NON INTERPOLATION IN MORREY–CAMPANATO AND BLOCK SPACES

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Abstract. We prove non interpolation results for the family of Morrey spaces. We introduce a scale of block spaces, which are preduals of Morrey spaces in some range. Negative interpolation results are also obtained in this case.

1. Introduction

The spaces $\mathcal{L}^{p,\alpha}$, for the range $\alpha \in (0, n/p]$ and $p \in [1, \infty]$, were introduced by Morrey in order to study regularity questions which appear in the Calculus of Variations, later Campanato extended the definition to the range $\alpha \in (-1, n/p]$.

$\mathcal{L}^{p,\alpha}$ is defined as the set of functions f locally in $L^p(\mathbf{R}^n)$ and such that there exists a constant σ for which

$$\sup_Q r^\alpha \left(r^{-n} \int_Q |f(x) - \sigma|^p dx \right)^{1/p} < \infty, \tag{1.1}$$

where the sup is taken over all the cubes in $\mathbf{R}^n$ and r denotes the side length. The norm $\|.\|_{p,\alpha}$ is defined as the infimum of (1,1) when $\sigma \in \mathbf{R}$.

In the range defined by Morrey, functions in this space have been used as weights, to substitute the Lebesgue spaces L^p by weighted-L^2, in Sobolev-Poincare inequalities, unique continuation, potentials in wave and Schrödinger equations and some other problems in PDE, see [CS], [ChR], [FP], [Sc], [T], [W]. There are still some interesting open problems, for example in unique continuation and in the restriction properties of the Fourier transform, see [K], [RV1], [RV2]. In this range we can, without loss of generality, take $\sigma = 0$, the endpoint case $p = \frac{n}{\alpha}$ is just L^p, being in the other case, $p < n/\alpha$, L^p strictly included in $\mathcal{L}^{p,\alpha}$.

When $\alpha < 0$ (Campanato's extended range) it has been proved that $\mathcal{L}^{p,\alpha}$ is the space of $(-\alpha)$- Hölder continuous functions , see [C] and [M]. When $\alpha = 0$ we have BMO.

In this work we reduce ourselves to the range $\alpha \in (0, n/p]$, $p \in (1, \infty]$. Therefore (see, for instance [Ku]) we have $\mathcal{L}^{p,\alpha}$ is the set of functions f locally in $L^p(\mathbf{R}^n)$ and such that

$$\|f\|_{p,\alpha} = \sup_Q r^\alpha \left(r^{-n} \int_Q |f(x)|^p dx \right)^{1/p} < \infty, \tag{1.2}$$

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where the sup is taken over all the cubes in $\mathbf{R}^n$ and r denotes the side length. Our concerns are duality and interpolation properties of this two-parameters family of spaces. A few more historical comments are in order.

Interpolation properties of $\mathcal{L}^{p,\alpha}$ were the objects of attention in several works during the 60's. Stampacchia [St], and Campanato and Murthy [CM] proved that if T is a linear operator bounded from L^{q_i} to $\mathcal{L}^{p_i,\alpha_i}$, $i=1,2$, with operator norm K_i, then T is bounded from L^{q_θ} to $\mathcal{L}^{p_\theta,\alpha_\theta}$ with norm at most $CK_1^{1-\theta}K_2^{\theta}$, where $1/p_\theta = (1-\theta)/p_1 + \theta/p_2$, $1/q_\theta = (1-\theta)/q_1 + \theta/q_2$, $\alpha_\theta = (1-\theta)/\alpha_1 + \theta/\alpha_2$ and C only depends on θ, α_i, p_i and q_i. A similar property was proved by Peetre, see [P], for a extended family of spaces. Actually as J. Peetre points out -see [P], pg. 77, any interpolation theorem will do, in the sense that one can replace (L^{p_0}, L^{p_1}) by an abstract pair (A_0, A_1), and L^p by an abstract interpolation space A constructed from (A_0, A_1) and still have and inequality as before. In particular this gives us that $\mathcal{L}^{p_\theta,\alpha_\theta}$ contains the corresponding interpolated space. The main purpose of this paper is to prove that the other contained does not hold. In the range $\alpha \in (-1, n/p]$, this was proved by Stein and Zygmund [SteZ] by constructing a linear operator bounded from $(-\alpha)$-Hölder continuous functions to $(-\alpha)$-Hölder continuous functions and from L^2 to L^2 which is not bounded from BMO to BMO.

