Dyadic BMO, paraproducts and Haar multipliers

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Abstract. Some new proofs on the boundedness of dyadic paraproducts for functions in dyadic BMO are given.

1. Introduction

Let $\mathcal{D}$ denote the collection of dyadic intervals of the real line $\mathbb{R}$, say $\mathcal{D} = \cup_{k\in\mathbb{Z}}\mathcal{D}_k$, where $\mathcal{D}_k$ stands for the family of dyadic intervals in the k-generation, that is $|I| = 2^{-k}$. Let $(h_I)_{I\in\mathcal{D}}$ be the Haar basis of $L^2(\mathbb{R})$, i.e. $h_I = \frac{1}{|I|^{1/2}}(\chi_{I^+} - \chi_{I^-})$ where I^+ and I^- stand for the right and left halves of I and set x_I the center of I. For $I \in \mathcal{D}$ and $\phi \in L^2(\mathbb{R})$, let ϕ_I denote the Haar coefficient,

$$\phi_I = \langle \phi, h_I\rangle = \int_I \phi(t)h_I dt$$

and $m_I\phi$ the average of ϕ over I,

$$m_I\phi = \langle \phi, \frac{\chi_I}{|I|}\rangle = \frac{1}{|I|}\int_I \phi(t)dt.$$

Observe that

$$m_I(h_J) = \langle \frac{\chi_I}{|I|}, h_J\rangle = 0, \qquad J \subseteq I, \quad J, I \in \mathcal{D} \tag{1.1}$$

$$m_I(h_J) = h_J(x_I) = h_J(t), \qquad I \subsetneq J, \quad J, I \in \mathcal{D}, t \in I. \tag{1.2}$$

We say that $\phi \in L^2(\mathbb{R})$ belongs to dyadic BMO, written $\phi \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{R})$, if

$$\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I |\phi(t) - m_I\phi|^2 dt\Big)^{1/2} < \infty. \tag{1.3}$$

For each $I \in \mathcal{D}$ we write

$$P_I(\phi) = \sum_{J\subseteq I} \phi_J h_J,$$

which, due to (1.1) and (1.2), coincides with

$$P_I(\phi) = (\phi - m_I\phi)\chi_I.$$

2000 *Mathematics Subject Classification.* Primary 47B; Secondary 42C.
Key words and phrases. Dyadic BMO, dyadic paraproduct, Haar multipliers.
The author was supported in part by Proyecto MTM2005-08350-C03-03.

Therefore $b \in \mathrm{BMO^d}(\mathbb{R})$ if and only if

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}} \|P_I(\phi)\|_{L^2} < \infty, \tag{1.4}$$

which, due to orthogonality of the Haar system, can be described by the Carleson condition

$$\sup_{I\in\mathcal{D}} \Big(\frac{1}{|I|} \sum_{J\in\mathcal{D}, J\subseteq I} |\phi_J|^2\Big)^{1/2} < \infty. \tag{1.5}$$

Due to John-Nirenberg's lemma, one can replace the $L^2(\mathbb{R})$ norm in (1.3) and (1.4) by any L^p-norm. That is, for $1 \leq p < \infty$, we have $\phi \in \mathrm{BMO^d}(\mathbb{R})$ if and only if

$$\sup_{I\in\mathcal{D}} (\frac{1}{|I|}\int_I |\phi(t) - m_I\phi|^p dt)^{1/p} = \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/p}} \|P_I(\phi)\|_{L^p} < \infty. \tag{1.6}$$

Another equivalent formulation comes from the duality (see [**G, M**])

$$\mathrm{BMO^d}(\mathbb{R}) = (H^1_d(\mathbb{R}))^*, \tag{1.7}$$

where $H^1_d(\mathbb{R})$ consists of those functions $\phi \in L^1(\mathbb{R})$ such that $\mathcal{S}(\phi) \in L^1(\mathbb{R})$, where

$$\mathcal{S}(\phi) = (\sum_{I\in\mathcal{D}} |\phi_I|^2 \frac{\chi_I}{|I|})^{1/2}$$

stands for the dyadic square function.

Let us recall that Littlewood-Paley theory gives that $\phi \in L^p(\mathbb{R})$ implies that $\mathcal{S}(\phi) \in L^p(\mathbb{R})$ for $1 < p < \infty$ and that H^1_d is the correct replacement in the limiting case $p = 1$.

