Norm estimates for operators from H^p to ℓ^q.

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Abstract

We give upper and lower estimates of the norm of a bounded linear operator from the Hardy space H^p to ℓ^q in terms of the norm of the rows and the columns of its associated matrix in certain vector-valued sequence spaces.

Key words: Hardy spaces, vector-valued sequence spaces, vector-valued BMO, absolutely summing operators.

1 Introduction

Let $1 \leq p, q \leq \infty$ and let $T : H^p \to \ell^q$ be a linear and bounded operator where H^p denote the Hardy space in the unit disc. To such an operator we associate the matrix $(t_{kn})_{k,n}$, defined by

$$T(u_n) = \sum_{k \in \mathbb{N}} t_{kn} e_k$$

where $u_n(z) = z^n$, $n \geq 0$, and $(e_k)_{k \in \mathbb{N}}$ stands for the canonical basis of ℓ^q. We denote by $T_k = (t_{kn})_{n \geq 0}$ and $x_n = (t_{kn})_{k \in \mathbb{N}}$ its rows and its columns respectively. Although explicitly computing the norm is not possible (even for $p = q = 2$) several theorems concerning upper and lower estimates of the norm $\|T\|$ in terms of

$$\|(T_k)\|_{\ell^r(\ell^s)} = (\sum_{k=1}^{\infty} (\sum_{n=0}^{\infty} |t_{kn}|^s)^{r/s})^{1/r}$$

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for different values of r and s were proved by B. Osikiewicz in [23]. The following results are the content of Theorems 2.1, 2.2, 2.3 and 2.4 in [23]: If $1 \le p \le 2$, $1 \le q \le \infty$ and $1/r = (1/q - 1/2)^+$ then

$$\|(T_k)\|_{\ell^r(\ell^2)} \le \|T\| \le \|(T_k)\|_{\ell^q(\ell^p)}. \tag{1}$$

If $2 \le p < \infty$, $1 \le q \le \infty$ and $1/s = (1/q - 1/p')^+$ then

$$\|(T_k)\|_{\ell^s(\ell^p)} \le \|T\| \le \|(T_k)\|_{\ell^q(\ell^2)}. \tag{2}$$

Whilst the upper estimates were shown to be sharp in the scale of $\ell^r(\ell^s)$ spaces, it was left open whether the values of r and s in the lower estimates could be improved.

The reader is referred to [8] for some results in the same spirit in the cases $0 < p < 1$. In this paper we shall see (1) and (2) can actually be improved in different directions. On the one hand we shall use not only the norm of the rows (T_k) but also the norm of the columns (x_n), which, sometimes gives better estimates. On the other hand we shall consider $\ell(p,q)$-spaces instead of ℓ^q-spaces to produce more precise estimates. Our main tool will be the description of the boundedness of operators between H^p and ℓ^q by means of vector-valued functions which will allow us to use results from vector-valued Hardy spaces and absolutely summing operators to get our theorems.

Let X be a complex Banach space with dual space X^*. We denote by $\ell^s(X)$ and $\ell^s_{weak}(X)$ the spaces of bounded sequences in X for $s = \infty$, and, for $1 \le s < \infty$, the spaces of sequences $(A_j) \subset X$ such that

$$\|(A_j)\|_{\ell^s(X)} = (\sum_j \|A_j\|^s)^{1/s} < \infty$$

and

$$\|(A_j)\|_{\ell^s_{weak}(X)} = \sup_{\|x^*\|=1} (\sum_j |\langle A_j, x^*\rangle|^s)^{1/s} < \infty.$$

It is easy to see that, for $1 \le p \le \infty, 1/p + 1/p' = 1$,

$$\|(A_j)\|_{\ell^p_{weak}(X)} = \sup\{\|\sum_j \beta_j A_j\| : \|(\beta_j)\|_{\ell^{p'}} = 1\}.$$

Hence $\ell^p_{weak}(X)$ can be identified with $L(\ell^{p'}, X)$ for $1 < p < \infty$ and $L(c_0, X)$ for $p = 1$. Also, for reflexive Banach spaces X and $1 \le p < \infty$, $\ell^p_{weak}(X)$ can be identified with $L(X^*, \ell^p)$ by defining $T(x^*) = (\langle A_j, x^*\rangle)_j$ and $\|T\| = \|(A_j)\|_{\ell^p_{weak}(X)}$.

We denote by $\ell(s,r,X)$, $0<r,s\leq\infty$, the space of sequences $(x_n)_{n\geq 0}\subset X$ such that

$$\|(x_n)\|_{\ell(s,\infty,X)}=\max\{\|x_0\|,\sup_{k\in\mathbb{N}}((\sum_{n=2^{k-1}}^{2^k-1}\|x_n\|^s)^{1/s}\}<\infty,$$

or

$$\|(x_n)\|_{\ell(s,r,X)}=(\|x_0\|^r+\sum_{k\in\mathbb{N}}(\sum_{n=2^{k-1}}^{2^k-1}\|x_n\|^s)^{r/s})^{1/r}<\infty.$$

In particular, $\ell(s,s,X)=\ell^s(X)$.

We denote by $H^p(X)$ (resp. $H^p_{weak}(X)$) the vector-valued Hardy spaces consisting of analytic functions $F:\mathbb{D}\to X$ such that

$$\|F\|_{H^p(X)}=\sup_{0<r<1}(\int_0^{2\pi}\|F(re^{it})\|^p\frac{dt}{2\pi})^{1/p}<\infty,$$

(resp.

$$\|F\|_{H^p_{weak}(X)}=\sup_{\|x^*\|=1}\|\langle F,x^*\rangle\|_{H^p}<\infty.)$$

As usual we write $M_p(F,r)=(\int_0^{2\pi}\|F(re^{it})\|^p\frac{dt}{2\pi})^{1/p}$.

We shall use the notation $\ell^p=\ell^p(\mathbb{C})$, $\ell(p,q)=\ell(p,q,\mathbb{C})$, $L^p=L^p(\mathbb{T})$ and $H^p=H^p(\mathbb{C})$ where H^p will be sometimes understood as functions in L^p using the fact that H^p isometrically embeds into L^p for $1\leq p\leq\infty$. We also make use of the duality results $(H^1)^*=BMOA$ (see [17]) and $(H^p)^*=H^{p'}$ (see [16]) for $1<p<\infty$.