Recently two of the authors, see [RV3], obtained negative results on interpolation properties in the Morrey range. To be precise, the lack of convexity which characterises interpolation functors of exponent θ, see [BL, page 27], is proved. In that work they need the dimension $n > 1$.

In the present work we go further and give examples, in the one dimensional case, of operators which are bounded from $\mathcal{L}^{p_i,\alpha} \to L^{q_i}$, $i = 1, 2$, $0 < \alpha \leq n/p$,$p \in (0, \infty)$ and are not bounded from the intermediate $\mathcal{L}^{p_\theta,\alpha}$ to L^{q_θ}.

We also give a description of the predual spaces of $\mathcal{L}^{p,\alpha}$ in the context of "block spaces"(see [SOS] and [So] for this terminology in other situations). We say that a measurable function b is a (q, β)-block if it is supported in a cube Q of lenghtside r in such a way that

$$\left(\frac{1}{|Q|}\int_Q |b(x)|^q dx\right)^{1/q} \leq \frac{1}{r^\beta}. \tag{1.3}$$

Some identification of the preduals of Morrey spaces has been already obtained -see [Z], and [A]. But our arguments are a bit simpler since here the decompositions are into blocks while there she uses atoms, that is mean zero blocks. This fact is, of course, related to the two possible definitons of Morrey spaces mentioned above.

In section 2 we prove that the predual of $\mathcal{L}^{p,\alpha}$, when $0 < \alpha < n/p$ is the space

$$\mathcal{B}_{q,\beta} = \big\{\sum_{k=1}^{\infty} \lambda_k b_k : \sum |\lambda_k| < \infty \text{ and } b_k \text{ is a } (q,\beta)\text{-block}\big\} \tag{1.4}$$

for $\beta = n - \alpha$ and $1/p + 1/q = 1$.

Meanwhile the spaces defined by (1.2) reduce to $\{0\}$ when $\alpha > n/p$, the definition of $\mathcal{B}_{q,\beta}$ gives non trivial spaces in the corresponding range, beyond the preduality exponents, $\beta > n/q$. So it makes sense to study interpolation properties in this range; in section 3 we also give a negative result in this context which is not covered by preduality and which is a complement to the questions posed in the 60's.

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2. Preduals

Definitions 1. *Let $1 \leq q < \infty, 0 < \beta$. We say that a measurable function b is a (q,β)-block if it is supported in a cube Q of lenghtside r in such a way that*

$$\left(\frac{1}{|Q|}\int_Q |b(x)|^q dx\right)^{1/q} \leq \frac{1}{r^\beta}.$$

Let us know consider the space $\mathcal{B}_{q,\beta}$ of measuble functions f such that can be written as

$$f = \sum_{k=1}^{\infty} \lambda_k b_k \qquad a.e.$$

where $\sum |\lambda_k| < \infty$ and b_k are (q,β)-blocks.

Let us define

$$\|f\|_{\mathcal{B}_{q,\beta}} = \inf\{\sum |\lambda_k| \text{ such that} f = \sum \lambda_k b_k\} \tag{2.1}$$

where the infimum is taken over all possible decompositions of f into (q,β)-blocks.

Let us start with some elementary properties of block spaces.