Other equivalent characterization of $H^1_d(\mathbb{R})$ are given in terms of dyadic atoms or maximal dyadic functions. The reader is referred to [**M**] or [**G**] for the results concerning dyadic H^1 and BMO.

The reader is referred to [**M, Per, G, Ch**] for the basic notions on dyadic Harmonic Analysis.

Let E_k stand for the conditional expectation over the filtration generated by dyadic intervals in $\mathcal{D}_k$, that is

$$E_k(f) = \sum_{|I|=2^{-k}} (m_I f)\chi_I$$

and $\Delta_k = E_{k+1} - E_k$, that is

$$\Delta_k f = \sum_{|I|=2^{-k}} f_I h_I.$$

From this one has

$$E_k(f) = \sum_{|I|>2^{-k}} f_I h_I.$$

Now the "dyadic paraproduct" is defined by the formula

$$\pi_b(f) = \sum_{k\in\mathbb{Z}} E_k f \Delta_k b = \sum_{I\in\mathcal{D}} b_I (m_I f) h_I.$$

For real-valued functions b the adjoint of this operator becomes

$$\Delta_b(f) = \sum_{k\in\mathbb{Z}} \Delta_k f \Delta_k b = \sum_{I\in\mathcal{D}} b_I f_I \frac{\chi_I}{|I|}.$$

We shall give new proofs of the following known result (whose proof is usually based on Carleson's lemma).

THEOREM 1.1. *(see* [**M**, pag 273]*, or* [**Per2**]*) Let* $\phi \in L^2(\mathbb{R})$*. The following are equivalent:*
(i) $\phi \in \mathrm{BMO^d}(\mathbb{R})$
(ii) $\pi_\phi(f) = \sum_{I\in\mathcal{D}} \phi_I(m_I f)h_I$ *defines a bounded linear operator on* $L^2(\mathbb{R})$*.*
(ii) $\Delta_\phi(f) = \sum_{I\in\mathcal{D}} \phi_I f_I \frac{\chi_I}{|I|}$ *defines a bounded operator on* $L^2(\mathbb{R})$*.*

The objective of this note is to get some new proofs of the characterization of dyadic BMO in terms of dyadic "paraproducts" which makes use of the notions of Haar multipliers, interpolation and Lorentz spaces. Some ideas to be considered later on were used in the work by S. Pott and the author in [**BP1, BP2**] (see also [**Bl**]) when analyzing the boundedness of dyadic paraproducts in the bidisc or in the operator valued case where the classical tools were not valid any longer.

Recall (see [**Per**]) that a sequence of functions $(\Phi_I)_{I\in\mathcal{D}}$ is said to be a *Haar multiplier*, if there exists $C > 0$ such that

$$\|\sum_{I\in\mathcal{D}} \Phi_I f_I h_I\|_{L^2(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})} \text{ for all } f \in L^2(\mathbb{R}).$$

We write $T_{(\Phi_I)}$ for the operator

$$T_{(\Phi_I)}(f) = \sum_{I\in\mathcal{D}} \Phi_I f_I h_I.$$

Is is clear that for $\Phi_I(t) = \alpha_I$ for $I \in \mathcal{D}$ one has $\|T_{(\Phi_I)}\| = \sup_{I\in\mathcal{D}} |\alpha_I|$. In general, using the Haar functions as test functions, one gets that if $(\Phi_I)_{I\in\mathcal{D}}$ is a *Haar multiplier* then

$$\sup_{I\in\mathcal{D}} \frac{\|\Phi_I\|_{L^2(\mathbb{R})}}{|I|^{1/2}} \le \|T_{(\Phi_I)}\|. \tag{1.8}$$

Given $b \in L^2(\mathbb{R})$, we define the "dyadic sweep" of b, to be denoted S_b, by

$$S_b = \mathcal{S}(b)^2 = \Delta_b(b) = \sum_{I\in\mathcal{D}} |b_I|^2 \frac{\chi_I}{|I|}.$$

Hence $b \in L^{2p}(\mathbb{R})$ implies $S_b \in L^p(\mathbb{R})$ for $1 < p < \infty$ and $\|S_b\|_{L^p} \le C\|b\|^2_{L^{2p}}$.

Let us include now the proof of the following known fact.

PROPOSITION 1.2. Let $b \in L^2(\mathbb{R})$. Then $b \in \mathrm{BMO^d}(\mathbb{R})$ if and only if $S_b \in \mathrm{BMO^d}(\mathbb{R})$.

