

Complex convexity and vector-valued Littlewood–Paley inequalities

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Abstract

Let $2 \leq p < \infty$ and let X be a complex Banach space. It is shown that X is p -uniformly PL -convex if and only if there exists $\lambda > 0$ such that $\|f\|_{H^p(X)} \geq \left(\|f(0)\|^p + \lambda \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z) \right)^{1/p}$, for all $f \in H^p(X)$. Applications to embeddings between vector-valued $BMOA$ spaces defined via Poisson integral or Carleson measures are provided.

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1 Introduction

Throughout the paper X will always be a complex Banach space, $dA(z) = \frac{1}{\pi} r dr d\theta$ stands for the normalized Lebesgue measure in the unit disc \mathbb{D} , we write $M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{it})\|^p dt \right)^{1/p}$ for $1 \leq p < \infty$ and $H^p(X)$ stands for the space of analytic functions in \mathbb{D} with values in X such that $\sup_{0 < r < 1} M_p(r, f) < \infty$.

When extending the notions of modulus of convexity to complex Banach spaces, different complex convexities have been considered in the literature. Recall the following moduli: If $1 \leq p \leq \infty$

$$H_\infty^X(\epsilon) = \inf \left\{ \sup_{\theta} \|x + e^{i\theta} y\| - 1 : \|x\| = 1, \|y\| = \epsilon \right\}, \quad (1.1)$$

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$$H_p^X(\epsilon) = \inf\left\{\left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta\right)^{1/p} - 1 : \|x\| = 1, \|y\| = \epsilon\right\}, \quad (1.2)$$

A complex Banach space is called uniformly \mathbb{C} -convex (see [10]), also called uniformly H^∞ -convex (see [8]), if $H_\infty^X(\epsilon) > 0$ for all $\epsilon > 0$ and uniformly PL -convex (see [8]) if $H_1^X(\epsilon) > 0$ for all $\epsilon > 0$

It is known that $H_1^X(\epsilon) > 0$ for all $\epsilon > 0$ if and only if $H_\infty^X(\epsilon) > 0$ for all $\epsilon > 0$. Actually (see [9]) there exists $A > 0$ such that

$$A(H_\infty^X(\epsilon))^2 \leq H_1^X(\epsilon) \leq H_\infty^X(\epsilon).$$

When these moduli are of power type we have the following notions: A Banach space X is said to be p -uniformly \mathbb{C} -convex ($p \geq 2$) (see [10]) there is a positive constant λ such that

$$\max_{\theta} \|x + e^{i\theta}y\| \geq \left(\|x\|^p + \lambda\|y\|^p\right)^{1/p}, \quad (1.3)$$

for all $x, y \in X$.

X is called p -uniformly PL -convex (see [8]) if there is a positive constant λ such that

$$\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta \geq \|x\|^p + \lambda\|y\|^p, \quad (1.4)$$

for all $x, y \in X$ for some constant $\lambda > 0$. We write $I_p(X)$ for the largest possible value of λ .

It is known that, for each fixed p , these notions are equivalent (see [11]).

It is not difficult to see that X is p -uniformly PL -convex if and only if X has modulus of PL -convexity of power type p , i.e. there exists $\delta > 0$ such that $H_p^X(\epsilon) \geq \delta\epsilon^p$ for $\epsilon > 0$ (see the argument in Proposition 2.2 for a proof).

Let us also mention that if X is p -uniformly PL -convex then also $L^p(\Omega, X)$ has the same property (see [8]).

Another complex convexity, introduced in [15], is the following: A Banach space X is said to be uniformly H_p -convex if $h_p^X(\epsilon) > 0$ for all $\epsilon > 0$ where

$$h_p^X(\epsilon) = \inf\{\|f\|_{H^p(X)} - 1 : \|f(0)\| = 1, \|f - f(0)\|_{H^p(X)} \geq \epsilon\}. \quad (1.5)$$

Since $H_p^X(\epsilon) \geq h_p^X(\epsilon)$, any uniformly H_p -convex space is also uniformly PL -convex.

