EMBEDDINGS BETWEEN OPERATOR-VALUED DYADIC BMO SPACES

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Abstract. We investigate a scale of dyadic operator-valued BMO spaces, corresponding to the different yet equivalent characterizations of dyadic BMO in the scalar case. In the language of operator spaces, we investigate different operator space structures on the scalar dyadic BMO space which arise naturally from the different characterisations of scalar BMO. We also give sharp dimensional growth estimates for the sweep of functions and its bilinear extension in some of those different dyadic BMO spaces.

1. Introduction

Let $\mathcal{D}$ denote the collection of dyadic subintervals of the unit circle $\mathbb{T}$, and let $(h_I)_{I\in\mathcal{D}}$, where $h_I = \frac{1}{|I|^{1/2}}(\chi_{I^+} - \chi_{I^-})$, be the Haar basis of $L^2(\mathbb{T})$. For $I \in \mathcal{D}$ and $\phi \in L^2(\mathbb{T})$, let ϕ_I denote the formal Haar coefficient $\int_I \phi(t)h_I dt$, and $m_I\phi = \frac{1}{|I|}\int_I \phi(t)dt$ denote the average of ϕ over I. We write $P_I(\phi) = \sum_{J\subseteq I} \phi_J h_J$.

We say that $\phi \in L^2(\mathbb{T})$ belongs to dyadic BMO, written $\phi \in \mathrm{BMO^d}(\mathbb{T})$, if

$$\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I |\phi(t) - m_I\phi|^2 dt\Big)^{1/2} < \infty. \tag{1}$$

Using the identity $P_I(\phi) = (\phi - m_I\phi)\chi_I$, this can also be written as

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/2}}\|P_I(\phi)\|_{L^2} < \infty, \tag{2}$$

or

$$\sup_{I\in\mathcal{D}} \frac{1}{|I|}\sum_{J\in\mathcal{D}, J\subseteq I} |\phi_J|^2 < \infty. \tag{3}$$

Due to John-Nirenberg's lemma, one can replace the $L^2(\mathbb{T})$ norm in (1) and (2) by any L^p-norm. That is, for $0 < p < \infty$, we have $\phi \in \mathrm{BMO^d}(\mathbb{T})$ if and only if

$$\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I |\phi(t) - m_I\phi|^p dt\Big)^{1/p} = \sup_{I\in\mathcal{D}} \frac{1}{|I|^{1/p}}\|P_I(\phi)\|_{L^p} < \infty. \tag{4}$$

It is well-known that the space $\mathrm{BMO^d}(\mathbb{T})$ has the following equivalent formulation in terms of boundedness of dyadic paraproducts: The map

$$\pi_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} \phi_I(m_I f)h_I \tag{5}$$

defines a bounded linear operator on $L^2(\mathbb{T})$, if and only if $\phi \in \mathrm{BMO^d}(\mathbb{T})$.

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For real-valued functions, we can also replace the boundedness of the dyadic paraproduct π_ϕ by the boundedness of its adjoint operator

$$\Delta_\phi : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} \phi_I f_I \frac{\chi_I}{|I|}. \tag{6}$$

Another equivalent formulation comes from the duality

$$\mathrm{BMO}^{\mathrm{d}}(\mathbb{T}) = (H^1_d(\mathbb{T}))^*, \tag{7}$$

where the dyadic Hardy space $H^1_d(\mathbb{T})$ consists of those functions $\phi \in L^1(\mathbb{T})$ for which the dyadic square function $\mathcal{S}\phi = (\sum_{I\in\mathcal{D}} |\phi_I|^2 \frac{\chi_I}{|I|})^{1/2}$ is also in $L^1(\mathbb{T})$. Let us recall that $H^1_d(\mathbb{T})$ can also be described in terms of dyadic atoms. That is, $H^1_d(\mathbb{T})$ consists of functions $\phi = \sum_{k\in\mathbb{N}} \lambda_k a_k, \lambda_k \in \mathbb{C}$, $\sum_{k\in\mathbb{N}} |\lambda_k| < \infty$, where the a_k are dyadic atoms, i.e. $\mathrm{supp}(a_k) \subset I_k$ for some $I_k \in \mathcal{D}$, $\int_{I_k} a_k(t)dt = 0$, and $\|a_k\|_\infty \le \frac{1}{|I_k|}$. The reader is referred to [M] or to [G] for standard results about H^1_d and $\mathrm{BMO}^{\mathrm{d}}$.

Let

$$S_\phi = (\mathcal{S}\phi)^2 = \sum_{I\in\mathcal{D}} |\phi_I|^2 \frac{\chi_I}{|I|}$$

denote the sweep of the function ϕ. Using John-Nirenberg's lemma, one easily verifies the well-known fact that

$$\phi \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{T}) \text{ if and only if } S_\phi \in \mathrm{BMO}^{\mathrm{d}}(\mathbb{T}). \tag{8}$$

The reader is referred to [B3] for a proof of (8) independent of John-Nirenberg's lemma.

The aim of this paper is twofold. Firstly, it is to investigate the spaces of operator-valued BMO functions corresponding to characterizations (1)-(7). In the operator-valued case, these characterizations are in general no longer equivalent. In the language of operator spaces, we investigate the different operator space structures on the scalar space $\mathrm{BMO}^{\mathrm{d}}$ which arise naturally from the different yet equivalent characterisations of $\mathrm{BMO}^{\mathrm{d}}$. The reader is referred to [B4, BP1, BP2, PSm] for some recent results on dyadic BMO and Besov spaces connected to the ones in this paper. The second aim is to give sharp dimensional estimate for the operator sweep and its bilinear extension, of which more will be said below, in these operator BMO^d norms.