We shall prove, among other things, the following estimates.

Theorem 1 *Let $1<p<\infty$, $1\leq q<\infty$ and let $T:H^p\to\ell^q$ be a bounded operator. Then, for $p_1=\min\{p,2\}$, $p_2=\max\{p,2\}$, $1/r=(1/q-1/p_1)^+$ and $1/s_u=(1/q-1/p_2'-(1/u-1/2)^+)^+$, we have*

$$\|T\|\leq\min\{\|(T_k)\|_{\ell^q(\ell^{p_1})},\|(x_n)\|_{\ell^{p_1}(\ell^q)}\}. \tag{3}$$

For each $u\geq q$ there exists $C>0$ such that

$$\max\{\|(T_k)\|_{\ell^r(\ell^{p_2})},\|(x_n)\|_{\ell^{s_u}(\ell^u)}\}\leq C\|T\|. \tag{4}$$

Remark 2 *Note that the use of columns in Theorem 1 provides sometimes better results than the use of rows. Indeed, taking into account that, for $q\leq p$,*

$$(\sum_{n=0}^{\infty}(\sum_{k=1}^{\infty}|a_{kn}|^q)^{p/q})^{1/p}\leq(\sum_{k=1}^{\infty}(\sum_{n=0}^{\infty}|a_{kn}|^p)^{q/p})^{1/q},$$

we obtain, for instance, in the case $p > 2$, $q = 1$ and $u = 2$, that $s_u = p$ and (4) improves (2) because

$$\|(T_k)\|_{\ell^p(\ell^p)} \leq \|(x_n)\|_{\ell^p(\ell^2)}.$$

Also in the case $1 < p < \infty$, $1 \leq q \leq \min\{p, 2\} = p_1$ we obtain that (3) improves (1) because

$$\|(x_n)\|_{\ell^{p_1}(\ell^q)} \leq \|(T_k)\|_{\ell^q(\ell^{p_1})}.$$

Selecting special values of u in Theorem 1 we obtain some new lower estimates of $\|T\|$.

Corollary 3 *Let $1 \leq q \leq 2$ and let $T : H^p \to \ell^q$ be a bounded operator.*

(i) If $1 \leq q \leq p \leq 2$, $1/r = 1/q - 1/p$ and $1/s = 1/q - 1/2$ then

$$C^{-1} \max\{\|(T_k)\|_{\ell^r(\ell^2)}, \|(x_n)\|_{\ell^s(\ell^2)}, \|(x_n)\|_{\ell^r(\ell^p)}\} \leq \|T\|.$$

(ii) Let $1 \leq q \leq p' \leq 2 \leq p < \infty$ such that $1/q - 1/p' \geq 1/p' - 1/2$. If $1/r = 1/q - 1/2$, $1/s = 1/q - 1/p'$ and $1/t = 1/q - 2/p' + 1/2$ then

$$C^{-1} \max\{\|(T_k)\|_{\ell^r(\ell^p)}, \|(x_n)\|_{\ell^s(\ell^2)}, \|(x_n)\|_{\ell^t(\ell^{p'})}\} \leq \|T\|.$$

In particular, for $1 \leq q \leq 2$, $p = 2$ and $1/r = 1/q - 1/2$, we have

$$\max\{\|(T_k)\|_{\ell^r(\ell^2)}, \|(x_n)\|_{\ell^r(\ell^2)}\} \leq C\|T\|. \tag{5}$$

Proof. (i) Let $1 \leq q \leq p \leq 2$. For each $p \leq u \leq 2$, we write $1/u = (1-\theta)/p + \theta/2$ for some $0 \leq \theta \leq 1$. Hence the values in Theorem 1 become $p_1 = p$, $p_2 = 2$, $1/r = 1/q - 1/p$ and $1/s_u = 1/q - 1/u = 1/r + \theta(1/p - 1/2)$. Now select $\theta = 0$ and $\theta = 1$ and apply (4) to get the desired estimates.

(ii) Let $1 \leq q \leq p' \leq 2 \leq p < \infty$ such that $1/q - 1/p' \geq 1/p' - 1/2$. For each $p' \leq u \leq 2$ now we obtain $p_1 = 2$, $p_2 = p$, $1/r = 1/q - 1/2$ and $1/s_u = (1/q - 1/p' - (1/u - 1/2))^+$. Our assumption implies that $s_u = t$ for $u = p'$ and $s_u = s$ for $u = 2$. Apply again (4) to finish the proof. □

Remark 4 *Assume $1 \leq q \leq p' < 2 < p < \infty$. Then (ii) in Corollary 3 gives $\|(T_k)\|_{\ell^r(\ell^p)} \leq C\|T\|$ for $1/r = 1/q - 1/2$ (which produces a better lower estimate than (2) since $r \leq s$ for $1/s = 1/q - 1/p'$).*

Actually, for $p \geq 2$, the value $v = r$ given by $1/r = 1/q - 1/2$ is the smallest value in the scale $\ell^v(\ell^p)$ to get the estimate $\|(T_k)\|_{\ell^v(\ell^p)} \leq C\|T\|$ as the *following example shows: Consider a lacunary multiplier $T : H^p \to \ell^q$ given by*

$$T(f)(z) = \sum_{k=0}^{\infty} \lambda_k a_{2^k} e_{2^k}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

In such a case $\|(T_k)\|_{\ell^v(\ell^p)} = \|(\lambda_k)\|_{\ell^v}$ and $\|\sum_k a_{2^k} z^{2^k}\|_{H^p} \approx (\sum_k |a_{2^k}|^2)^{1/2}$. This shows that $\|T\| \approx \|(\lambda_k)\|_{\ell^r}$ for $1/r = 1/q - 1/2$.

To present further improvements we shall replace the scale of ℓ^p-spaces by the $\ell(p,q)$-spaces (see [19]) when computing the norm of the rows and the columns of the matrix associated to the operator.

Our first result will be the following extension of Theorem 1.

Theorem 5 *Let $1 < p < \infty, 1 \leq q < \infty$, $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$ and let $T : H^p \to \ell^q$ be a bounded operator. Then*

$$\|T\| \leq \min\{\|(T_k)\|_{\ell^q(\ell(p_1,2))}, \|(x_n)\|_{\ell(p_1,2,\ell^q)}\} \tag{6}$$

For each $u \geq q$ there exists $C > 0$ such that

$$\max\{\|(T_k)\|_{\ell^r(\ell(p_2,2))}, \|(x_n)\|_{\ell(s_u,2,\ell^u)}\} \leq C\|T\|, \tag{7}$$

where $1/r = (1/q - 1/p_1)^+$ and $1/s_u = (1/q - 1/p_2' - (1/u - 1/2)^+)^+$.