A Banach space X is called p -uniformly H_p -convex (see [14, 12, 16, 17]) if there is a positive constant λ such that

$$\|f\|_{H^p(X)} \geq \left(\|f(0)\|^p + \lambda \|f - f(0)\|_{H^p(X)} \right)^{1/p}, \quad (1.6)$$

for all $f \in H^p(X)$. We denote by $J_p(X)$ the largest possible value of λ .

Of course, if X is p -uniformly H_p -convex then X is p -uniformly PL -convex and $J_p(X) \leq I_p(X)$.

Vector-valued versions of Littlewood-Paley theory have been considered by several authors and several reasons. Let us mention the notion we shall be using in this paper.

Let $2 \leq p < \infty$. A complex Banach space is said to have analytic Lusin cotype p (see [17]) (called also property $(H)_p$ in [1]) if there exist $C > 0$ such that

$$\left(\int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z) \right)^{1/p} \leq C \|f\|_{H^p(X)} \quad (1.7)$$

for all X -valued polynomials f . For the use of this property and related ones we refer the reader to [1, 2, 3, 4, 5, 6, 17].

It is known that analytic Lusin cotype p implies cotype p (see [1]). A remarkable fact, proved by Q. Xu, is its connection with complex convexity.

Theorem 1.1 (see [17], Theorem 5.1)

- (i) If X is p -uniformly H^p -convex then X has analytic Lusin cotype p .
- (ii) If X has analytic Lusin cotype p then X can be renormed to have an equivalent p -uniformly PL -convex norm.

It was conjectured there that actually analytic Lusin cotype p should coincide with p -uniformly H^p -convexity.

Our Corollary 2.4 establishes that this is not the case, actually we show that if X has a p -uniformly PL -convex norm then X has analytic Lusin cotype .

It should be pointed out that while Xu's arguments are based upon some martingale estimates, our proof of the converse of (ii) is mainly based upon subharmonic functions. In our arguments Riesz measures of the subharmonic functions $\|f\|^p$ where $f : \mathbb{D} \rightarrow X$ are analytic play an important role. The reader is referred to the recent paper by G. Blower and T. Ransford (see [7])

for the use of Riesz measures to determine the q -uniform PL -convexity of the spaces.

Let us now introduce a new modulus of complex convexity that we need for our proof.

Definition 1.2 *Let X be a complex Banach space and let $2 \leq p < \infty$. We define*

$$A_p^X(\epsilon) = \inf\{\|f\|_{H^p(X)} - 1 : \|f(0)\| = 1, \left(\int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z)\right)^{1/p} \geq \epsilon\},$$

Of course there exists $C > 0$ such that $H_p^X(\epsilon) \geq A_p^X(C\epsilon)$ for all $\epsilon > 0$.

Our main theorem establishes that A_p^X is of power type p if and only if the same happens to H_p^X . As a simple consequence we get that p -uniformly PL -convex spaces do have analytic Lusin cotype p .

The paper is divided into two sections. The main theorem and its proof is done in section 2 while the last section is devoted to get some applications. It is shown that if X is a p -uniformly PL -convex space then $H^p(X)$ satisfies certain lacunary p -lower estimates (in the sense introduced by G. Pisier in [12]) and also that for 2-uniformly PL -convex spaces we have the embedding between vector-valued BMOA spaces defined by Carleson measures and the one defined by the condition $\sup_{z \in \mathbb{D}} P(\|f\|^2)(z) - \|f(z)\|^2 < \infty$.

2 The main theorem

Let us start by characterizing p -uniformly PL -convex spaces in terms of vector valued Hardy spaces.

Proposition 2.1 *Let X be a complex Banach space and $2 \leq p < \infty$. The following are equivalent.*

- (i) X is p -uniformly PL -convex.
- (ii) There exists a constant $c > 0$ such that

$$\|f\|_{H^\infty(X)} \geq \left(\|f(0)\|^p + c\|f'(0)\|^p\right)^{1/p}, \quad (2.1)$$

for all $f \in H^\infty(X)$.