We require some further notation for the operator-valued case. Let $\mathcal{H}$ be a separable, finite or infinite-dimensional Hilbert space. Let $\mathcal{F}_{00}$ denote the subspace of $\mathcal{L}(\mathcal{H})$-valued functions on $\mathbb{T}$ with finite formal Haar expansion. Given $e, f \in \mathcal{H}$ and $B \in L^2(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ we denote by B_e the function in $L^2(\mathbb{T}, \mathcal{H})$ defined by $B_e(t) = B(t)(e)$ and by $B_{e,f}$ the function in $L^2(\mathbb{T})$ defined by $B_{e,f}(t) = \langle B(t)(e), f\rangle$. As in the scalar case, let B_I denote the formal Haar coefficients $\int_I B(t)h_I dt$, and $m_I B = \frac{1}{|I|}\int_I B(t)dt$ denote the average of B over I for any $I \in \mathcal{D}$. Observe that for B_I and $m_I B$ to be well-defined operators, we shall be assuming that the $\mathcal{L}(\mathcal{H})$- valued function B is *weak**-integrable. That means, using the duality $\mathcal{L}(\mathcal{H}) = (\mathcal{H}\hat{\otimes}\mathcal{H})^*$, that $\langle B(\cdot)(e), f\rangle \in L^1(\mathbb{T})$ for $e, f \in \mathcal{H}$ and for any measurable set A, there exist $B_A \in \mathcal{L}(\mathcal{H})$ such that $\langle B_A(e), f\rangle = \langle \int_A B(t)(e)dt, f\rangle$ for $e, f \in \mathcal{H}$.

We denote by $\mathrm{BMO^d}(\mathbb{T},\mathcal{H})$ the space of Bochner integrable $\mathcal{H}$-valued functions $b:\mathbb{T}\to\mathcal{H}$ such that

$$\|b\|_{\mathrm{BMO^d}}=\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I\|b(t)-m_Ib\|^2dt\Big)^{1/2}<\infty \tag{9}$$

and by $\mathrm{wBMO^d}(\mathbb{T},\mathcal{H})$ the space of Pettis integrable $\mathcal{H}$-valued functions $b:\mathbb{T}\to\mathcal{H}$ such that

$$\|b\|_{\mathrm{wBMO^d}}=\sup_{I\in\mathcal{D},e\in\mathcal{H},\|e\|=1}\Big(\frac{1}{|I|}\int_I|\langle b(t)-m_Ib,e\rangle|^2dt\Big)^{1/2}<\infty. \tag{10}$$

In the operator-valued case we define the following notions corresponding to the previous formulations: We denote by $\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of Bochner integrable $\mathcal{L}(\mathcal{H})$-valued functions B such that

$$\|B\|_{\mathrm{BMO^d_{norm}}}=\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\int_I\|B(t)-m_IB\|^2dt\Big)^{1/2}<\infty, \tag{11}$$

by $\mathrm{SBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of $\mathcal{L}(\mathcal{H})$-valued functions B such that $B_e\in\mathrm{BMO^d}(\mathbb{T},\mathcal{H})$ for all $e\in\mathcal{H}$ and

$$\|B\|_{\mathrm{SBMO^d}}=\sup_{I\in\mathcal{D},e\in\mathcal{H},\|e\|=1}\Big(\frac{1}{|I|}\int_I\|(B(t)-m_IB)e\|^2dt\Big)^{1/2}<\infty, \tag{12}$$

and, finally, by $\mathrm{WBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of *weak**-integrable $\mathcal{L}(\mathcal{H})$-valued functions B such that $B_{e,f}\in\mathrm{BMO^d}$ for all $e,f\in\mathcal{H}$ and

$$\|B\|_{\mathrm{WBMO^d}}=\sup_{I\in\mathcal{D},\|e\|=\|f\|=1}\Big(\frac{1}{|I|}\int_I|\langle(B(t)-m_IB)e,f\rangle|^2dt\Big)^{1/2}<\infty, \tag{13}$$

or, equivalently, such that

$$\|B\|_{\mathrm{WBMO^d}}=\sup_{e\in\mathcal{H},\|e\|=1}\|B_e\|_{\mathrm{wBMO^d}(\mathbb{T},\mathcal{H})}=\sup_{A\in S_1,\|A\|_1\le 1}\|\langle B,A\rangle\|_{\mathrm{BMO^d}(\mathbb{T})}<\infty.$$

Here, S_1 denotes the ideal of trace class operators in $\mathcal{L}(\mathcal{H})$, and $\langle B,A\rangle$ stands for the scalar-valued function given by $\langle B,A\rangle(t)=\mathrm{trace}(B(t)A^*)$.

The space $\mathrm{BMO^d_{Carl}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ is the space of *weak**-integrable operator-valued functions for which

$$\|B\|_{\mathrm{BMO^d_{Carl}}}=\sup_{I\in\mathcal{D}}\Big(\frac{1}{|I|}\sum_{J\in\mathcal{D},J\subseteq I}\|B_J\|^2\Big)^{1/2}<\infty. \tag{14}$$

We would like to point out that while B belongs to one of the spaces $\mathrm{BMO^d_{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})),\mathrm{WBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H})))$ or $B\in\mathrm{BMO^d_{Carl}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ if and only if B^* does, this is not the case for the space $\mathrm{SBMO}^d(\mathbb{T},\mathcal{L}(\mathcal{H}))$. This leads to the following notion (see [GPTV, Pet, PXu]): We say that $B\in\mathrm{BMO^d_{so}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$, if B and B^* belong to $\mathrm{SBMO^d}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. We define

$$\|B\|_{\mathrm{BMO^d_{so}}}=\|B\|_{\mathrm{SBMO^d}}+\|B^*\|_{\mathrm{SBMO^d}}. \tag{15}$$

We now define another operator-valued BMO space, using the notion of Haar multipliers.

A sequence $(\Phi_I)_{I\in\mathcal{D}}$, $\Phi_I\in L^2(I,\mathcal{L}(\mathcal{H}))$ for all $I\in\mathcal{D}$, is said to be an *operator-valued Haar multiplier* (see [Per, BP1]), if there exists $C>0$ such that

$$\|\sum_{I\in\mathcal{D}}\Phi_I(f_I)h_I\|_{L^2(\mathbb{T},\mathcal{H})}\le C\Big(\sum_{I\in\mathcal{D}}\|f_I\|^2\Big)^{1/2}\text{ for all }(f_I)_{I\in\mathcal{D}}\in l^2(\mathcal{D},\mathcal{H}).$$

We write $\|(\Phi_I)\|_{mult}$ for the norm of the corresponding operator on $L^2(\mathbb{T},\mathcal{H})$.

Letting again as in the scalar-valued case $P_I B = \sum_{J\subseteq I} h_J B_J$, we denote the space of those *weak**-integrable $\mathcal{L}(\mathcal{H})$-valued functions for which $(P_I B)_{I\in\mathcal{D}}$ defines a bounded operator-valued Haar multiplier on $L^2(\mathbb{T},\mathcal{H})$ by $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ and write

$$\|B\|_{\mathrm{BMO}_{\mathrm{mult}}} = \|(P_I B)_{I\in\mathcal{D}}\|_{mult}. \tag{16}$$

We shall use the notation $\Lambda_B(f) = \sum_{I\in\mathcal{D}}(P_I B)(f_I)h_I$.