PROOF. Note that $P_I(\chi_J) = 0$ for $I \subseteq J$. This shows that

$$P_I(S_b) = P_I(S_{P_I b}).$$

Therefore

$$\begin{aligned}
\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}}\|P_I(S_b)\|_{L^2(\mathbb{R})} &= \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}}\|P_I(S_{P_I b})\|_{L^2(\mathbb{R})} \\
&\le \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}}\|S_{P_I b}\|_{L^2(\mathbb{R})} \\
&= \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|^{1/4}}\|P_I b\|_{L^4(\mathbb{R})}\right)^2
\end{aligned}$$

From John-Nirenberg's Lemma one has

$$\begin{aligned}\sup_{I\in\mathcal{D}}\left(\frac{1}{|I|^{1/4}}\|P_Ib\|_{L^4(\mathbb{R})}\right)^2 &\leq C\sup_{I\in\mathcal{D}}\frac{1}{|I|}\|P_Ib\|^2_{L^2(\mathbb{R})}\\ &= C\sup_{I\in\mathcal{D}}\frac{1}{|I|}\sum_{J\subseteq I}|b_J|^2\\ &= C\|b\|^2_{BMO^d}.\end{aligned}$$

Hence $\|S_b\|_{BMO^d}\leq C\|b\|^2_{BMO^d}$.

Conversely, let $I\in\mathcal{D}$ and $c_I=\sum_{I\subsetneq J}|b_J|^2\frac{1}{|J|}$. Note that $(S_b-c_I)\chi_I=\sum_{J\subseteq I}|b_J|^2\frac{\chi_J}{|J|}$. Hence

$$\begin{aligned}\sum_{J\subseteq I}|b_J|^2 &= \|(S_b-c_I)\chi_I\|_{L^1(\mathbb{R})}\\ &\leq \|(S_b-c_I)\chi_I\|_{L^2(\mathbb{R})}|I|^{1/2}\leq C_1\|S_b\|_{BMO^d}|I|.\end{aligned}$$

Hence $\|b\|^2_{BMO^d}\leq C_1\|S_b\|_{BMO^d}$.

□

Note that only the implication $b\in \mathrm{BMO^d}(\mathbb{R})$ implies $S_b\in\mathrm{BMO^d}(\mathbb{R})$ makes use of John-Nirenberg's lemma. We shall give another approach later independent of it.

We start mentioning the following formula:

$$\pi_b(f)=\sum_{I\in\mathcal{D}}[P_{I^+}(b)+P_{I^-}(b)]f_Ih_I. \tag{1.9}$$

Indeed,

$$\begin{aligned}\pi_b(f) &= \sum_{J\in\mathcal{D}}b_J(m_Jf)h_J\\ &= \sum_{J\in\mathcal{D}}b_J(\sum_{J\subsetneq I}m_J(h_I)f_I)h_J\\ &= \sum_{I\in\mathcal{D}}(\sum_{J\subseteq I^+}b_Jh_J+\sum_{J\subseteq I^-}b_Jh_J)f_Ih_I\\ &= \sum_{I\in\mathcal{D}}[P_{I^+}(b)+P_{I^-}(b)]f_Ih_I.\end{aligned}$$

Therefore

$$(\pi_b+\Delta_b)(f)=\sum_{I\in\mathcal{D}}P_I(b)f_Ih_I. \tag{1.10}$$

Therefore $\pi_b+\Delta_b$ being bounded in $L^2(\mathbb{R})$ is the same as $(P_Ib)_{I\in\mathcal{D}}$ being a Haar multiplier. Hence from (1.8) one has the estimate $\|b\|_{BMO^d}\leq\|\pi_b+\Delta_b\|$.

This means that only $(i)\Longrightarrow(ii)$ in Theorem 1.1 needs a proof. Note that, using that $\pi_b^*=\Delta_b$ one sees that $(ii)\Longleftrightarrow(iii)$. Hence (ii) in Theorem 1.1 implies $\pi_b+\Delta_b$ is bounded and, due to the previous remark, (i) holds.

We shall present a proof the following list of equivalent formulations.

THEOREM 1.3. *Let $b\in L^2(\mathbb{R})$be real-valued. The following are equivalent:*

(1) $\pi_b(1)=b\in\mathrm{BMO^d}(\mathbb{R})$
(2) π_b *is bounded on* $L^2(\mathbb{R})$.