Of course, Theorem 1 follows from Theorem 5 using the inclusions $\ell^q(\ell^{p_1}) \subset \ell^q(\ell(p_1, 2))$, $\ell^{p_1}(\ell^q) \subset \ell(p_1, 2, \ell^q)$, $\ell^r(\ell(p_2, 2)) \subset \ell^r(\ell^{p_2})$ and, since $s_u \geq 2$, also $\ell(s_u, 2, \ell^u) \subset \ell^{s_u}(\ell^u)$.

Using the inequalities(see Lemma 13 below)

$$\|(x_n)\|_{\ell(p,q,\ell^r)} \leq \|(T_k)\|_{\ell^r(\ell(p,q))}, \quad \min\{p,q\} \geq r,$$

$$\|(x_n)\|_{\ell^r(\ell(p,q))} \leq \|(T_k)\|_{\ell(p,q,\ell^r)}, \quad \max\{p,q\} \leq r,$$

we can formulate the following corollaries of Theorem 5.

Corollary 6 *Let $1 \leq q < p \leq 2$ and $T : H^p \to \ell^q$ be a bounded operator. If $1/s = 1/q - 1/p$ then there exists $C > 0$ such that*

$$C^{-1}\|(x_n)\|_{\ell(s,2,\ell^p)} \leq \|T\| \leq \|(x_n)\|_{\ell(p,2,\ell^q)}. \tag{8}$$

Corollary 7 *Let $1 \leq q \leq p' \leq 2 \leq p < \infty$ and $T : H^p \to \ell^q$ be a bounded operator. If $1/r = 1/q - 1/2$ and $1/s = 1/q - 1/p'$ then there exists $C > 0$ such that*

$$C^{-1} \max\{\|(T_k)\|_{\ell^r(\ell(p,2))}, \|(x_n)\|_{\ell(s,2,\ell^2)}\} \leq \|T\| \leq \|(x_n)\|_{\ell^2(\ell^q)}. \tag{9}$$

Theorem 5 will follow from very general arguments valid for many other spaces relying upon some geometrical properties which are shared by other spaces. However in the case $1 \leq p < 2$ other tools are at our disposal and allow us to get better estimates. For instance, in the case $p = 1$ we can produce new upper estimates using results on Taylor coefficients of functions in $BMOA$.

Theorem 8 *Let $T : H^1 \to \ell^q$ be a bounded operator.*

(i) For $q = 1$ we have

$$\|T\| \leq C \min\{\|(x_n)\|_{\ell(1,2,\ell^1)}, \|((n+1)^{1/2} x_n)\|_{\ell(2,\infty,\ell^1)}\}.$$

(ii) For $1 \leq q \leq 2$ we have

$$\|T\| \leq C \min\{\|(T_k)\|_{\ell^q(\ell(1,2))}, \|(x_n)\|_{\ell(1,2,\ell^q)}, \|((n+1)^{1/2} x_n)\|_{\ell(2,\infty,\ell^q)}\}.$$

(iii) For $q \geq 2$ we have

$$\|T\| \leq C \min\{\|(T_k)\|_{\ell^q(\ell(1,2))}, \|(A_k)\|_{\ell^q(\ell(2,\infty))}, \|((n+1)^{1/2} x_n)\|_{\ell(2,\infty,\ell^q)}\},$$

where $A_k = ((n+1)^{1/2} t_{kn})_n$.

Also new lower estimates can be achieved for $1 < p < 2$ using the factorization $H^p = H^2 H^t$ where $1/2 + 1/t = 1/p$.

Theorem 9 *Let $1 \leq p < 2$, $1 \leq q \leq 2$, $1/r = 1/q - 1/2$ and $1/t = 1/p - 1/2$ and let $T : H^p \to \ell^q$ be a bounded operator. Then there exists $C > 0$ such that*

$$\sup_{\|(\alpha_l)\|_{\ell(t',2)}=1} \max\{\|(\sum_{l=0}^{\infty} \alpha_l t_{k,l+n})_n\|_{\ell^r(\ell^2)}, \|(\sum_{l=0}^{\infty} \alpha_l t_{k,l+n})_k\|_{\ell^r(\ell^2)}\} \leq C\|T\|.$$

Finally the special behavior of the inclusion map $\ell^1 \to \ell^2$ allows to get further extensions in the case $q = 1$.

Theorem 10 *Let $1 \leq p < 2$, $1/t = 1/p - 1/2$ and $T : H^p \to \ell^1$ be a bounded operator. There exists $C > 0$ such that*

$$\max\{\sup_{\|\sum_l \alpha_l z^l\|_{H^t}=1} \|(\sum_{l=0}^{\infty} \alpha_l t_{k,n+l})_n\|_{\ell^2(\ell^2)}, \sup_{\|(\alpha_l)\|_{\ell(t',2)}=1} \|(\sum_{l=0}^{\infty} \alpha_l t_{k,n+l})_k\|_{\ell^2(\ell^2)}\} \leq C\|T\|.$$

As a simple application of Theorem 8 and Theorem 10 (selecting sequences $\alpha_j = \frac{1}{\sqrt{N}}$ for $0 \leq j \leq N$ and $\alpha_j = 0$ for $j \geq N+1$) we get the following new estimates, that can be compared with the known ones for particular types of operators such as multipliers, composition operators and so on.

Corollary 11 *Let $T : H^1 \to \ell^1$ be a bounded operator. There exists $C > 0$ such that*

$$\sup_{N\in\mathbb{N}} \|(\frac{1}{\sqrt{N}} \sum_{l=n}^{n+N} x_l)_n\|_{\ell^2(\ell^2)} \leq C\|T\|$$

$$\|T\| \leq C \min\{\|((n+1)^{1/2} x_n)\|_{\ell(2,\infty,\ell^1)}, \|(x_n)\|_{\ell(1,2,\ell^1)}\}.$$

The paper is organized as follows. Section 2 contains some preliminary results concerning the reformulation of the boundedness of operators from H^p to ℓ^q and some facts on the spaces $\ell(p,q,X)$ to be used in the sequel. Some tools from the theory of vector-valued Hardy and $BMOA$ spaces are presented in Section 3. The proof of Theorem 5 is postponed to Section 4. Last section is devoted to the case $1 \leq p < 2$ and to present the proofs of Theorems 8, 9 and 10.