Let us mention that there is a further BMO space, defined in terms of paraproducts, which is very much connected with $\mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ and was studied in detail in [BP2]. Operator-valued paraproducts are of particular interest, because they can be seen as dyadic versions of vector Hankel operators or of vector Carleson embeddings, which are important in the real and complex analysis of matrix valued functions and its applications in the theory of infinite-dimensional linear systems (see e.g. [JPa], [JPaP]).

Let $B \in \mathcal{F}_{00}$. We define the dyadic operator-valued paraproduct with symbol B,

$$\pi_B : L^2(\mathbb{T},\mathcal{H}) \to L^2(\mathbb{T},\mathcal{H}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I(m_I f)h_I,$$

and

$$\Delta_B : L^2(\mathbb{T},\mathcal{H}) \to L^2(\mathbb{T},\mathcal{H}), \quad f = \sum_{I\in\mathcal{D}} f_I h_I \mapsto \sum_{I\in\mathcal{D}} B_I(f_I)\frac{\chi_I}{|I|}.$$

It is easily seen that $(\pi_B)^* = \Delta_{B^*}$.

We denote by $\mathrm{BMO}_{\mathrm{para}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of *weak**-integrable operator-valued functions for which $\|\pi_B\| < \infty$ and write

$$\|B\|_{\mathrm{BMO}_{\mathrm{para}}} = \|\pi_B\|. \tag{17}$$

We refer the reader to [B4, BP2] and [Me1, Me2] for results on this space. It is elementary to see that

$$\Lambda_B(f) = \sum_{I\in\mathcal{D}} B_I(m_I f)h_I + \sum_{I\in\mathcal{D}} B_I(f_I)\frac{\chi_I}{|I|} = \pi_B f + \Delta_B f. \tag{18}$$

Hence $\Lambda_B = \pi_B + \Delta_B$ and $(\Lambda_B)^* = \Lambda_{B^*}$. This shows that $\|B\|_{\mathrm{BMO}_{\mathrm{mult}}} = \|B^*\|_{\mathrm{BMO}_{\mathrm{mult}}}$.

Let us finally denote by $\mathrm{BMO}_{\mathrm{spara}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$ the space of symbols B such that π_B and π_{B^*} are bounded operators, and define

$$\|B\|_{\mathrm{BMO}_{\mathrm{spara}}} = \|\pi_B\| + \|\pi_{B^*}\|. \tag{19}$$

Since $\Delta_B = \pi^*_{B^*}$, one concludes that $\mathrm{BMO}_{\mathrm{spara}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subseteq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$.

We write $\approx$ for equivalence of norms up to a constant (independent of the dimension of the Hilbert space $\mathcal{H}$, if this appears), and similarly $\lesssim, \gtrsim$ for the corresponding one-sided estimates up to a constant.

Recall that for a given Banach space $(X, \|\cdot\|)$, a family of norms $(M_n(X), \|\cdot\|_n)$ on the spaces $M_n(X)$ of X-valued $n\times n$ matrices defines an *operator space structure* on X, if $\|\cdot\|_1 \approx \|\cdot\|$,

(M1) $\|A\oplus B\|_{n+m} \leq \max\{\|A\|_n, \|B\|_m\}$ for $A\in M_n(X)$, $B\in M_m(X)$

(M2) $\|\alpha A\beta\|_m \leq \|\alpha\|_{M_{n,m}(\mathbb{C})}\|A\|_n\|\beta\|_{M_{m,n}(\mathbb{C})}$ for all $A\in M_n(X)$ and all scalar matrices $\alpha\in M_{n,m}(\mathbb{C})$, $\beta\in M_{m,n}(\mathbb{C})$.

(see e. g. [ER]). One verifies easily that all the $\mathrm{BMO}^{\mathrm{d}}$-norms on $\mathcal{L}(\mathcal{H})$-valued functions defined above, except $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ and $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}$, define operator space structures on $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})$ when taken for n-dimensional $\mathcal{H}$, $n \in \mathbb{N}$.

The aim of the paper is to show the following strict inclusions for infinite-dimensional $\mathcal{H}$:

$$\begin{aligned}(20)\qquad \mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \\ \subsetneq \mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO}^{\mathrm{d}}(\mathbb{T},\mathcal{L}(\mathcal{H}))\end{aligned}$$

and

$$(21)\qquad \mathrm{BMO}^{\mathrm{d}}_{\mathrm{Carl}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{spara}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H})).$$

This means that the corresponding inclusions of operator spaces over $\mathrm{BMO}^{\mathrm{d}}(\mathbb{T})$, where they apply, are completely bounded, but not completely isomorphic (for the notation, see again e. g. [ER]). We will also consider the preduals for some of the spaces shown. Finally, we will give sharp estimates for the dimensional growth of the sweep and its bilinear extension on $\mathrm{BMO}_{\mathrm{para}}$, $\mathrm{BMO}_{\mathrm{mult}}$ and $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$, completing results in [BP2] and [Me2].

The paper is organized as follows. In Section 2, we prove the chains of strict inclusions (20) and (21). Actually the only nontrivial inclusion to be shown is $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T},\mathcal{L}(\mathcal{H})) \subset \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T},\mathcal{L}(\mathcal{H}))$. For this purpose, we introduce a new Hardy space H^1_Λ adapted to the problem, and then the result can be shown from an estimate on the dual side. The remaining inclusions are immediate consequences of the definition, and only the counterexamples showing that none of the spaces are equal need to be found.

The reader is referred to [Me1] for more on the theory of operator-valued Hardy spaces.

Section 3 deals with dimensional growth properties of the *operator sweep* and its bilinear extension. We define the operator sweep for $B \in \mathcal{F}_{00}$,

$$S_B = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} B_I^* B_I,$$

and its bilinear extension

$$\Delta[U^*,V] = \sum_{I\in\mathcal{D}} \frac{\chi_I}{|I|} U_I^* V_I \qquad (U,V \in \mathcal{F}_{00}).$$

These maps are of interest for several reasons. They are closely connected with the paraproduct and certain bilinear paraproducts, they provide a tool to understand the dimensional growth in the John-Nirenberg lemma, and they are useful to understand products of paraproducts and products of certain other operators (see [BP2], [PSm]).