Lemma 1.
(a) $\mathcal{B}_{q,\beta} \subset L^{n/\beta}$ if $n/\beta \leq q$.
(b) $L^q \subset \mathcal{B}_{q,\beta}$ if $q \leq n/\beta$.
(c) For any cube Q and $f \in L^q_{loc}$ we have

$$\|\chi_Q f\|_{\mathcal{B}_{q,\beta}} \leq |Q|^{\beta/n-1/q}\|\chi_Q f\|_q.$$

Proof:.
(a) follows from Hölder and Minkowsky inequalities and the condition $\sum |\lambda_k| < \infty$.
(b) Denote by $Q_k = \{x : |x_i| \leq 2^k, i = 1, ..., n\}$. Assume first that $q < n/\beta$. Then

$$f = \sum_1^{\infty} |Q_k|^{\beta/n-1/q}\|f\|_q b_k,$$

where $b_k = \frac{f(\chi_{Q_k}-\chi_{Q_{k-1}})}{|Q_k|^{\beta/n-1/q}\|f\|_q}$ is a (q,β)-block.

Assume now that $q = n/\beta$ and $f \in L^q$. Since $f\chi_{Q_k}$ is a Cauchy sequence in L^q then we can find n_k such that $\|f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}\|_q < 2^{-k}$. Now write

$$f = f\chi_{Q_{n_1}} + \sum_{k=1}^{\infty} f\chi_{Q_{n_{k+1}}} - f\chi_{Q_{n_k}}.$$

This clearly shows that $f \in \mathcal{B}_{q,\beta}$ and $\|f\|_{\mathcal{B}_{q,\beta}} \leq 2\|f\|_q$.

(c) Just observe that

$$(f\chi_Q)(x) = |Q|^{\beta/n-1/q}\|\chi_Q f\|_q b$$

with b a (q,β)-block.

Remarks
1. $\mathcal{B}_{q,\beta} = L^q$ for $\beta = n/q$. Then its dual is $L^p = \mathcal{L}^{p,\alpha}$ for $\alpha = n/p$.
2. As we observe in the introduction $\mathcal{B}_{q,\beta}$ is a meaningful space in the case (b) of lemma 1 and it contains L^q.

Theorem 1. *Let $1 < p < \infty$ and $\alpha \in (0, n/p)$, then if we take β and q such that $\alpha + \beta = n$ and $1/p + 1/q = 1$ we have*

$$(\mathcal{B}_{q,\beta})^* = \mathcal{L}^{p,\alpha}.$$

Proof. Assume $f \in \mathcal{L}^{p,\alpha}$ and take b a (q,β)-block supported in a cube Q of side r, then

$$\begin{aligned}\int_{\mathbf{R}^n} |fb| &\le (\int_Q |f|^p)^{1/p} (\int_Q |b|^q)^{1/q} \\ &= r^{\alpha} (\frac{1}{|Q|} \int_Q |f|^p)^{1/p} r^{\beta} (\frac{1}{|Q|} \int_Q |b|^q)^{1/q} \le \|f\|_{p,\alpha}.\end{aligned}$$

Take now $g = \sum \lambda_k b_k$, then

$$\int_{\mathbf{R}^n} |fg| \le \sum |\lambda_k| \int_{\mathbf{R}^n} |f b_k| \le \sum |\lambda_k| \|f\|_{p,\alpha} \le \|g\|_{\mathcal{B}_{q,\beta}} \|f\|_{p,\alpha}.$$

This proves that if $f \in \mathcal{L}^{p,\alpha}$ then $\Phi(g) = \int fg \in (\mathcal{B}_{q,\beta})^*$. Then $\mathcal{L}^{p,\alpha} \subset (\mathcal{B}_{q,\beta})^*$.

To prove the other inclusion take $\Phi \in (\mathcal{B}_{q,\beta})^*$ and a cube Q, from (c) of Lemma 1 we have that Φ restricted to the subset $L^q(Q)$ is in $L^q(Q)^*$, and hence there exists a $f_Q \in L^p(Q)$ such that

$$\int f_Q g = \Phi(g) \text{ for any } g \in L^q(Q).$$

Write $\mathbf{R}^n = \cup_1^\infty Q_k$, Q_k increasing, define $f(x) = f_{Q_k}(x)$ if $x \in Q_k$, which makes sense since $\int_E f_{Q_k} = \int_E f_{Q_{k+1}}$ for any Borel subset of Q_k and hence $f_{Q_k}(x) = f_{Q_{k+1}}(x)$ a.e. $x \in Q_k$.