(iii) There exists a constant $c > 0$ such that

$$\|f\|_{H^p(X)} \geq \left(\|f(0)\|^p + c\|f'(0)\|^p \right)^{1/p}, \quad (2.2)$$

for all $f \in H^p(X)$.

Proof. (i) \Rightarrow (ii). Let $\|f\|_{H^\infty(X)} = 1$ and $\phi \in X^*$, $\|\phi\| = 1$. By Schwarz' lemma,

$$|\phi(f'(0))| \leq 1 - |\phi(f(0))|^2 \leq 2(1 - |\phi(f(0))|).$$

Hence for all θ

$$|\phi(f(0) + e^{i\theta}f'(0)/2)| \leq |\phi(f(0))| + (1/2)|\phi(f'(0))| \leq 1,$$

and, therefore $\|f(0) + e^{i\theta}f'(0)/2\| \leq 1$.

Now since X is p -uniformly PL -convex we get (ii) with $c = I_p(X)/2^p$.

(ii) \Rightarrow (i). It is obvious that (ii) implies that X is p -uniformly \mathbb{C} -convex.

Hence also p -uniformly PL -convex using [11].

(i) \Rightarrow (iii) Assume that $f \in H^p(X)$ and define the analytic function $g : \mathbb{D} \mapsto L^p(\partial\mathbb{D}, X) =: Y$ by $g(z)(e^{i\theta}) = f(ze^{i\theta})$ ($z \in \mathbb{D}$). Then $g \in H^\infty(L^p(X))$ and $\|g\|_{H^\infty(L^p(X))} = \|f\|_{H^p(X)}$. Hence we get (iii) with $c = I_p(L^p(X))/2^p$.

(iii) \Rightarrow (i). This is obvious. \blacksquare

Let us now present an equivalent formulation for the modulus A_p^X to be of power type p .

Proposition 2.2 *Let X be complex Banach space and $2 \leq p < \infty$. The following are equivalent.*

(i) There exists $\lambda > 0$ such that

$$\|f\|_{H^p(X)} \geq \left(\|f(0)\|^p + \lambda \int_{\mathbb{D}} (1 - \|z\|^2)^{p-1} \|f'(z)\|^p dA(z) \right)^{1/p}, \quad (2.3)$$

for all $f \in H^p(X)$.

(ii) There exists $\delta > 0$ such that $A_p^X(\epsilon) \geq \delta\epsilon^p$ for all $\epsilon > 0$.

Proof. Assume (i). For each function $f \in H^p(X)$ such that $\|f(0)\| = 1$ and $\int_{\mathbb{D}} (1 - \|z\|^2)^{p-1} \|f'(z)\|^p dA(z) \geq \epsilon^p$ we have

$$\|f\|_{H^p(X)} - 1 \geq (1 + \lambda\epsilon^p)^{1/p} - 1 \geq (\lambda/p)\epsilon^p.$$

Hence $A_p^X(\epsilon) \geq (\lambda/p)\epsilon^p$.

On the other hand if $A_p^X(\epsilon) \geq \delta\epsilon^p$ for $\epsilon > 0$ then for any $x \in X \setminus \{0\}$ one has

$$\begin{aligned} \|f\|_{H^p(X)} - \|f(0)\| &= \|f(0)\| \left(\left\| \frac{f}{\|f(0)\|} \right\|_{H^p(X)} - 1 \right) \\ &\geq \delta \|f(0)\|^{1-p} \left(\int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z) \right). \end{aligned}$$

This shows that

$$\begin{aligned} \|f\|_{H^p(X)} &\geq \|f(0)\| \left(1 + \delta \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \frac{\|f'(z)\|^p}{\|f(0)\|^p} dA(z) \right) \\ &\geq (\|f(0)\|^p + \delta \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z))^{1/p}. \end{aligned}$$

■

This shows that if $A_p^X(\epsilon) \geq \delta\epsilon^p$ for some $\delta > 0$ then X is p -uniformly PL -convex. Our main result establishes that the converse is also true.