Considering (8) in the operator valued case, it was shown in [BP2] that

$$(22)\qquad \|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{mult}}} + \|B\|^2_{\mathrm{SBMO}^{\mathrm{d}}} \approx \|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{para}}}.$$

Here, we prove the bilinear analogue

$$(23)\qquad \|\Delta[U^*,V]\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{mult}}} + \sup_{I\in\mathcal{D}} \frac{1}{|I|} \|\sum_{J\subset I} U_J^* V_J\| \approx \|\pi_U^* \pi_V\|.$$

It was also shown in [BP2] that

$$\|S_B\|_{\mathrm{SBMO^d}} \le C\log(n+1)\|B\|^2_{\mathrm{SBMO^d}} \tag{24}$$

for $dim(\mathcal{H}) = n$, where C is a constant independent of n, and that this estimate is sharp.

We extend this by proving sharp estimates of $\|S_B\|$ and $\|\Delta[U^*, V]\|$ in terms of $\|B\|, \|U\|, \|V\|$ with respect to the norms in SBMO^d, $\mathrm{BMO_{para}}$, $\mathrm{BMO_{mult}}$ and $\mathrm{BMO^d_{norm}}$.

2. Strict inclusions

Let us start by stating the following characterizations of SBMO to be used later on. Some of the equivalences can be found in [GPTV], we give the proof for the convenience of the reader.

Proposition 2.1. *Let $B \in \mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Then*

$$\begin{aligned}
\|B\|^2_{\mathrm{SBMO^d}} &= \sup_{e\in\mathcal{H},\|e\|=1} \|B_e\|^2_{\mathrm{BMO^d}(\mathbb{T},\mathcal{H})} \\
&= \sup_{I\in\mathcal{D},\|e\|=1} \frac{1}{|I|}\|P_I(B_e)\|^2_{L^2(\mathcal{H})} \\
&= \sup_{I\in\mathcal{D}} \frac{1}{|I|}\|\sum_{J\subseteq I} B^*_J B_J\| \\
&= \sup_{I\in\mathcal{D}} \left\| \frac{1}{|I|}\int_I (B(t)-m_IB)^*(B(t)-m_IB)dt \right\| \\
&= \sup_{I\in\mathcal{D}} \|m_I(B^*B) - m_I(B^*)m_I(B)\|.
\end{aligned}$$

Proof. The two first equalities are obvious from the definition. Now observe that

$$\|\sum_{J\subseteq I} B^*_J B_J\| = \sup_{\|e\|=1,\|f\|=1} \sum_{J\subseteq I}\langle B_J(e), B_J(f)\rangle = \sup_{\|e\|=1}\sum_{J\subseteq I}\|B_J(e)\|^2 = \|P_I(B_e)\|^2_{L^2(\mathcal{H})}.$$

The other equalities follow from

$$\begin{aligned}
\|m_I(B^*B) - m_I(B^*)m_I(B)\| &= \left\| \frac{1}{|I|}\int_I (B(t)-m_IB)^*(B(t)-m_IB)dt \right\| \\
&= \sup_{e\in\mathcal{H},\|e\|=1} \frac{1}{|I|}\int_I \langle (B(t)-m_IB)^*(B(t)-m_IB)e, e\rangle dt \\
&= \sup_{e\in\mathcal{H},\|e\|=1} \frac{1}{|I|}\int_I \|P_IBe\|^2 dt.
\end{aligned}$$

□

Lemma 2.2. *Let $B = \sum_{k=1}^N B_k r_k$ where $r_k = \sum_{|I|=2^{-k}} |I|^{1/2}h_I$ denote the Rademacher functions. Then*

$$\|B\|_{\mathrm{SBMO^d}} = \sup_{\|e\|=1} (\sum_{k=1}^N \|B_ke\|^2)^{1/2} \tag{25}$$

$$\|B\|_{\mathrm{BMO_{so}}} = \sup_{\|e\|=1} (\sum_{k=1}^{N} \|B_k e\|^2)^{1/2} + \sup_{\|e\|=1} (\sum_{k=1}^{N} \|B_k^* e\|^2)^{1/2} \tag{26}$$

$$\|B\|_{\mathrm{WBMO}^d} = \sup_{\|f\|=\|e\|=1} (\sum_{k=1}^{N} |\langle B_k e, f\rangle|^2)^{1/2}. \tag{27}$$

Proof. This follows from standard Littlewood-Paley theory. $\square$

For $x, y \in \mathcal{H}$ we denote by $x \otimes y$ the rank 1 operator in $\mathcal{L}(\mathcal{H})$ given by $(x \otimes y)(h) = \langle h, y\rangle x$. Clearly $(x \otimes y)^* = (y \otimes x)$.

Proposition 2.3. *Let* $\dim \mathcal{H} = \infty$. *Then*

$$\mathrm{BMO_{mult}} \subsetneq \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{WBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})).$$

Proof. Note that if $(\Phi_I)_{I\in\mathcal{D}}$ is a Haar multiplier then

$$\sup_{I\in\mathcal{D}, \|e\|=1} |I|^{-1/2} \|\Phi_I(e)\|_{L^2(\mathbb{T},\mathcal{H})} \leq \|(\Phi_I)\|_{mult}. \tag{28}$$

The first inclusion thus follows from (28) and Proposition 2.1. The other inclusions are immediate. Let us see that they are strict. It was shown in [GPTV] that $\mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \neq \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Let (e_k) is an orthonormal basis of $\mathcal{H}$ and $h \in \mathcal{H}$ with $\|h\| = 1$. Hence by (25), $B = \sum_{k=1}^{\infty} h \otimes e_k \ r_k \in \mathrm{SBMO^d}$ and $B^* = \sum_{k=1}^{\infty} e_k \otimes h \ r_k \notin \mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Thus $B \in \mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \setminus \mathrm{BMO^d_{so}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. Similarly by (25) and (27), $B \in \mathrm{WBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \setminus \mathrm{SBMO^d}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. $\square$

Note that

$$\Lambda_B f = Bf - \sum_{I\in\mathcal{D}} (m_I B)(f_I) h_I \tag{29}$$

which allows to conclude immediately that $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Our next objective is to see that $\mathrm{BMO^d_{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. For that, we need again some more notation.

Let S_1 denote the ideal of trace class operators on $\mathcal{H}$ and recall that $S_1 = \mathcal{H}\hat{\otimes}\mathcal{H}$ and $(S_1)^* = \mathcal{L}(\mathcal{H})$ with the pairing $\langle U, (e \otimes d)\rangle = \langle U(e), d\rangle$.