Throughout the paper, as usual, $L(X,Y)$ stands for the space of bounded linear operators, $a^+ = \max\{a, 0\}$, p' for the conjugate exponent of p and C denotes a constant that may vary from line to line.

2 Preliminary results

As it was mentioned in the introduction for each $1 \leq p, q \leq \infty$ and each bounded operator $T : H^p \to \ell^q$ we define the matrix $(a_{kn}(T)) = (t_{kn})$ given by

$$T(u_n) = (t_{kn})_{k\in\mathbb{N}} \quad \text{for} \quad u_n(z) = z^n, n \geq 0. \tag{10}$$

Observe that for each $k \in \mathbb{N}$ the functional $\xi_k T(f) = \langle T(f), e_k\rangle$, which belongs to $(H^p)^*$, is represented by an analytic function, say $g_k = g_k(T)$. We denote by $F_T(z) = (g_k(z))_{k\in\mathbb{N}}$ the ℓ^q-valued analytic function associated to T.

Clearly each row $T_k = (t_{kn})_{n\geq 0}$ coincides with the sequence of Taylor coefficients of the function g_k, that is

$$g_k(z) = \sum_{n=0}^{\infty} t_{kn} z^n \tag{11}$$

and each column $x_n = (t_{kn})_{k\in\mathbb{N}}$ coincides with the n-Taylor coefficient of the vector-valued analytic function $F_T : \mathbb{D} \to \ell_q$ given by

$$F_T(z) = \sum_{n=0}^{\infty} x_n z^n, \ x_n = \sum_{k=1}^{\infty} t_{kn} e_k. \tag{12}$$

With this notation, for a polynomial $f(z)$ with Taylor coefficients (a_n), we have the expressions

$$T(f) = \sum_{n=0}^{\infty} a_n x_n = \lim_{r\to 1} \int_{-\pi}^{\pi} F_T(re^{i\theta})\overline{f(re^{i\theta})}\frac{d\theta}{2\pi}, \tag{13}$$

$$T(f) = (\sum_{n=0}^{\infty} a_n t_{kn})_{k\in\mathbb{N}} = \Big(\lim_{r\to 1} \int_{-\pi}^{\pi} g_k(re^{i\theta})\overline{f(re^{i\theta})}\frac{d\theta}{2\pi}\Big)_{k\in\mathbb{N}}. \tag{14}$$

Let us make explicit the conditions describing that a function belongs to the vector-valued Hardy spaces for $X = \ell^s$. If $1 \le r, s < \infty$, (f_k) is a sequence in H^r and $\sum_k |f_k(z)|^s < \infty$, $|z| < 1$, then $F(z) = (f_k(z))_{k\in\mathbb{N}}$ is a well defined ℓ^s-valued analytic function in the unit disc. Moreover

$$\|F\|_{H^r_{weak}(\ell^s)} = \sup\{\|\sum_{k=0}^{\infty} \lambda_k f_k\|_{H^r} : \|(\lambda_k)\|_{\ell^{s'}} = 1\} \tag{15}$$

and

$$\|F\|_{H^r(\ell^s)} = \|(\sum_{k=0}^{\infty} |f_k|^s)^{1/s}\|_{L^r}, \tag{16}$$

where in (16) f_k stands also for the boundary values of the same analytic function. Note that (16) follows from the fact that ℓ^s has the Radon-Nikodym property (see [15] and [9]) and therefore functions in $H^r(\ell^s)$ have radial boundary values in $L^r(\ell^s)$.

The following useful reformulation of the boundedness of operators from H^p to ℓ^q is straightforward.

Proposition 12 *Let $1 < p < \infty, 1 \le q < \infty$ and let $T : H^p \to \ell^q$ be a linear operator. The following are equivalent:*

(i) T is bounded.

(ii) $F_T \in H^{p'}_{weak}(\ell^q)$.

(iii) $(g_k(T))_k \in \ell^q_{weak}(H^{p'})$.

Moreover

$$\|T\| = \|F_T\|_{H^{p'}_{weak}(\ell^q)} = \|(g_k(T))\|_{\ell^q_{weak}(H^{p'})}. \tag{17}$$

Let us now mention some facts about the spaces $\ell(p,q,X)$ which will be needed later on: If $1 \le p_1, p_2, q_1, q_2 \le \infty$, $1/p = (1/p_2 - 1/p_1)^+$ and $1/q = (1/q_2 - 1/q_1)^+$ then

$$\ell(p,q) = \{(\lambda_n) : (\lambda_n \beta_n) \in \ell(p_2, q_2) \text{ for any } (\beta_n) \in \ell(p_1, q_1)\}. \tag{18}$$

Let $q, \beta > 0$. Then (see [16] and [4,21] respectively)

$$\|((n+1)^{-\beta}\alpha_n)_n\|_{\ell(1,\infty)} \approx \sup_{0<r<1} (1-r)^{\beta} (\sum_n |\alpha_n| r^n), \tag{19}$$

$$\|((n+1)^{-\beta}\alpha_n)_n\|_{\ell(1,q)} \approx \Big(\int_0^1 (1-r)^{\beta q - 1} (\sum_n |\alpha_n| r^n)^q dr \Big)^{1/q}. \tag{20}$$

For any Banach space X and $1 \le p, q < \infty$ we have

$$\ell(p,q,X)^* = \ell(p', q', X^*). \tag{21}$$

We finish the section with the following application of Minkowski's inequality.