(3) π_b *is bounded on* $L^p(\mathbb{R})$ *for some (or for all)* $1 < p < \infty$.
(4) Δ_b *is bounded on* $L^2(\mathbb{R})$.
(5) Δ_b *is bounded on* $L^p(\mathbb{R})$ *for some (or for all)* $1 < p < \infty$.
(6) $\pi_b + \Delta_b$ *is bounded on* $L^2(\mathbb{R})$.
(7) $\pi_b + \Delta_b$ *is bounded on* $L^p(\mathbb{R})$ *for some (or for all)* $1 < p < \infty$.
(8) $(P_I b)_{I\in\mathcal{D}}$ *is a Haar multiplier.*
(9) $(b_I h_I)_{I\in\mathcal{D}}$ *is a Haar multiplier.*
(10) $S_b = \Delta_b(b) \in \mathrm{BMO^d}(\mathbb{R})$.

2. First proof of Theorem 1.1

We shall need the following lemma which is essentially contained in [**Per**, Lemma 2.10] and whose modification is included here.

Lemma 2.1. *Let* $1 < p < \infty$ *and let* T *be a linear (or sublinear) operator of weak type (p,p) such that*

$$supp\ T(h_I) \subseteq I$$

for all $I \in \mathcal{D}$. *Then* T *is strong type* (q, q) *for* $1 < q < p$.

Proof. If suffices to see that T is weak type $(1, 1)$ and then use interpolation. Assume $f \in L^1(\mathbb{R})$ and let $\lambda > 0$. Apply Calderón-Zygmund decomposition (see [**GR**] or [**Per**, Lemma 2.7]) to decompose $f = g + b$ where $g \in L^\infty(\mathbb{R})$, $\|g\|_\infty \leq \lambda$, $\|g\|_1 \leq 2\|f\|_1$, $b = \sum_j b_j$ where $b_j = (f - m_{I_j} f)\chi_{I_j} = P_{I_j} f$ and $\{I_j\}$ form a disjoint sequence of dyadic intervals such that $\sum_j |I_j| \leq \frac{\|f\|_1}{\lambda}$. Now, as usual,

$$|\{|T(f)| > \lambda\}| \leq |\{|T(g)| > \lambda/2\}| + |\{|T(b)| > \lambda/2\}|.$$

Note that

$$|\{|T(g)| > \lambda/2\}| \leq 2^p C \frac{\|g\|_p^p}{\lambda^p} \leq 2^{p+1} C \frac{\|g\|_\infty^{p-1}\|f\|_1}{\lambda^p} \leq 2^{p+1} C \frac{\|f\|_1}{\lambda}.$$

On the other hand, since $supp\ T(b_j) \subseteq I_j$,

$$|\{|T(b)| > \lambda/2\}| \leq | \cup_j I_j| \leq \frac{\|f\|_1}{\lambda}.$$

□

$(i) \Longrightarrow (ii)$: Let us show that $b \in \mathrm{BMO^d}(\mathbb{R})$ implies π_b is bounded on $L^2(\mathbb{R})$.

Our first step is to see that $b \in \mathrm{BMO^d}(\mathbb{R})$ implies that there exists $C > 0$ such that

$$\|\pi_b(\chi_A)\|_{L^2(\mathbb{R})} \leq C|A|^{1/2} \tag{2.1}$$

for any open set $A \subset \mathbb{T}$.

Given an open set A we write $P_A(f) = \sum_{I\subseteq A} f_I h_I$. Note that condition (1.5) gives

$$|b_J|^2 \leq \|b\|_{BMO^d}|J|, J \in \mathcal{D}$$

and also that, since A is a disjoint union of dyadic intervals, one has

$$\|P_A b\|^2_{L^2(\mathbb{R})} = \sum_{I\subseteq A} |b_I|^2 \leq \|b\|^2_{BMO^d}|A|$$

The first observation is that

$$\pi_b(\chi_A) = P_A(b) + \sum_{|A\cap J|<|J|} b_J \frac{|A \cap J|}{|J|} h_J.$$

Then

$$\begin{aligned}\|\pi_b(\chi_A)\|^2_{L^2(\mathbb{R})} &= \|P_A(b)\|^2_{L^2(\mathbb{R})} + \sum_{|A\cap J|<|J|} |b_J|^2 \frac{|A\cap J|^2}{|J|^2} \\ &\leq \|b\|^2_{BMO^d}|A| + C(\sup_{J\in\mathcal{D}} \frac{|b_J|^2}{|J|}) \sum_{|A\cap J|<|J|} |A\cap J| \\ &\leq C\|b\|^2_{BMO^d}|A|.\end{aligned}$$

Now (2.1) implies (see [**SW**]) that $\pi_b : L^{2,1}(\mathbb{R}) \to L^2(\mathbb{R})$ where $L^{2,1}(\mathbb{R})$ is the corresponding Lorentz space. This shows that the adjoint operator $\Delta_b : L^2(\mathbb{R}) \to L^{2,\infty}(\mathbb{R})$ is bounded.