It is easy to see that the space $\mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ can be embedded isometrically into the dual of a certain H^1 space of S_1 valued functions:

Definition 2.4. *Let* $f, g \in L^2(\mathbb{T}, \mathcal{H})$. *Define*

$$f \circledast g = \sum_{I\in\mathcal{D}} h_I (f_I \otimes m_I g + m_I f \otimes g_I).$$

Let $H^1_\Lambda(\mathbb{T}, S_1)$ *be the space of functions* $f = \sum_{k=1}^{\infty} \lambda_k f_k \circledast g_k$ *such that* $f_k, g_k \in L^2(\mathbb{T}, \mathcal{H})$, $\|f_k\|_2 = \|g_k\|_2 = 1$ *for all* $k \in \mathbb{N}$, *and* $\sum_{k=1}^{\infty} |\lambda_k| < \infty$.

We endow the space with the norm given by the infimum of $\sum_{k=1}^{\infty} |\lambda_k|$ *for all possible decompositions.*

With this notation, $B \in \mathrm{BMO_{mult}}$ acts on $f \circledast g$ by

$$\langle B, f \circledast g\rangle = \int_{\mathbb{T}} \langle B(t), (f \circledast g)(t)\rangle dt = \langle \Lambda_B f, g\rangle.$$

By definition of $H^1_\Lambda(\mathbb{T}, S_1)$, $\|B\|_{(H^1_\Lambda(\mathbb{T},S_1))^*} = \|\Lambda_B\|$.

We will now define a further H^1 space of S_1-valued functions. For $F \in L^1(\mathbb{T}, S_1)$, define the dyadic Hardy-Littlewood maximal function F^* of F in the usual way,

$$F^*(t) = \sup_{I \in \mathbb{D}, t \in I} \frac{1}{|I|} \int_I \|F(s)\|_{S_1} ds.$$

Then let $H^1_{\max,\mathrm{d}}(\mathbb{T}, S_1)$ be given by functions $F \in L^1(\mathbb{T}, S_1)$ such that $F^* \in L^1(\mathbb{T})$. By a result of Bourgain ([Bou], Th.12), $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ embeds continuously into $(H^1_{\max,\mathrm{d}}(\mathbb{T}, S_1))^*$ (see also [B1, B2]).

Lemma 2.5. $H^1_\Lambda(\mathbb{T}, S_1) \subseteq H^1_{\max,\mathrm{d}}(\mathbb{T}, S_1)$.

Proof. It is sufficient to show that there is a constant $C > 0$ such that for all $f, g \in L^2(\mathbb{T}, \mathcal{H})$, $f \circledast g \in H^1_{\max,\mathrm{d}}(\mathbb{T}, S_1)$, and $\|f \circledast g\|_{H^1_{\max,\mathrm{d}}(\mathbb{T},S_1)} \leq C\|f\|_2\|g\|_2$. One verifies that

$$f \circledast g = \sum_{I \in \mathcal{D}} h_I(f_I \otimes m_I g + m_I f \otimes g_I) = f \otimes g - \sum_{I \in \mathcal{D}} \frac{\chi_I}{|I|} f_I \otimes g_I.$$

Towards the estimate of the maximal function, let E_k denote the expectation with respect to the σ-algebra generated by dyadic intervals of length 2^{-k},

$$E_k F = \sum_{I \in \mathcal{D}, |I| > 2^{-k}} h_I F_I,$$

for each $k \in \mathbb{N}$. Then we have

$$E_k(f \circledast g) = (E_k f) \circledast (E_k g), \tag{30}$$

as

$$\sum_{I \in \mathcal{D}, |I| > 2^{-k}} h_I(f_I \otimes m_I g + m_I f \otimes g_I) = \sum_{I \in \mathcal{D}} h_I((E_k f)_I \otimes m_I(E_k g) + m_I(E_k f) \otimes (E_k g)_I).$$

Thus

$$\begin{aligned}(f \circledast g)^*(t) = \sup_{k \in \mathbb{N}} \|E_k(f \circledast g)(t)\|_{S_1} &\leq \sup_{k \in \mathbb{N}} \|(E_k f)(t)\| \|(E_k g)(t)\| + \sum_{I \in \mathcal{D}} \frac{\chi_I(t)}{|I|} \|f_I\| \|g_I\| \\ &\leq \|f^*(t)\| \|g^*(t)\| + \sum_{I \in \mathcal{D}} \frac{\chi_I(t)}{|I|} \|f_I\| \|g_I\|,\end{aligned}$$

and

$$\|(f \circledast g)^*\|_1 \leq \|f^*\|_2 \|g^*\|_2 + \|f\|_2 \|g\|_2 \leq C\|f\|_2 \|g\|_2$$

by the Cauchy-Schwarz inequality and boundedness of the dyadic Hardy-Littlewood maximal function on $L^2(\mathbb{T}, \mathcal{H})$. □

In particular, $H^1_\Lambda(\mathbb{T}, S_1) \subseteq L^1(\mathbb{T}, S_1)$.

We can now prove our inclusion result:

Theorem 2.6. $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO}_{\mathrm{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Proof. The inclusion follows by Lemma 2.5, duality and Bourgain's result.

To see that the spaces do not coincide, use the fact that $\mathrm{BMO}^{\mathrm{d}}(\ell_\infty) \subsetneq \ell_\infty(\mathrm{BMO}^{\mathrm{d}})$ to find for each $N \in \mathbb{N}$ functions $b_k \in \mathrm{BMO}$, $k = 1, ..., N$, such that $\sup_{1 \leq k \leq N} \|b_k\|_{\mathrm{BMO}^{\mathrm{d}}} \leq 1$, but $\|(b_k)_{k=1,\ldots,N}\|_{\mathrm{BMO}^d(\mathbb{T}, l^\infty_N)} \geq c_N$, $c_N \to \infty$ as $N \to \infty$.