Theorem 2.3 *Let $2 \leq p < \infty$. The following are equivalent.*

- (i) X is p -uniformly PL -convex.
- (ii) There exists a constant $C > 0$ such that for each $r < \frac{1}{2}$

$$\|f'(z)\|^p (1 - |z|^2)^p \leq \frac{C}{r^p (1 - r^2)^p} \int_{D_{2r}(z)} d\mu(w),$$

where $d\mu$ is the Riesz measure of the subharmonic function $\|f\|^p$ (see [13]) and $D_r(z)$ denotes the ball centred at z and radius r for the pseudo-hyperbolic distance $d(w, z) = |\phi_z(w)| = \left| \frac{z-w}{1-\bar{z}w} \right|$.

- (iii) There exists $\lambda > 0$ such that

$$\|f\|_{H^p(X)} \geq \left(\|f(0)\|^p + \lambda \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \|f'(z)\|^p dA(z) \right)^{1/p},$$

for all $f \in H^p(X)$.

Proof. (i) \Rightarrow (ii) Let us fix $0 < r < 1$. If u is an arbitrary subharmonic function on \mathbb{D} then (Poisson–Jensen) formula states that

$$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta - u(0) = \frac{1}{2\pi} \int_{|w|<r} \log \frac{r}{|w|} d\mu(w).$$

where μ stands for the Riesz measure associated to the subharmonic function u (see [13]).

Applying this to $u(z) = \|f(z)\|^p$ and using (2.2) in Proposition 2.1 for the function $f_r(z) = f(rz)$ we get

$$r^p \|f'(0)\|^p \leq \frac{1}{\lambda} \int_{|w|<r} \log \frac{r}{|w|} d\mu(w). \quad (2.4)$$

Applying (2.4) to $f(\phi_\xi(w))$ for $\phi_\xi(w) = \frac{\xi-w}{1-\xi w}$ we obtain

$$r^p (1 - |\xi|^2)^p \|f'(\xi)\|^p \leq \frac{1}{\lambda} \int_{D_r(\xi)} \log \frac{r}{|\phi_\xi(w)|} d\mu(w), \quad (2.5)$$

On the other hand $\|f'(0)\|^p \leq \frac{1}{r^2} \int_{|w|<r} \|f'(w)\|^p dA(w)$, which implies

$$\begin{aligned} (1 - |z|^2)^p \|f'(z)\|^p &\leq \frac{1}{r^2} \int_{|w|<r} \|f'(\phi_z(w))\|^p \frac{(1 - |\phi_z(w)|^2)^{p-2}}{(1 - |w|^2)^{p-2}} |\phi'_z(w)|^2 dA(w) \\ &\leq \frac{1}{r^{2+p}(1 - r^2)^{p-2}} \int_{D_r(z)} r^p (1 - |\xi|^2)^{p-2} \|f'(\xi)\|^p dA(\xi). \end{aligned}$$

Now using (2.5) together with the previous estimate we obtain that again

$$\begin{aligned} (1 - |z|^2)^p \|f'(z)\|^p &\leq \frac{1}{\lambda} \frac{1}{r^{2+p}(1 - r^2)^{p-2}} \int_{D_r(z)} \left(\int_{D_r(\xi)} \log \frac{r}{|\phi_\xi(w)|} d\mu(w) \right) \frac{dA(\xi)}{(1 - |\xi|^2)^2} \\ &\leq \frac{1}{\lambda} \frac{1}{r^{2+p}(1 - r^2)^{p-2}} \int_{D_{2r}(z)} \left(\int_{D_r(w)} \log \frac{r}{|\phi_w(\xi)|} \frac{dA(\xi)}{(1 - |\xi|^2)^2} \right) d\mu(w) \\ &\leq \frac{1}{\lambda} \frac{1}{r^{2+p}(1 - r^2)^{p-2}} \int_{D_{2r}(z)} \left(\int_{|\xi|<r} \log \frac{r}{|\xi|} \frac{dA(\xi)}{(1 - |\xi|^2)^2} \right) d\mu(w) \end{aligned}$$