Only remains to prove that $f \in \mathcal{L}^{p,\alpha}$. Take a cube Q and j such that $Q \subset Q_j$, then

$$\begin{aligned}|Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f|^p \right)^{1/p} &= |Q|^{\alpha/n - 1/p} \sup_{\|h\|_q = 1} \int_Q f h dx \\ &\le \sup_{\|h\|_q = 1} \int_{Q_j} f_{Q_j} (h \chi_Q |Q|^{\alpha/n - 1/p}) dx \\ &\le \|\Phi\| \|h \chi_Q |Q|^{\alpha/n - 1/p}\|_{\mathcal{B}_{q,\beta}} \le \|\Phi\|,\end{aligned}$$

since $h\chi_Q |Q|^{\alpha/n-1/p} = h\chi_Q |Q|^{\beta/n-1/q}$ is a (q,β)-block.

3. Interpolation

In [RV3] it was proved the lack of logarithmic convexity of the operator norm of an operator bounded from $\mathcal{L}^{p_i,\alpha}$ to L^1 , $i = 1, 2$ with $1 \le p_2 \le \frac{n-1}{2} \le p_1 < \infty$, $0 < \alpha < n$. This operator requires the dimension > 1. We start by exhibiting an example in dimension 1 of non-boundedness in an intermediate space. Similar examples can be contructed in higher dimension.

Theorem 2. *Take p_1, p_2, and p_3 such that $1 < p_2 < p_3 < p_1 \le 1, \alpha = \frac{1}{p_1}$. Then there exists $q_1, q_2 \in (1, \infty)$ and a linear operator T such that, we have*

$$T : \mathcal{L}^{p_i,\alpha} \to L^{q_i}, i = 1, 2, \tag{3.1}$$

and

$$T : \mathcal{L}^{p_3,\alpha} \not\to L^{q_3}, \tag{3.2}$$

where $\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{1}{q_3} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Lemma 2. *Let $0 < p_3, \alpha, \beta$ be positive numbers such that*

$$max\{p_3, 1\} \le \frac{\beta}{(\beta+1)\alpha} \tag{3.3}$$

and let N_0 such that $(\beta+1) < \frac{N_0}{\log N_0}$. Define

$$I_j^N = [2^N + jN^\beta, 2^N + jN^\beta + 1]$$

for $N > N_0$, $N \in \mathbf{N}$,and $j = 0, 1, ..., N-1$. Then

$$\| \sum_{N > N_0} \sum_{j=0}^{N-1} \chi_{I_j^N} \|_{p_3,\alpha} \le C,$$

where C is a universal constant.

Proof. By inspection one can see that the biggest value of $|I|^\alpha \left(\frac{1}{|I|}|I \cap (\cup I_j^N)|\right)^{\frac{1}{p_3}}$ is achieved when the inf I is at a point 2^N and $N^{\beta+1} \approx |I|$.

Proof of theorem 2. Choose β and q_1 such that

$$p_3 < \frac{\beta}{\alpha(\beta+1)} < p_1, \tag{3.4}$$

$$\frac{2}{q_1} = min\{\frac{2}{p_2} + \alpha(1+\beta) - \frac{\beta}{p_1}, 2\}, \tag{3.5}$$

and $q_2 = p_2$.

Hence $q_1 < q_2 = p_2$.

Define $E_N = \cup_{j=0}^{N-1} I_j^N$ with I_j^N as in lemma 2 and

$$Tf(x) = \sum_{N > N_0} \lambda_N \chi_{E_N}(x) f(x), \tag{3.6}$$

with $\lambda_N = \frac{1}{N^\gamma}$, γ such that

$$\frac{2}{p_2} < \gamma < \frac{2}{q_3} = \frac{2(1-\theta)}{q_1} + \frac{2\theta}{q_2}. \tag{3.7}$$