Lemma 13 *Let $(a_{kn})_{k,n} \subset \mathbb{C}$ and write $A_k = (a_{kn})_{n \ge 0}$ and $B_n = (a_{kn})_{k \in \mathbb{N}}$. Then*

$$\|(A_k)\|_{\ell(q,s,\ell^p)} \le \|(B_n)\|_{\ell^p(\ell(q,s))}, \quad 1 \le p \le \min\{q,s\} \le \infty. \tag{22}$$

$$\|(A_k)\|_{\ell^p(\ell(q,s))} \le \|(B_n)\|_{\ell(q,s,\ell^p)}, \quad 1 \le \max\{q,s\} \le p < \infty. \tag{23}$$

Proof. Assume $1 \le p \le \min\{q,s\} \le \infty$. Since $\ell(q/p, s/p)$ is a normed space (because $q/p \ge 1$ and $s/p \ge 1$) using Minkowski's inequality we have

$$\begin{aligned}
\|(A_k)\|_{\ell(q,s,\ell^p)} &= \|(\sum_{n=0}^{\infty} |a_{kn}|^p)_k\|^{1/p}_{\ell(q/p,s/p)} \\
&\le \Big(\sum_{n=0}^{\infty} \|(|a_{kn}|^p)_k\|_{\ell(q/p,s/p)} \Big)^{1/p} \\
&= \Big(\sum_{n=0}^{\infty} \|B_n\|^p_{\ell(q,s)} \Big)^{1/p} = \|(B_n)\|_{\ell^p(\ell(q,s))}.
\end{aligned}$$

Assume now that $1 \le \max\{q,s\} \le p < \infty$. Observe that applying (22) to the adjoint matrix, we conclude that for any matrix (a'_{kn}) we also have

$$\|(B'_n)\|_{\ell(q',s',\ell^{p'})} \le \|(A'_k)\|_{\ell^{p'}(\ell(q',s'))}.$$

Now use (21) to conclude (23). □

3 Some results for vector-valued Hardy and BMOA

One of the first uses of Hausdorff-Young's inequality for vector-valued Lebesgue spaces goes back to [25]. The next lemma is well known and its proof is sketched here for completeness.

Lemma 14 *Let $1 < p \leq 2$, $p \leq q \leq p'$ and $F(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\ell^q)$. Then*

$$(\sum_{n=0}^{\infty} \|x_n\|_{\ell^q}^{p'})^{1/p'} \leq \|F\|_{H^p(\ell^q)}.$$

Proof. For $p = 2$ and $q = 2$ Plancherel's theorem holds and gives

$$(\sum_{n=0}^{\infty} \|x_n\|_{\ell^2}^{2})^{1/2} = \|F\|_{L^2(\ell^2)}.$$

On the other hand for $q = 1$ or $q = \infty$ we trivially have

$$\sup_{n \geq 0} \|x_n\|_{\ell^q} \leq \|F\|_{L^1(\ell^q)}.$$

Hence it follows, by interpolation, that

$$(\sum_{n=0}^{\infty} \|x_n\|_{\ell^s}^{p'})^{1/p'} \leq \|F\|_{H^p(\ell^s)}$$

for $s = p$ or $s = p'$. Now interpolating again between ℓ^p and $\ell^{p'}$ we get the general case. □

Actually there exists a generalization of Hausdorff-Young's inequalities to the setting on $\ell(p, q, X)$ spaces valid for some Banach spaces X. We present here a self contained proof of the following result, although the reader should be aware that the proof relies upon certain vector-valued Littlewood-Paley inequalities (see [6,5]) and it can be extended to other spaces.

Lemma 15 *Let $1 \leq p, q < \infty$ and $F(z) = \sum_{n=0}^{\infty} x_n z^n \in H^p(\ell^q)$.*

(i) If $1 < p \leq 2$ and $p \leq q \leq 2$ then $\|(x_n)\|_{\ell(p',2,\ell^q)} \leq \|F\|_{H^p(\ell^q)}$.

(ii) If $2 \leq p < \infty$ and $2 \leq q \leq p$ then $\|F\|_{H^p(\ell^q)} \leq \|(x_n)\|_{\ell(p',2,\ell^q)}$.

Proof. (i) It was shown in [1, Proposition 1.4] that, for $1 \leq p \leq q \leq 2$, we have

$$(\int_0^1 (1-r) M_p^2(F', r) dr)^{1/2} \leq C \|F\|_{H^p(\ell^q)}.$$

Using Lemma 14 we obtain

$$\int_0^1 (1-r)(\sum_{n=1}^{\infty} n^{p'} \|x_n\|_{\ell^q}^{p'} r^{(n-1)p'})^{2/p'} dr \leq C\|F\|^2_{H^p(\ell^q)}.$$

Applying now (20) to $\alpha_n = n^{p'}\|x_n\|_{\ell^q}^{p'}$, $\beta = p'$ and $q = 2/p'$ we get

$$\int_0^1 (1-r)(\sum_{n=1}^{\infty} n^{p'} \|x_n\|_{\ell^q}^{p'} r^{(n-1)p'})^{2/p'} dr \approx \|(\|x_n\|_{\ell^q}^{p'})\|_{\ell(1,2/p')} \approx \|(\|x_n\|_{\ell^q})\|^2_{\ell(p',2)},$$

which finishes this part.

(ii) follows from the dualities $(H^p(\ell^q))^* = H^{p'}(\ell^{q'})$ for $1 < p, q < \infty$ and $\ell(r,s,X)^* = \ell(r',s',X^*)$ for $1 < r, s < \infty$. □

Let us now use the embedding $\ell^1 \to \ell^2$ and its properties.

Lemma 16 *Let $1 \leq p < \infty$. If $F \in H^p_{weak}(\ell^1)$ then $F \in H^p(\ell^2)$ and*

$$\|F\|_{H^p(\ell^2)} \leq C\|F\|_{H^p_{weak}(\ell^1)}.$$

Proof. Write $F(z) = (f_k(z))_{k\in\mathbb{N}}$ where $f_k \in H^p$ and

$$\sup_{|\epsilon_k|=1} \|\sum_{k=1}^{\infty} \epsilon_k f_k\|_{H^p} = \|F\|_{H^p_{weak}(\ell^1)}.$$

Now considering $\epsilon_k = r_k(t)$ for $t \in [0,1]$ where r_k are the Rademacher functions, we obtain

$$\int_0^1 \|\sum_{k=1}^{\infty} r_k(t) f_k\|_{L^p} dt \leq \sup_{t\in[0,1]} \|\sum_{k=1}^{\infty} r_k(t) f_k\|_{L^p}.$$

Hence Kintchine's inequality implies

$$\|(\sum_{k=1}^{\infty} |f_k|^2)^{1/2}\|_{L^p} \leq C\|F\|_{H^p_{weak}(\ell^1)}.$$

The result now follows from (16). □

Let us now introduce the vector-valued versions of $BMOA$ that we shall use in the paper. The reader is referred to [5,**?**] for other possible definitions and their connections. We write $BMOA_{\mathcal{C}}(X)$ (resp. $BMOA_{weak}(X)$) for the space of analytic functions $F : \mathbb{D} \to X$ such that

$$\|F\|_{BMOA_{\mathcal{C}}(X)} = \|F(0)\| + \sup_{|z|<1} (\int_{\mathbb{D}} (1-|w|^2)\|F'(w)\|^2 P_z(w) dA(w))^{1/2} < \infty,$$

(resp.

$$\|F\|_{BMOA_{weak}(X)} = \sup_{\|x^*\|=1} \|\langle F, x^*\rangle\|_{BMOA} < \infty)$$

where, as usual, $P_z(w) = \frac{1-|z|^2}{|1-z\bar{w}|^2}$ is the Poisson kernel and dA stands for the normalized Lebesgue measure on the unit disc $\mathbb{D}$.