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and consider the operator-valued function $B(t) = \sum_{k=1}^N b_k(t) e_k \otimes e_k \in L^2(\mathbb{T}, \mathcal{L}(\ell_2))$. Clearly $B_I = \sum_{k=1}^N (b_k)_I e_k \otimes e_k$, and for each $\mathbb{C}^N$-valued function $f = \sum_{k=1}^N f_k e_k$, $f_1, \dots, f_N \in L^2(\mathbb{T})$, we have

$$\Lambda_B(f) = \sum_{k=1}^N \Lambda_{b_k}(f_k) e_k.$$

Choosing the f_k such that $\|f\|_2^2 = \sum_{k=1}^N \|f_k\|^2_{L^2(\mathbb{T})} = 1$, we find that

$$\|\Lambda_B(f)\|^2_{L^2(\mathbb{T},\ell_2)} = \sum_{k=1}^N \|\Lambda_{b_k}(f_k)\|^2_{L^2(\mathbb{T})} \le C \sum_{k=1}^N \|b_k\|^2_{\mathrm{BMO^d}} \|f_k\|^2_{L^2(\mathbb{T})} \le C,$$

where C is a constant independent of N. Therefore, Λ_B is bounded.

But since $\|B\|_{\mathrm{BMO^d_{norm}}} = \|(b_k)_{k=1,\dots,N}\|_{\mathrm{BMO^d}(\mathbb{T}, l^\infty_N)} \ge c_N$, it follows that $\mathrm{BMO_{mult}}(\mathbb{T})$ is not continuously embedded in $\mathrm{BMO^d_{norm}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. From the open mapping theorem, we obtain inequality of the spaces. □

The next proposition shows that the space $\mathrm{BMO^d_{Carl}}$ belongs to a different scale than $\mathrm{BMO^d_{norm}}$ and $\mathrm{BMO_{mult}}$.

Proposition 2.7. $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO^d_{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Proof. This follows from the result $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO_{para}}$ in [Me2] (see Lemma 3.1 below) and next proposition. We give a simple direct argument. Choose an orthonormal basis of $\mathcal{H}$ indexed by the elements of $\mathcal{D}$, say $(e_I)_{I\in\mathcal{D}}$, and let $\Phi_I = e_I \otimes e_I$, $\Phi_I h = \langle h, e_I\rangle e_I$. Let $\lambda_I = |I|^{1/2}$ for $I \in \mathcal{D}$, and define $B = \sum_{I\in\mathcal{D}} h_I \lambda_I \Phi_I$. Then $\sum_{I\in\mathcal{D}} \|B_I\|^2 = \sum_{I\in\mathcal{D}} |I| = \infty$, so in particular $B \notin \mathrm{BMO^d_{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. But the operator function B is diagonal with uniformly bounded diagonal entry functions $\phi_I(t) = \langle B(t)e_I, e_I\rangle = |I|^{1/2} h_I(t)$, so $B \in L^\infty(\mathcal{L}(\mathcal{H}))$. □

Proposition 2.8.

$$\mathrm{BMO^d_{Carl}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO_{spara}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subsetneq \mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H})).$$

Proof. The inclusion $\mathrm{BMO^d_{Carl}} \subseteq \mathrm{BMO_{spara}}$ is easy, since (14) implies that for $B \in \mathrm{BMO^d_{Carl}}$, the $\mathrm{BMO^d_{Carl}}$ norm equals the norm of the scalar $\mathrm{BMO^d}$ function given by $|B| := \sum_{I\in\mathcal{D}} h_I \|B_I\|$. For $f \in L^2(\mathcal{H})$, let $|f|$ denote the function given by $|f|(t) = \|f(t)\|$. Thus

$$\|\pi_B f\|_2^2 = \sum_{I\in\mathcal{D}} \|B_I m_I f\|^2 \le \sum_{I\in\mathcal{D}} (\|B_I\| m_I |f|)^2 = \|\pi_{|B|} |f|\|.$$

The boundedness of π_{B^*} follows analogously.

To show that $\mathrm{BMO^d_{Carl}} \neq \mathrm{BMO_{spara}}$, we can use the diagonal operator function B constructed in Proposition 2.7. There, it is shown that $B \notin \mathrm{BMO^d_{Carl}}$, and that the diagonal entry functions $\phi_I = \langle Be_I, e_I\rangle$ are uniformly bounded. Since the paraproduct of each scalar-valued L^∞ function is bounded, we see that $\pi_B = \bigoplus_{I\in\mathcal{D}} \pi_{\phi_I}$ is bounded. Similarly, π_{B^*} is bounded. Thus $B \in \mathrm{BMO_{spara}}$. It is clear from (18) that $\mathrm{BMO_{spara}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \subseteq \mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$.

Using that $L^\infty(\mathbb{T}, \mathcal{L}(\mathcal{H})) \not\subseteq \mathrm{BMO_{spara}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$ (see [Me2]), one concludes that $\mathrm{BMO_{spara}}(\mathbb{T}, \mathcal{L}(\mathcal{H})) \neq \mathrm{BMO_{mult}}(\mathbb{T}, \mathcal{L}(\mathcal{H}))$. □

3. Sharp dimensional growth of the sweep

We begin with the following lower estimate of the $\mathrm{BMO}_{\mathrm{para}}$ norm in terms of the L^∞ norm of certain $\mathrm{Mat}(\mathbb{C}, n \times n)$-valued functions from [Me2].

Lemma 3.1. *(see [Me2], Thm 1.1.) There exists an absolute constant $c > 0$ such that for each $n \in \mathbb{N}$, there exists a measurable function $F : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$ with $\|F\|_\infty \leq 1$ and $\|\pi_F\| \geq c \log(n+1)$.*

Here are our dimensional estimates of the sweep.

Theorem 3.2. *There exists an absolute constant $C > 0$ such that for each $n \in \mathbb{N}$ and each measurable function $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$,*

$$\|S_B\|_{\mathrm{BMO}_{\mathrm{para}}} \leq C \log(n+1) \|B\|^2_{\mathrm{BMO}_{\mathrm{para}}}, \tag{31}$$

$$\|S_B\|_{\mathrm{BMO}_{\mathrm{mult}}} \leq C (\log(n+1))^2 \|B\|^2_{\mathrm{BMO}_{\mathrm{mult}}}, \tag{32}$$

$$\|S_B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} \leq C (\log(n+1))^2 \|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}}, \tag{33}$$

and the dimensional estimates are sharp.

Proof. Let $B : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$ be measurable. Since $\|B\|_* = \lim_{k \to \infty} \|E_k B\|_*$ in all of the above BMO norms and $E_k S_B = E_k S_{E_k B}$ for $k \in \mathbb{N}$, it suffices to consider the case $B \in \mathcal{F}_{00}$.