Notice that from (3.5) $\frac{1}{q_3} > \frac{1}{p_2}$. Hence we trivially have

$$\begin{aligned} \|Tf\|_{q_1} &= \left(\sum_N \lambda_N^{q_1} \int_{E_N} |f|^{q_1}\right)^{\frac{1}{q_1}} \\ &\le \left(\sum_N \lambda_N^{q_1} N^{q_1(\frac{1}{q_1}-\frac{1}{p_1})} \|f\|_{L^{p_1}(E_N)}^{q_1}\right)^{\frac{1}{q_1}} \\ &\le \left(\sum_N \lambda_N^{q_1} N^{1-q_1(\alpha(1+\beta)-\frac{\beta}{p_1})}\right)^{\frac{1}{q_1}} \|f\|_{p_1,\alpha}, \end{aligned} \tag{3.8}$$

since

$$\|f\|_{L^{p_1}(E_N)} \le N^{\frac{(\beta+1)}{p_1}-\alpha(\beta+1)}\|f\|_{p_1,\alpha}.$$

Then $\|Tf\|_{q_1} \le C\|f\|_{p_1,\alpha}$, follows from (3.5) and (3.7).

We have from (3.7) that $1-\gamma q_2 < -1$, and for $q_2 = p_2$:

$$\begin{aligned}\|Tf\|_{q_2} &\le (\sum_N \lambda_N^{q_2} \sum_{j=1}^{N-1} \|f\|^{p_2}_{L^{p_2}(I_j^N)})^{1/p_2}\\ &\le (\sum_N \lambda_N^{q_2} N)^{\frac{1}{q_2}} \|f\|_{p_2,\alpha} \le C\|f\|_{p_2,\alpha}.\end{aligned}$$

On the other hand we know from lemma 2 that $f = \sum_N \chi_{E_N} \in \mathcal{L}^{p_3,\alpha}$ and from (3.7) $1-\gamma q_3 > -1$, hence

$$\|Tf\|_{q_3} = (\sum_N \lambda_N^{q_3} N)^{\frac{1}{q_3}} = \left(\sum N^{1-\gamma q_3}\right)^{\frac{1}{q_3}} = \infty.$$

The proof is over.

The next theorem states that interpolation for the blocks spaces $\mathcal{B}_{q,\beta}$ does not hold between points at both sides of the line $q = \frac{1}{\beta}$. In fact we give an operator bounded $L^{p_i} \to \mathcal{B}_{q_i,\beta}$, $i = 1, 2$ which is not bounded from $L^{p_\theta} \to \mathcal{B}_{q_\theta,\beta}$, and such that $\mathcal{B}_{q_2,\beta}$ is on the predual range and $\mathcal{B}_{q_1,\beta}$ is out of it .

Theorem 3. *Let $1 \le q_1 < q_2 \le 2$. There exist $\beta \in (\frac{1}{q_2}, \frac{1}{q_1}), \theta \in (0,1), p_1, p_2$ and a linear operator T such that we have*

$$T : L^{p_1} \to \mathcal{B}_{q_1,\beta} \tag{3.9}$$

$$T : L^{p_2} \to \mathcal{B}_{q_2,\beta} \tag{3.10}$$

and

$$T : L^{p_\theta} \nrightarrow \mathcal{B}_{q_\theta,\beta} \tag{3.11}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ and $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

In the proof of theorem 3 the following lemma will be used:

Lemma 3. *Let $\{E_k\} \subset \mathbf{R}$ and let $B_k = co(E_k)$ denotes its convex hull, let $q_1 < p_1$, $\beta > 0$ and $f \in L^{p_1}(\mathbf{R})$ and $\|f\|_{p_1} = 1$. If $\{B_k\}$ are disjoint, and*

$$\lambda_k |E_k|^{\frac{1}{q_1}-\frac{1}{p_1}} |B_k|^{\beta-\frac{1}{q_1}} \in l^1, \tag{3.12}$$

then $\sum \lambda_k f \chi_{E_k} \in \mathcal{B}_{q_1,\beta}$

Proof. Just observe that from Hölder

$|E_k|^{\frac{1}{p_1}-\frac{1}{q_1}} |B_k|^{\frac{1}{q_1}-\beta} f\chi_{E_k}$ is a (q_1,β)-block.