Note that $BMOA_{weak}(X) = L(H^1, X)$. Therefore if $T : H^1 \to \ell^q$ is a bounded linear operator for $1 < q < \infty$ we have

$$\|(g_k(T))\|_{\ell^q_{weak}(BMOA)} = \|T^*\| = \|T\| = \|F_T\|_{BMOA_{weak}(\ell^q)}. \tag{24}$$

In the case $q = 1$ we have that if $T : H^1 \to \ell^1$ is bounded then

$$\|(g_k(T))\|_{\ell^1_{weak}(BMOA)} \leq \|T^*\| = \|T\| = \|F_T\|_{BMOA_{weak}(\ell^1)}. \tag{25}$$

Let us see that the following limiting case for $p = \infty$ of Lemma 16 also holds.

Lemma 17 *If $F \in BMOA_{weak}(\ell^1)$ then $F \in BMOA_{\mathcal{C}}(\ell^2)$. Moreover*

$$\|F\|_{BMOA_{\mathcal{C}}(\ell^2)} \leq C\|F\|_{BMOA_{weak}(\ell^1)}.$$

Proof. Recall first that the inclusion map $i : \ell^1 \to \ell^2$ is 2-summing (it is even 1-summing from Grothendieck's theorem [14,?]), i.e. if $(A_n) \in \ell^2_{weak}(\ell^1)$ then $(A_n) \in \ell^2(\ell^2)$ with $\|(A_n)\|_{\ell^2(\ell^2)} \leq C\|(A_n)\|_{\ell^2_{weak}(\ell^1)}$. This implies (see [27]) that there exists $C > 0$ such that, for any finite measure space (Ω, Σ, μ), if $f : \Omega \to \ell^1$ is measurable and $\sup_{\|x^*\|_{\ell^\infty}=1} \|\langle f, x^*\rangle\|_{L^2(\mu)} \leq 1$ then $f \in L^2(\mu, \ell^2)$ and $\|f\|_{L^2(\mu,\ell^2)} \leq C$.

Let us fix $z \in \mathbb{D}$ and consider the probability measure on $\mathbb{D}$ given by $d\mu_z(w) = P_z(w)dA(w)$. Consider now $f(w) = (1 - |w|^2)^{1/2}F'(w)$ and note that, since $F \in BMOA_{weak}(\ell^1)$, we have

$$\sup_{|z|<1} \sup_{\|x^*\|=1} \|\langle f(w), x^*\rangle\|_{L^2(d\mu_z)} \leq \|F\|_{BMO_{weak}(\ell^1)}.$$

Hence $f \in L^2(d\mu_z, \ell^1)$ for all $z \in \mathbb{D}$ with $\|f\|_{L^2(d\mu_z,\ell^2)} \leq C\|F\|_{BMO_{weak}(\ell^1)}$. This implies $F \in BMOA_{\mathcal{C}}(\ell^2)$ and $\|F\|_{BMOA_{\mathcal{C}}(\ell^2)} \leq C\|F\|_{BMOA_{weak}(\ell^1)}$. □

4 Proof of Theorem 5

We start by showing the following general fact.

Proposition 18 *Let* $1 < p < \infty, 1 \leq q < \infty$*,* $p_1 = \min\{2, p\}$ *and* $1/r = (1/q - 1/p_1)^+$*. Let* $T : H^p \to \ell^q$ *be a bounded linear operator and* $g_k = g_k(T)$ *be given by (11). Then there exists* $C > 0$ *such that*

$$\|T\| \leq \min\{(\sum_{k=0}^{\infty} \|g_k\|_{H^{p'}}^q)^{1/q}, \|(\sum_{k=0}^{\infty} |g_k|^q)^{1/q}\|_{L^{p'}}\}. \tag{26}$$

$$C^{-1} \max\{\sup_{\|(\lambda_k)\|_{q'}=1} \|(\sum_{k=0}^{\infty} |\lambda_k|^2 |g_k|^2)^{1/2}\|_{L^{p'}}, (\sum_{k=0}^{\infty} \|g_k\|_{H^{p'}}^r)^{1/r} \leq \|T\| \tag{27}$$

Proof. (26) follows by Proposition 12 using (16) and the facts $\|(g_k)\|_{\ell^p_{weak}(X)} \leq \|(g_k)\|_{\ell^p(X)}$ and $\|F\|_{H^p_{weak}(X)} \leq \|F\|_{H^p(X)}$.

Let us show (27). For each $\lambda = (\lambda_k) \in \ell^{q'}$, denote $T_\lambda : H^p \to \ell^1$ given by

$$T_\lambda(f) = \sum_{k=0}^{\infty} \lambda_k \langle T(f), e_k \rangle.$$

Since $\|T\| = \sup\{\|T_\lambda\| : \|(\lambda_k)\|_{q'} = 1\}$, and $g_k(T_\lambda) = \lambda_k g_k(T)$, from (17), we have to get lower estimates of $\|(\lambda_k g_k)\|_{\ell^1_{weak}(H^{p'})}$.

Using that $H^{p'}$ has cotype $u = \max\{p', 2\}$ (see for instance [14]),we have

$$(\sum_{k=0}^{\infty} |\lambda_k|^u \|g_k\|_{H^{p'}}^u)^{1/u} \leq C\|(\lambda_k g_k)\|_{\ell^1_{weak}(H^{p'})}$$

and, taking the supremum over (λ_k) in the unit ball of $\ell^{q'}$, we obtain that $\|(g_k)\|_{\ell^r(H^{p'})} \leq C\|T\|$ for $1/r = (1/u - 1/q')^+ = (1/q - 1/p_1)^+$.