Using now that $\int_{|\xi|<r} \log \frac{r}{|\xi|} \frac{dA(\xi)}{(1 - |\xi|^2)^2} \leq \frac{r^2}{(1 - r^2)^2} \int_{|\xi|<1} \log \frac{1}{|\xi|} dA(\xi) = \frac{r^2}{2(1 - r^2)^2}$ we get (ii) with $C = \frac{1}{2\lambda}$.

(ii) \Rightarrow (iii) Fix $r = \frac{1}{4}$ and let $f \in H^p(X)$. Integrating and applying Fubini we get

$$\int_{\mathbb{D}} \|f'(z)\|^p (1 - |z|^2)^{p-1} dA(z) \leq C \int_{\mathbb{D}} \left(\int_{D_{1/2}(w)} (1 - |z|^2)^{-1} dA(z) \right) d\mu(w).$$

Since $\int_{D_{1/2}(w)} (1 - |z|^2)^{-1} dA(z) \leq C(1 - |w|)$, we get

$$\begin{aligned} \int_{\mathbb{D}} \|f'(z)\|^p (1 - |z|^2)^{p-1} dA(z) &\leq C \int_{\mathbb{D}} (1 - |w|) d\mu(w) \\ &\leq C \int_{\mathbb{D}} \log \frac{1}{|w|} d\mu(w) \\ &= C \lim_{r \rightarrow 1} \int_{|w| < r} \log \frac{r}{|w|} d\mu(w) \\ &= C \lim_{r \rightarrow 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|^p d\theta - \|f(0)\|^p \right). \end{aligned}$$

(iii) \Rightarrow (i) This is obvious. ■

Corollary 2.4 *Let $2 \leq p < \infty$. Then X has analytic Lusin cotype p if and only if X has an equivalent norm with respect to which it is p -uniformly PL -convex.*

Proof. Theorem 2.3 gives one implication and the other corresponds to Theorem 5.1(ii) in [17]. ■

Corollary 2.5 *Let $p \geq 2$, $1 \leq q \leq p$ and let X be a p -uniformly PL -convex space. If $f \in H^q(X)$, then*

$$\int_0^1 \{M_{q,X}(r, f')\}^p (1 - r)^{p-1} dr \leq C \left(\|f\|_{H^q(X)}^p - \|f(0)\|^p \right). \quad (2.6)$$

Proof. The space $Y = L^q(\partial\mathbb{D}, X)$ is p -uniformly PL -convex. Applying Theorem 2.3 to the function $g(z)(e^{i\theta}) = f(e^{i\theta}z)$ we get the result. ■

It should be noted that inequality (2.6) contains, besides the Littlewood–Paley inequality, the inequality of Hardy and Littlewood,

$$\int_0^1 \{M_q(r, f')\}^2 (1 - r) dr \leq C (\|f\|_q^p - |f(0)|^p) \quad (1 \leq q \leq 2),$$

which holds for scalar-valued H^q -functions.

Theorem 2.3 can be generalized, with the same proof, to obtain the following.

Theorem 2.6 *Suppose that X is p -uniformly PL -convex and that $p \leq q < \infty$. If $f \in H^q(X)$ then*

$$\int_{\mathbb{D}} \|f'(z)\|^p \|f(z)\|^{q-p} (1 - |z|)^{p-1} dA(z) \leq C (\|f\|_{q,X}^q - \|f(0)\|^q).$$

3 Applications

Let us recall the following notion introduced by Pisier in [12] and closely related to p -uniform convexity.