Notice that

$$\Delta_b(h_I) = b_I \frac{\chi_I}{|I|} \tag{2.2}$$

and

$$\pi_b(h_I) = (P_{I^+}b + P_{I^-}b)h_I \tag{2.3}$$

Due to (2.2) we can apply Lemma 2.1 to get that $\Delta_b : L^q(\mathbb{R}) \to L^q(\mathbb{R})$ is bounded for any $1 < q < 2$. Now take again the adjoint to obtain that $\pi_b : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded for $2 < p < \infty$.

Now use (2.3) and apply Lemma 2.1 again to obtain $\pi_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded.

3. Second proof of Theorem 1.1

A fundamental tool is the following result which can be found in [**BP1**] or [**B**] for operator valued functions.

LEMMA 3.1. *Let $b \in L^2(\mathbb{R})$. Then*

$$\Delta_b \pi_b = \pi_{S_b} + \Delta_{S_b} - \Gamma_b,$$

where $\Gamma_b(f) = \sum_{J\in\mathcal{D}} \frac{b_J^2}{|J|} f_J h_J$.

PROOF. Note that $P_J(S_b) = (S_b - m_J S_b)\chi_J = \sum_{I\subseteq J} b_I^2 \frac{\chi_I}{|I|}$. Then

$$\begin{aligned}\Delta_b\pi_b(f) &= \Delta_b(\sum_{I\in\mathcal{D}} b_I m_I(f) h_I) \\ &= \sum_{I\in\mathcal{D}} b_I^2 m_I(f) \frac{\chi_I}{|I|} \\ &= \sum_{I\in\mathcal{D}} b_I^2 \sum_{I\subsetneq J} f_J h_J \frac{\chi_I}{|I|} \\ &= \sum_{J\in\mathcal{D}} (\sum_{I\subsetneq J} b_I^2 \frac{\chi_I}{|I|}) f_J h_J \\ &= \sum_{J\in\mathcal{D}} P_J(S_b) f_J h_J - \sum_{J\in\mathcal{D}} \frac{b_J^2}{|J|} f_J h_J \\ &= \pi_{S_b}(f) + \Delta_{S_b}(f) - \Gamma_b(f).\end{aligned}$$

□

Now we first show some apparently weaker result.

LEMMA 3.2. *If* $b \in BMO^d(\mathbb{R})$ *then* $\pi_b + \Delta_b$ *is bounded in* $L^2(\mathbb{R})$.
Moreover $\|\pi_b + \Delta_b\| \le C\|b\|_{BMO^d}$.

PROOF. For $f, g \in L^2(\mathbb{R})$,

$$\begin{aligned}\langle \pi_b(f) + \Delta_b(f), g\rangle &= \sum_{I\in\mathcal{D}} [(m_I f) g_I + (m_I g) f_I] b_I \\ &= \langle b, \sum_{I\in\mathcal{D}} [(m_I f) g_I + (m_I g) f_I] h_I\rangle\end{aligned}$$

Hence it suffices to see, using the duality $(H^1(\mathbb{R}))^* = BMO^d(\mathbb{R})$, that

$$h = \sum_{I\in\mathcal{D}} [(m_I f) g_I + (m_I g) f_I] h_I \in H^1(\mathbb{R})$$

and $\|h\|_{H^1} \le C\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}$.

