|^2)r||(F*f)'(r^2e^{2it})||^2}{|1-\bar{z}r^2e^{2it}|^2}\frac{dt}{2\pi}dr \\
&\le C\int_0^1\int_0^{2\pi} \frac{(1-|z|^2)}{|1-\bar{z}r^2e^{2it}|^2}\left(\int_0^r M_{1,X}^2(f', s)ds\right)\frac{dt}{2\pi}dr \\
&\le C\int_0^1\int_0^r M_{1,X}^2(f', s)dsdr = C\int_0^1 (1-s)M_{1,X}^2(f_1, s)ds.
\end{aligned}$$

Of course

$$||\int_0^{2\pi} F*f(e^{it})\frac{dt}{2\pi}|| = ||T_0(x_0)|| \le ||T_0||||x_0|| \le ||F||_{Bloch(B(X))}||f||_{1,X}.$$

Therefore combining both estimates and using Lemma 1.1 the proof is finished. □

## References


[1] J.M. Anderson, J. Clunie. Ch Pommerenke, *On Bloch functions and normal functions*, J. Reine Angew. Math. **270** (1974), 12-37.

[2] J.M. Anderson, A.L. Shields, *Coefficient multipliers of Bloch functions*, Trans. Amer. Math. Soc. **224** (1976), 256-265.

[3] P. Duren, *Theory of $H_p$-spaces*, Academic Press, New York, 1970.

[4] J. B. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.

[5] G.H. Hardy, J.E. Littlewood, *Theorems concerning mean values of analytic or harmonic functions*, Quart. J. Math. **12** (1941), 221-256.

[6] J. Lindenstrauss, L Tzafriri, *Classical Banach spaces II*, Springer Verlag, New York, 1979.

[7] M. Mateljevic, M. Pavlovic, *Multipliers of $H^p$ and $BMO$*, Pacific J. Math. **146** (1990), 71-84.

[8] B. Maurey, G Pisier, *Séries de variables aléatories vectorialles independantes et proprietés geometriques des espaces de Banach*, Studia Math. **58** (1976), 45-90.

[9] S. Kwapien, *Isomorphic characterizations of inner product spaces by orthogonal series with vector coefficients*, Studia math. **44** (1972), 583-595.

[10] G. Pisier, *Les inegalités de Kintchine-Kahane d'apres C. Borel*, Exposé VII, Ecole Polytechnique, Centre de Matematiques., Séminaire sr le Géometrie d'Espaces de Banach (1977-1978).

[11] A. Zygmund, *Trigonometric series*, Cambrigde Univ. Press., New York, 1959.