We start by proving (31). Since

$$\|\pi_B\| \leq C' \log(n+1) \|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \tag{34}$$

for some absolute constant $C' > 0$ (see [NTV], [K]) and

$$\|B\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \leq \|B\|_{\mathrm{BMO}_{\mathrm{mult}}}, \tag{35}$$

we have

$$\|S_B\|_{\mathrm{BMO}_{\mathrm{para}}} \leq C' \log(n+1) \|S_B\|_{\mathrm{BMO}_{\mathrm{mult}}} \leq C \log(n+1) \|B\|^2_{\mathrm{BMO}_{\mathrm{para}}}$$

by (22).

For the sharpness of the estimate, take F as in Lemma 3.1. Again, approximating by $E_k F$, we can assume that $F \in \mathcal{F}_{00}$. Since each function in $L^\infty(\mathbb{T}, \mathrm{Mat}(\mathbb{C}, n \times n))$ is the linear combination of 4 nonnegative-matrix valued functions, the L^∞-norm of which is controlled by the norm of the original function, we can (by replacing c with a smaller constant) assume that F is a nonnegative matrix-valued function. Each such nonnegative matrix-valued function F can be written as $F = S_B$ with $B \in \mathcal{F}_{00}$, for example by choosing $B = \sum_{I \in \mathcal{D}, |I| = 2^{-k}} h_I B_I$, where $B_I = |I|^{1/2} (F^I)^{1/2}$, $F = \sum_{I \in \mathcal{D}, |I| = 2^{-k}} \chi_I F^I$. It follows that

$$\begin{aligned} \|S_B\|_{\mathrm{BMO}_{\mathrm{para}}} &\geq c \log(n+1) \|S_B\|_\infty \\ &\geq c/2 \log(n+1) (\|S_B\|_{\mathrm{BMO}_{\mathrm{mult}}} + \|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}}) \gtrsim \log(n+1) \|B\|^2_{\mathrm{BMO}_{\mathrm{para}}} \end{aligned}$$

again by (22). Here, we use the estimate $\|B\|^2_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{so}}} \leq \|S_B\|_\infty$, which can easily be obtained by

$$\|P_I B e\|_2^2 = \|S_{P_I B e}\|_1 \leq |I| \|S_{P_I B e}\|_\infty \leq |I| \|S_{P_I B}\|_\infty \leq |I| \|S_B\|_\infty \text{ for } e \in \mathcal{H}, \|e\| = 1.$$

This proves that (31) is sharp.

Let us now show (32). Note that by (18) and (34), for $B \in \mathcal{F}_{00}$,

$$\|S_B\|_{\mathrm{BMO_{mult}}} \lesssim \|B\|^2_{\mathrm{BMO_{para}}} \le C'^2 \log(n+1)^2 \|B\|^2_{\mathrm{BMO_{mult}}}.$$

For sharpness, choose $B \in \mathcal{F}_{00}$, $\|B\|_\infty \le 1$, $\|\pi_B\| \ge c\log(n+1)$ as above, to obtain

$$\begin{aligned}\|S_B\|_{\mathrm{BMO_{mult}}} + \|B\|^2_{\mathrm{BMO^d_{so}}} &\gtrsim \|B\|^2_{\mathrm{BMO_{para}}} \\ &\ge c^2 \log(n+1)^2 \|B\|^2_\infty \ge c^2 \log(n+1)^2 \|B\|^2_{\mathrm{BMO_{mult}}}\end{aligned}$$

and thus

$$\|S_B\|_{\mathrm{BMO_{mult}}} \gtrsim \log(n+1)^2 \|B\|^2_{\mathrm{BMO_{mult}}},$$

as $\|B\|_{\mathrm{BMO^d_{so}}} \le \|B\|_{\mathrm{BMO_{mult}}}$.

Finally, let us show (33). Again, we can restrict ourselves to the case $B \in \mathcal{F}_{00}$ by an approximation argument. We use the fact that the UMD constant of $\mathrm{Mat}(\mathbb{C}, n\times n)$ is equivalent to $\log(n+1)$ and the representation

$$S_B(t) = \int_\Sigma (T_\sigma B)^*(t)(T_\sigma B)(t)d\sigma \qquad (B \in \mathcal{F}_{00})$$

(see [BP2], [GPTV]), where T_σ denotes the dyadic martingale transform $B \mapsto T_\sigma B = \sum_{I\in\mathcal{D}} \sigma_I h_I B_I$, $\sigma = (\sigma_I)_{I\in\mathcal{D}} \in \{-1,1\}^{\mathcal{D}}$, and $d\sigma$ the natural product probability measure on $\Sigma = \{-1,1\}^{\mathcal{D}}$ assigning measure 2^{-n} to cylinder sets of length n, to prove that

$$\begin{aligned}\|P_I S_B\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} &= \|P_I S_{P_I B}\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \le 2\|S_{P_I B}\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\ &\lesssim (\log(n+1))^2 \|P_I B\|^2_{L^2(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \le (\log(n+1))^2 |I| \|B\|^2_{\mathrm{BMO^d_{norm}}},\end{aligned}$$

which gives the desired inequality.

To prove sharpness, choose $B \in \mathcal{F}_{00}$, $\|B\|_\infty \le 1$, $\|\pi_B\| \ge c\log(n+1)$ and note that by Theorem 2.6,

$$\begin{aligned}\|S_B\|_{\mathrm{BMO^d_{norm}}} + \|B\|^2_{\mathrm{BMO^d_{so}}} &\gtrsim \|S_B\|_{\mathrm{BMO_{mult}}} + \|B\|^2_{\mathrm{BMO^d_{so}}} \\ &\gtrsim \|B\|^2_{\mathrm{BMO_{para}}} \ge c^2\log(n+1)^2\|B\|^2_\infty \ge c^2 \log(n+1)^2 \|B\|^2_{\mathrm{BMO^d_{norm}}}.\end{aligned}$$

Since $\|B\|_{\mathrm{BMO^d_{so}}} \le \|B\|_{\mathrm{BMO^d_{norm}}}$, this implies

$$\|S_B\|_{\mathrm{BMO^d_{norm}}} \gtrsim \log(n+1)^2 \|B\|^2_{\mathrm{BMO^d_{norm}}}.$$

□

We now consider the bilinear extension of the sweep. By [PSm], [BP2] or [B4]

$$\pi^*_U \pi_V = \Lambda_{\Delta[U^*,V]} + D_{U^*,V} \qquad (U, V \in \mathcal{F}_{00}), \tag{36}$$

where $D_{U^*,V}$ is given by $D_{U^*,V} h_I e = h_I \frac{1}{|I|} \sum_{J\subset I} U^*_J V_J e$ for $I \in \mathcal{D}$, $e \in \mathcal{H}$.