Proof of theorem 3 :.

Observe that $\frac{1}{q_2'} < \frac{1}{q_1}$. Take $p_2 = q_2$ and $p_2 < p_1$ such that

$$\frac{1}{q_2'} < \frac{1}{q_1} - \frac{1}{p_1}. \tag{3.13}$$

Now choose θ such that

$$\frac{1}{q_2'} < (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1}), \tag{3.14}$$

and now β such that

$$\frac{1}{q_\theta} < \beta < \frac{1}{q_1} \tag{3.15}$$

where $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Consider now $E_k = [2^k, 2^k+1) \cup [2^{k+1}-1, 2^{k+1})$ and define

$$T(f) = (\sum_{k=1}^{\infty} k^{-\gamma}\chi_{E_k})f$$

for γ chosen such that

$$\frac{1}{p_2'} = \frac{1}{q_2'} < \gamma < (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1}). \tag{3.16}$$

Since $\beta < \frac{1}{q_1}$, $|E_k| = 2$ and $|B_k| = 2^k$ then Lemma 3 implies (3.9).

Now from condition (3.15) we have that $\frac{1}{q_2} < \beta$ what allows us to use Theorem 1 and write (remember that $q_2 = p_2$) for $\alpha = 1-\beta$:

$$\begin{aligned}
\|\sum k^{-\gamma} f\chi_{E_k}\|_{\mathcal{B}_{q_2,\beta}} &= \sup_{\|g\|_{q'_2,\alpha}\le 1} \int (\sum k^{-\gamma} f\chi_{E_k})g \\
&= \sup_{\|g\|_{p'_2,\alpha}\le 1} \int (\sum k^{-\gamma} g\chi_{E_k})f \\
&\le \|f\|_{p_2} \sup_{\|g\|_{p'_2,\alpha}\le 1} \|\sum k^{-\gamma} g\chi_{E_k}\|_{p_2'} \\
&= \|f\|_{p_2} \sup_{\|g\|_{p'_2,\alpha}\le 1} \left(\sum k^{-\gamma p'_2} \int_{E_k} |g|^{p'_2}\right)^{\frac{1}{p_2'}} \\
&\le \|f\|_{p_2} 2^{\frac{1}{p_2'}} \left(\sum k^{-\gamma p'_2}\right)^{\frac{1}{p_2'}}
\end{aligned}$$

The last inequality follows from the fact that $\int_{E_k} |g|^{p'_2} \le 2\|g\|^{p_2'}_{p_2',\alpha}$. The above series converges from (3.16). This proves (3.10).

Finally, since $\frac{1}{q_\theta} < \beta$, then (3.11) is equivalent to see that $T^* = T$ is not bounded from $\mathcal{L}^{q_\theta',1-\beta}$ to $L^{p_\theta'}$.

Take now $A_N = \cup_{k=1}^N E_k$ and $f_N = |A_N|^{-\frac{1}{q_\theta'}}\chi_{A_N}$.

It is elementary to see that $||f_N||_{q_\theta',1-\beta} \le 1$.

On the other hand

$$||T(f_N)||_{p_\theta'} = 2^{\frac{1}{p_\theta'}}(2N)^{-\frac{1}{q_\theta'}}(\sum_{k=1}^N k^{-\gamma p_\theta'})^{\frac{1}{p_\theta'}}.$$

Note that

$$\frac{1}{q_\theta} - \frac{1}{p_\theta} = (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1}) \tag{3.17}$$

and then (3.16) gives that $\gamma p'_\theta < 1$ what allows to write

$$\sum_{k=1}^{N} k^{-\gamma p'_\theta} \geq C N^{-\gamma p'_\theta + 1}. \tag{3.18}$$

Using (3.17) and (3.18) we get that

$$||T(f_N)||_{p'_\theta} \geq C N^{-\gamma + (1-\theta)(\frac{1}{q_1} - \frac{1}{p_1})}.$$

Now (3.16) gives that $sup_N ||T(f_N)||_{p'_\theta} = \infty$ and the proof is completed.

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