On the other hand, Khinchine's inequality implies that

$$\|(\sum_{k=0}^{\infty} |\lambda_k|^2 |g_k|^2)^{1/2}\|_{L^{p'}} \leq C\|(\lambda_k g_k)\|_{\ell^1_{weak}(H^{p'})},$$

and the proof of the proposition is finished. □

We now proceed to the proof of Theorem 5. Let $1 < p < \infty, 1 \leq q < \infty$, $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$. Let $T : H^p \to \ell^q$ be a bounded linear operator and $F_T(z) = (g_k(z))_k = \sum_{n=0}^{\infty} x_n z^n$ be defined by the formulas (11) and (12).

Let us first show that $\|T\| \leq \min\{\|(T_k)\|_{\ell^q(\ell(p_1,2))}, \|(x_n)\|_{\ell(p_1,2,\ell^q)}\}$.

Our proof will be based upon the following extension of Hausdorff-Young's inequalities (see [19]): If $p_1 = \min\{p, 2\}$ and $p_2 = \max\{p, 2\}$ then

$$\|g\|_{H^{p'}} \leq \|(\alpha_n)\|_{\ell(p_1,2)}, \quad \|(\alpha_n)\|_{\ell(p_2,2)} \leq \|g\|_{H^{p'}}$$

for any $g(z) = \sum_{n=0}^{\infty} \alpha_n z^n$.

Therefore (26) in Proposition 18 implies

$$\|T\| \leq C\|(g_k)\|_{\ell^q(H^{p'})} \leq C\|(T_k)\|_{\ell^q(\ell(p_1,2))}.$$

On the other hand, $\|T\| = \|F_T\|_{H^{p'}_{weak}(\ell^q)}$ and we have

$$\begin{aligned}
\|F_T\|_{H^{p'}_{weak}(\ell^q)} &= \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|\langle \lambda, F_T\rangle\|_{H^{p'}} \\
&\leq \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|(\langle \lambda, x_n\rangle)_n\|_{\ell(p_1,2)} \\
&\leq \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|(\sum_{k=1}^{\infty} \lambda_k t_{kn})_n\|_{\ell(p_1,2)} \\
&\leq \|(\sum_{k=1}^{\infty} |t_{kn}|^q)_n^{1/q}\|_{\ell(p_1,2)} \\
&\leq \|(x_n)\|_{\ell(p_1,2,\ell^q)}.
\end{aligned}$$

Let us now show that for each $u \geq q$ there exists $C > 0$ such that

$$\max\{\|(T_k)\|_{\ell^r(\ell(p_2,2))}, \|(x_n)\|_{\ell(s_u,2,\ell^u)}\} \leq C\|T\|,$$

where $1/r = (1/q - 1/p_1)^+$ and $1/s_u = (1/q - 1/p_2' - (1/u - 1/2)^+)^+$.

Note that (27) in Proposition 18 together with the Hausdorff-Young's inequalities give

$$\|(T_k)\|_{\ell^r(\ell(p_2,2))} \leq \|(g_k)\|_{\ell^r(H^{p'})} \leq C\|T\|.$$

On the other hand, as above $\|T\| = \|F_T\|_{H^{p'}_{weak}(\ell^q)}$ and combining Hausdorff-Young and (18), we obtain

$$\begin{aligned}
\|F_T\|_{H^{p'}_{weak}(\ell^q)} &= \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|\langle \lambda, F_T\rangle\|_{H^{p'}} \\
&\geq \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|(\langle \lambda, x_n\rangle)_n\|_{\ell(p_2,2)} \\
&= \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1, \|(\beta_n)\|_{\ell(p_2',2)}=1} \sum_{n=0}^{\infty} |\beta_n \sum_{k=1}^{\infty} \lambda_k t_{kn}| \\
&\geq \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1, \|(\beta_n)\|_{\ell(p_2',2)}=1} |\sum_{n=0}^{\infty} \langle \beta_n x_n, \lambda\rangle|
\end{aligned}$$

Therefore $(\beta_n x_n) \in \ell^1_{weak}(\ell^q)$ for any $(\beta_n) \in \ell(p'_2, 2)$ and

$$\sup_{\|(\beta_n)\|_{\ell(p'_2,2)}=1} \|(\beta_n x_n)\|_{\ell^1_{weak}(\ell^q)} \leq \|F_T\|_{H^{p'}_{weak}(\ell^q)} = \|T\|.$$

We now use the fact (due to B. Carl in [12] and G. Bennett in [3] independently) that the inclusion map $\ell^q \to \ell^u$ is $(a,1)$-summing for $1/a = 1/q - (1/u - 1/2)^+$ (see [14, pg. 209]) to conclude that $(\beta_n x_n) \in \ell^a(\ell^u)$ for any $(\beta_n) \in \ell(p'_2, 2)$. Now (18) implies $(x_n) \in \ell(s, 2, \ell^u)$ for $1/s = (1/a - 1/p'_2)^+$. The proof is then complete. □

5 Improvements for $1 \leq p < 2$

We first recall some known facts about $BMOA$-functions. It was shown in [10] that $M_2(f', r) = O(\frac{1}{(1-r)^{1/2}})$ implies $f \in BMOA$. Moreover

$$\|f\|_{BMOA} \leq C(|f(0)| + \sup_{0<r<1} (1-r)^{1/2} M_2(f', r)).$$

Using this estimate and (19) we conclude that

$$\|g\|_{BMOA} \leq C\|((n+1)^{1/2}\alpha_n)\|_{\ell(2,\infty)} \tag{28}$$

Also, using duality together with Paley's inequality for functions in H^1 (see [16]) we obtain

$$\|g\|_{BMOA} \leq C\|(\alpha_n)\|_{\ell(1,2)}. \tag{29}$$

The reader should notice that these two sufficient conditions on the Taylor coefficients to define $BMOA$-function are of independent nature. It suffices to take $\alpha_n = \frac{1}{n+1}$ to have an example satisfying $((n+1)^{1/2}\alpha_n) \in \ell(2,\infty)$ but $(\alpha_n) \notin \ell(1,2)$ and to take $\alpha_{2^k} = \frac{1}{k}$ and zero otherwise to have $(\alpha_n) \in \ell(1,2)$ but $(n+1)^{1/2}\alpha_n \notin \ell(2,\infty)$.