Let $p \geq 2$ and X be a complex Banach space. We say that $H^p(X)$ satisfies a radial lower p -estimate if there exist $\delta > 0$ and $C \geq 1$ such that for all increasing sequences $0 < r_0 < r_1 < \dots < r_n \dots < 1$ and for all f in $H^p(X)$ we have

$$(\|f_{r_0}\|_{H^p(X)}^p + \delta \sum_{n=1}^{\infty} \|f_{r_n} - f_{r_{n-1}}\|_{H^p(X)}^p)^{1/p} \leq C \|f\|_{H^p(X)}^p \quad (3.1)$$

It was shown in [12] that if X is p -uniformly convex then $H^p(X)$ satisfies a radial lower p -estimate with constant 1.

We say that $H^p(X)$ satisfies a lacunary lower p -estimate (see [4]) if the previous estimate holds only for $r_n = 1 - 2^{-n}$.

Now we can prove the following counterpart for p -uniformly PL -spaces.

Theorem 3.1 *If X is p -uniformly PL -convex then $H^p(X)$ satisfies a lacunary lower p -estimate with constant 1.*

Proof. Let $r_n = 1 - 2^{-n}$ and let f be a function in $H^p(X)$. Since $f_{r_n}(e^{it}) - f_{r_{n-1}}(e^{it}) = \int_{r_{n-1}}^{r_n} f'(se^{it}) ds$ we have

$$\|f_{r_n} - f_{r_{n-1}}\|_{H^p(X)} \leq \int_{r_{n-1}}^{r_n} M_p(f', s) ds \leq (r_n - r_{n-1})^{1/p'} \left(\int_{r_{n-1}}^{r_n} M_p^p(f', s) ds \right)^{1/p}.$$

Hence, due to the equality $r_n - r_{n-1} = 1 - r_n$, we obtain

$$\sum_{n=0}^{\infty} \|f_{r_n} - f_{r_{n-1}}\|_{H^p(X)}^p \leq \sum_{n=0}^{\infty} \int_{r_{n-1}}^{r_n} (1-s)^{p-1} M_p^p(f', s) ds.$$

Therefore

$$\sum_{n=0}^{\infty} \|f_{r_n} - f_{r_{n-1}}\|_{H^p(X)}^p \leq \int_{|w|>r_0} (1-|w|)^{p-1} |f'(w)|^p \frac{dA(w)}{|w|}. \quad (3.2)$$

Now Theorem 2.3 leads to

$$\left(\|f(0)\|^p + \lambda \int_{\mathbb{D}} (1-|w|^{p-1}) \|f'(w)\|^p dA(w) \right)^{1/p} \leq \|f\|_{H^p(X)}$$

for some $\lambda > 0$.

Composing with the Möbius transform $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$ we get

$$\begin{aligned} & \left(\|f(z)\|^p + \lambda \int_{\mathbb{D}} (1 - |w|^2)^{p-1} \|f'(\phi_z(w))\|^p |\phi'_z(w)|^p dA(w) \right)^{1/p} \leq \\ & \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(\phi_z(e^{it}))\|^p dt \right)^{1/p}. \end{aligned}$$

The change of variable $e^{i\theta} = \phi_z(e^{it})$ (hence $dt = P_z(e^{i\theta})d\theta$) and the fact $|\phi'_z(w)| = \frac{1-|\phi_z(w)|^2}{1-|w|^2}$ imply

$$\begin{aligned} & \|f(re^{iu})\|^p + \lambda \int_{\mathbb{D}} (1 - |w|^2)^{p-1} \|f'(w)\|^p P_{re^{iu}}(w) dA(w) \leq \\ & \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|^p P_{re^{iu}}(e^{i\theta}) d\theta \right). \end{aligned}$$

Hence

$$\|f_r\|_{H^p(X)}^p + \lambda \int_{\mathbb{D}} (1 - |w|^2)^{p-1} \|f'(w)\|^p \frac{1 - r^2}{1 - r^2|w|^2} dA(w) \leq \|f\|_{H^p(X)}^p. \quad (3.3)$$