Notice that $|m_I f \chi_I| \le m_I|f|\chi_I \le f^*\chi_I$ where $f^*(t) = \sup_{t\in I\in\mathcal{D}} \frac{1}{|I|}\int_I |f(t)|dt$ stands for the dyadic maximal function. Therefore

$$\mathcal{S}(h) = (\sum_{I\in\mathcal{D}} [(m_I f) g_I + (m_I g) f_I]^2 \frac{\chi_I}{|I|})^{1/2} \le C(f^*\mathcal{S}(g) + g^*\mathcal{S}(f))$$

what gives

$$\|\mathcal{S}(h)\|_{L^1(\mathbb{R})} \le C(\|f^*\|_{L^2(\mathbb{R})}\|\mathcal{S}(g)\|_{L^2(\mathbb{R})} + \|f^*\|_{L^2(\mathbb{R})}\|\mathcal{S}(g)\|_{L^2(\mathbb{R})}) \le C\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$

□

$(i) \implies (ii)$ Since $b \in \mathrm{BMO^d}(\mathbb{R})$ one has invoking Lemma 1.2 that $S_b \in \mathrm{BMO^d}(\mathbb{R})$. Now Lemma 3.2 implies $\pi_{S_b} + \Delta_{S_b}$ is bounded in $L^2(\mathbb{R})$. On the other hand the assumption $b \in \mathrm{BMO^d}(\mathbb{R})$ guarantees that $|b_J|^2 \le C|J|$ for all $J \in \mathcal{D}$. Hence Γ_b is bounded on $L^2(\mathbb{R})$. Finally using Lemma 3.1 one gets that $\pi_b^*\pi_b$, and hence π_b is bounded on $L^2(\mathbb{R})$.

4. Final remarks

To get the proof of the complete list of equivalences we can use the following two propositions.

PROPOSITION 4.1. Let $b \in L^2(\mathbb{R})$ be real-valued. The following are equivalent:

(1) π_b is bounded on $L^2(\mathbb{R})$.
(2) π_b is bounded on $L^p(\mathbb{R})$ for some (for all) $1 < p < \infty$.
(3) Δ_b is bounded on $L^2(\mathbb{R})$.
(4) Δ_b is bounded on $L^p(\mathbb{R})$ for some (for all) $1 < p < \infty$.
(5) $(b_I h_I)_{I\in\mathcal{D}}$ is a Haar multiplier.
(6) $(P_{I^+}b + P_{I^-}b)_{I\in\mathcal{D}}$ is a Haar multiplier.

PROOF. (1) $\iff$ (3) $\iff$ (5) $\iff$ (6) and (2) $\iff$ (4) are immediate.

(1) $\implies$ (2) From (2.3) and Lemma 2.1 one obtains that π_b is bounded on $L^p(\mathbb{R})$ for $1 < p < 2$. Thus Δ_b is bounded on $L^q(\mathbb{R})$ for $2 < q < \infty$. Now again Lemma 2.1 and (2.2) gives Δ_b is bounded on $L^q(\mathbb{R})$ for $1 < q < \infty$. Finally take adjoints to get that π_b is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.

(2) $\Longrightarrow$ (1). If π_b is bounded on some $L^p(\mathbb{R})$ for $p > 2$ the result follows from (2.3) and Lemma 2.1. In the case $1 < p \leq 2$ one can repeat the argument in the previous implication. □

PROPOSITION 4.2. If $\Delta_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is bounded then $\Delta_b : \mathrm{BMO^d}(\mathbb{R}) \to \mathrm{BMO^d}(\mathbb{R})$ is also bounded.

In particular if $b \in BMO^d(\mathbb{R})$ then $S_b = \Delta_b(b) \in BMO^d(\mathbb{R})$.

PROOF. Let $f \in \mathrm{BMO^d}(\mathbb{R})$. Notice that $P_I(\chi_J) = 0$ if $I \subseteq J$, hence, for a given $I \in \mathcal{D}$ one has

$$P_I(\Delta_b(f)) = P_I \Delta_b(P_I f).$$

Hence

$$\|P_I(\Delta_b(f))\|_{L^2(\mathbb{R})} \leq \|\Delta_b(P_I f)\|_{L^2(\mathbb{R})} \leq \|\Delta_b\| \|(P_I f)\|_{L^2(\mathbb{R})} \leq \|\Delta_b\| \|f\|_{BMO^d} |I|^{1/2}.$$

This gives that $\|\Delta_b(f)\|_{BMO^d} \leq \|\Delta_b\| \|f\|_{BMO^d}$. □

The reader should notice that this last argument gives a proof of the fact $b \in BMO^d(\mathbb{R})$ implies $S_b \in BMO^d(\mathbb{R})$ with no use of John-Nirenberg's lemma. This idea was important in the bidisc where the equivalence between the norms for different values of p was not at our disposal (see [**BP1**]).

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