Proof of Theorem 8

Using (28) and (29) together with (24) we have the estimate

$$\|T\| \leq \|(g_k)\|_{\ell^q(BMOA)} \leq C \min\{\|(T_k)\|_{\ell^q(\ell(1,2))}, \|(A_k)\|_{\ell^q(\ell(2,\infty))}\}.$$

On the other hand

$$\|T\| = \|F_T\|_{BMOA_{weak}(\ell^q)}$$

$$= \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \|\langle \lambda, F_T\rangle\|_{BMO}$$
$$\leq \sup_{\|(\lambda_k)\|_{\ell^{q'}}=1} \min\{\|(\langle\lambda, x_n\rangle)\|_{\ell(1,2)}, \|(\langle\lambda, (n+1)^{1/2}x_n\rangle)\|_{\ell(2,\infty)}\}$$
$$\leq \min\{\|(x_n)\|_{\ell(1,2,\ell^q)}, \|((n+1)^{1/2}x_n)\|_{\ell(2,\infty,\ell^q))}\}.$$

Invoking Lemma 13 we obtain the following estimates

$$\|(x_n)\|_{\ell(1,2,\ell^1)} \leq \|(T_k)\|_{\ell^1(\ell(1,2))},$$

$$\|((n+1)^{1/2}x_n)\|_{\ell(2,\infty,\ell^1)} \leq \|(A_k)\|_{\ell^1(\ell(2,\infty))},$$
$$\|((n+1)^{1/2}x_n)\|_{\ell(2,\infty,\ell^q)} \leq \|(A_k)\|_{\ell^q(\ell(2,\infty))}, \quad q \leq 2,$$
$$\|(T_k)\|_{\ell^q(\ell(1,2))} \leq \|(x_n)\|_{\ell(1,2,\ell^q)}, \quad q \geq 2.$$

Hence (i), (ii) and (iii) follow from these estimates. □

Proof of Theorem 9

Take $t \geq 2$ such that $1/t + 1/2 = 1/p$ and $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^t$ with $\|\phi\|_{H^t} = 1$. Define $T_\phi : H^2 \to \ell^q$ given by

$$T_\phi(f) = T(\phi f).$$

Due to the factorization result (see [16]) $H^p = H^2 H^t$ we can write

$$\|T\| = \sup\{\|T_\phi\| : \|\phi\|_{H^t} = 1\}.$$

Observe that

$$x_n(T_\phi) = T(u_n\phi) = \sum_{l=0}^{\infty} \alpha_l T(u_{n+l}) = \sum_{l=0}^{\infty} \alpha_l x_{n+l}.$$

Therefore the matrix associated to T_ϕ is given by $a_{kn}(T_\phi) = (t'_{kn})$ where

$$t'_{kn} = \sum_{l\geq n} \alpha_{l-n} t_{kl} = \sum_{l=0}^{\infty} \alpha_l t_{k,l+n}.$$

Now using (5) one can write, for $1/r = 1/q - 1/2$,

$$\begin{aligned} \max\{\|((T_\phi)_k)\|_{\ell^r(\ell^2)}, \|(x_n(T_\phi))\|_{\ell^r(\ell^2)}\} &\leq C\|T_\phi\| \\ &\leq C\|T\|\|\phi\|_{H^t} \\ &\leq C\|T\|\|(\alpha_l)\|_{\ell(t',2)}. \end{aligned}$$

This shows the result. □

Proof of Theorem 10

Assume $1 \leq p < 2$ and let $T : H^p \to \ell^1$ be bounded. The estimate

$$\sup_{\|(\alpha_l)\|_{\ell(t',2)}=1} \|(\sum_{l=0}^{\infty} \alpha_l t_{k,n+l})_k\|_{\ell^2(\ell^2)} \leq C\|T\|$$

was obtained in Theorem 9 in the case $q = 1$.

Let us show

$$\sup_{\|\sum_l \alpha_l z^l\|_{H^t}=1} \|(\sum_{l=0}^{\infty} \alpha_l t_{k,n+l})_n\|_{\ell^2(\ell^2)} \leq C\|T\|. \tag{30}$$

In the case $1 < p < 2$, we can use (17) to conclude that $F_T \in H^{p'}_{weak}(\ell^1)$ and, due to Lemma 16, $F_T \in H^{p'}(\ell^2)$.

In the case $p = 1$, we can use (25) to obtain $F_T \in BMOA_{weak}(\ell^1)$ and Lemma 17 to conclude that $F_T \in BMOA_{\mathcal{C}}(\ell^2))$.

Using the dualities $(H^p(\ell^2))^* = H^{p'}(\ell^2)$ for $1 < p < 2$ and $(H^1(\ell^2))^* = BMOA_{\mathcal{C}}(\ell^2))$ for $p = 1$, we can write, for $1 \leq p < 2$, that

$$\sup\{|\sum_{n=0}^{\infty} \langle x_n, x'_n\rangle| : G(z) = \sum_{j=0}^{\infty} x'_n z^n, \|G\|_{H^p(\ell^2)} = 1\} \leq C\|T\|.$$

In particular, for each $g(z) = \sum_{n=0}^{\infty} y_n z^n \in H^2(\ell^2)$ and $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^t$ where $1/t + 1/2 = 1/p$, the function $G(z) = g(z)\phi(z) = \sum_n x'_n z^n \in H^p(\ell^2)$ satisfies $x'_n = \sum_{l=0}^{n} y_l \alpha_{n-l}$ and $\|G\|_{H^p(\ell^2)} \leq \|g\|_{H^2(\ell^2)}\|\|\phi\|_{H^t}$. Therefore, in such a case, we obtain

$$\sum_{n=0}^{\infty} \langle x_n, x'_n\rangle = \sum_{l=0}^{\infty}\sum_{n=l}^{\infty} \langle x_n, y_l \alpha_{n-l}\rangle = \sum_{l=0}^{\infty}\langle \sum_{n=0}^{\infty} \alpha_l x_{n+l}, y_l\rangle.$$

Finally, taking the supremum over $\|(y_j)\|_{\ell^2(\ell^2)} = 1$ and $\|\phi\|_{H^t} = 1$ we get (30). □

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