Combining (3.2) and (3.3) and noticing that $\frac{1-r_0^2}{1-r_0^2|w|^2} > \frac{1}{1+r_0^2}$ for $|w| > r_0$ we obtain

$$\|f_{r_0}\|_{H^p(X)}^p + \lambda \frac{1+r_0^2}{r_0} \sum_{n=0}^{\infty} \|f_{r_n} - f_{r_{n-1}}\|_{H^p(X)}^p \leq \|f\|_{H^p(X)}^p.$$

Now the fact that $r_0 + \frac{1}{r_0} \geq 2$ gives the result with $\delta = 2\lambda$. ■

Let us now apply the result to some get some embeddings between different vector valued BMOA spaces.

A X -valued analytic function $f \in H^1(\mathbb{T}, X)$ is said to belong to $BMOA(X)$ if

$$\sup_{|z|<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(e^{it}) - f(z)\|^2 P_z(e^{it}) dt < \infty \quad (3.4)$$

where $P_z(w) = \frac{1-|z|^2}{|1-\bar{z}w|}$.

The norm is given by $\|f\|_{H^1(X)} + \sup_{|z|<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{it}) - f(z)\|^2 P_z(e^{it}) dt \right)^{1/2}$.

A X -valued analytic function f is said to belong to $BMOA_C(X)$ (see [3]) if

$$\sup_{|z|<1} \int_{\mathbb{D}} (1 - |w|^2) \|f'(w)\|^2 P_z(w) dA(w) < \infty \quad (3.5)$$

The norm is given by $\|f(0)\| + \sup_{|z|<1} \left(\int_{\mathbb{D}} (1 - |w|^2) \|f'(w)\|^2 P_z(w) dA(w) \right)^{1/2}$.

It is known that if $BMOA(X) \subset BMOA_C(X)$ then X has cotype 2 (see [3])

Recall that in the scalar case we have

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{it}) - f(z)|^2 P_z(e^{it}) dt = P(|f|^2)(z) - |f(z)|^2$$

Using this, we still can define another $BMOA$ space.

A X -valued analytic function f is said to belong to $BMOA_P(X)$ if

$$\sup_{|z|<1} P(\|f\|^2)(z) - \|f(z)\|^2 < \infty \quad (3.6)$$

The norm is given by $\|f(0)\| + \sup_{|z|<1} (P(\|f\|^2)(z) - \|f(z)\|^2)^{1/2}$.

Theorem 3.2 *If X is 2-uniformly PL-convex space then the natural inclusion map from $BMOA_P(X)$ into $BMOA_C(X)$ is bounded.*

Proof. Using Theorem 2.3 we have that

$$\left(\|f(0)\|^2 + \lambda \int_{\mathbb{D}} (1 - |w|^2) \|f'(w)\|^2 dA(w) \right)^{1/2} \leq \|f\|_{H^2(X)}$$

for all function $f \in H^2(X)$.

Composing with the Moebius transform $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$ we get

$$\left(\|f(z)\|^2 + \lambda \int_{\mathbb{D}} (1 - |w|^2) \|f'(\phi_z(w))\|^2 |\phi'_z(w)|^2 dA(w) \right)^{1/2} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(\phi_z(e^{it}))\|^2 dt \right)^{1/2}.$$

The change of variable $e^{i\theta} = \phi_z(e^{it})$ gives $dt = P_z(e^{i\theta}) d\theta$ and also $|\phi'_z(w)| = \frac{1 - |\phi_z(w)|^2}{1 - |w|^2}$ we obtain

$$\left(\|f(z)\|^2 + \lambda \int_{\mathbb{D}} (1-|w|^2) \|f'(w)\|^2 P_z(w) dA(w)\right)^{1/2} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{i\theta})\|^2 P_z(e^{i\theta}) d\theta\right)^{1/2}$$

This gives the desired estimate of norms $\|f\|_{BMOA_c} \leq \frac{1}{\sqrt{\lambda}} \|f\|_{BMOA_{\mathcal{P}}}$. ■

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