Proposition 3.3.

$$\|\pi^*_U \pi_V\| \approx \|\Delta[U^*,V]\|_{\mathrm{BMO_{mult}}} + \sup_{I\in\mathcal{D}} \frac{1}{|I|} \|\sum_{J\subset I} U^*_J V_J\| \qquad (U, V \in \mathcal{F}_{00}).$$

Proof. Obviously $\|D_{U^*,V}\| = \sup_{I\in\mathcal{D}} \frac{1}{|I|}\|\sum_{J\subset I} U^*_J V_J\|$. Thus by (36),

$$\|\pi^*_U \pi_V\| \le \|\Delta[U^*,V]\|_{\mathrm{BMO_{mult}}} + \sup_{I\in\mathcal{D}} \frac{1}{|I|} \|\sum_{J\subset I} U^*_J V_J\|.$$

For the reverse estimate, it suffices to observe that $D_{U^*,V}$ is the block diagonal of the operator $\pi_U^*\pi_V$ with respect to the orthogonal subspaces $h_I\mathcal{H}$, $I \in \mathcal{D}$ and therefore $\|D_{U^*,V}\| \leq \|\pi_U^*\pi_V\|$. □

Here are the dimensional estimates of the bilinear map Δ.

Corollary 3.4. *There exists an absolute constant $C > 0$ such that for each $n \in \mathbb{N}$ and each pair of measurable functions $U, V : \mathbb{T} \to \mathrm{Mat}(\mathbb{C}, n \times n)$,*

$$\|\Delta[U^*, V]\|_{\mathrm{SBMO^d}} \leq C \log(n+1)\|U\|_{\mathrm{SBMO^d}}\|V\|_{\mathrm{SBMO^d}}, \tag{37}$$

$$\|\Delta[U^*, V]\|_{\mathrm{BMO_{para}}} \leq C \log(n+1)\|U\|_{\mathrm{BMO_{para}}}\|V\|_{\mathrm{BMO_{para}}}, \tag{38}$$

$$\|\Delta[U^*, V]\|_{\mathrm{BMO_{mult}}} \leq C(\log(n+1))^2\|U\|_{\mathrm{BMO_{mult}}}\|V\|_{\mathrm{BMO_{mult}}}, \tag{39}$$

$$\|\Delta[U^*, V]\|_{\mathrm{BMO^d_{norm}}} \leq C(\log(n+1))^2\|U\|_{\mathrm{BMO^d_{norm}}}\|V\|_{\mathrm{BMO^d_{norm}}}, \tag{40}$$

and the dimensional estimates are sharp.

Proof. Only the upper bounds need to be shown. For (37), use Proposition 2.1 to write $\|B\|_{\mathrm{SBMO^d}} = \sup_{I\in\mathcal{D},\|e\|=1} \|\Lambda_B(h_I e)\|$ and (36) to estimate

$$\|\Delta[U^*, V]\|_{\mathrm{SBMO^d}} \leq \sup_{I\in\mathcal{D},\|e\|=1} \|\pi_U^*\pi_V h_I e\| + \sup_{I\in\mathcal{D},\|e\|=1} \|D_{U^*,V}(h_I e)\|.$$

Now observe that for $e \in \mathcal{H}$, $I \in \mathcal{D}$, one has

$$\|\pi_U^*\pi_V h_I e\| \leq \|U\|_{\mathrm{BMO_{para}}}\|V\|_{\mathrm{SBMO^d}}\|e\| \leq C' \log(n+1)\|U\|_{\mathrm{SBMO^d}}\|V\|_{\mathrm{SBMO^d}}\|e\|$$

by (34). Since $D_{U^*,V} h_I e = \frac{1}{|I|}\sum_{J\subset I} U_J^* V_J e\, h_I$, one obtains

$$\begin{aligned}\|D_{U^*,V}(h_I e)\| &= \sup_{f\in\mathcal{H},\|f\|=1} |\langle D_{U^*,V}(h_I e), h_I f\rangle| \\ &= \sup_{f\in\mathcal{H},\|f\|=1} \frac{1}{|I|}\Big|\sum_{J\subset I}\langle V_J e, U_J f\rangle\Big| \leq \|V_e\|_{\mathrm{BMO^d}(\mathbb{T},\mathcal{H})}\|U\|_{\mathrm{SBMO^d}},\end{aligned}$$

and the proof of (37) if complete.

Using first (34) and (35) and then Proposition 3.3, we obtain (38). In a similar way, using first Proposition 3.3 and then (34), (35) yields (39).

Finally, for (40) observe first that for any $U, V \in \mathcal{F}_{00}$, $e, f \in \mathcal{H}$, $t \in \mathbb{T}$,

$$\begin{aligned} & |\langle \Delta[U^*, V](t)e, f\rangle| \\ = \ & \Big|\sum_{I\in\mathcal{D}} \left\langle \frac{\chi_I(t)}{|I|^{1/2}} V_I e, \frac{\chi_I(t)}{|I|^{1/2}} U_I f \right\rangle\Big| \\ \leq \ & \left(\sum_{I\in\mathcal{D}} \Big\|\frac{\chi_I(t)}{|I|^{1/2}} V_I e\Big\|^2\right)^{1/2} \left(\sum_{I\in\mathcal{D}} \Big\|\frac{\chi_I(t)}{|I|^{1/2}} U_I f\Big\|^2\right)^{1/2} \\ = \ & \langle S_U(t)e, e\rangle^{1/2}\langle S_V(t)f, f\rangle^{1/2} \leq \|S_U(t)\|^{1/2}\|S_V(t)\|^{1/2}\end{aligned}$$

and therefore

$$\|\Delta[U^*, V](t)\| \leq \|S_U(t)\|^{1/2}\|S_V(t)\|^{1/2} \quad (t \in \mathbb{T}). \tag{41}$$

Now consider the $\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}$ norm of $\Delta[U^*, V]$. For $I \in \mathcal{D}$,

$$
\begin{aligned}
& \|P_I \Delta[U^*, V]\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\
= \; & \|P_I \Delta[P_I U^*, P_I V]\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\
\leq \; & 2\|\Delta[P_I U^*, P_I V]\|_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\
\leq \; & 2\| \|S_{P_I U}(\cdot)\|^{1/2} \|S_{P_I V}(\cdot)\|^{1/2}\|_{L^1(\mathbb{T})} \\
\leq \; & 2\|S_{P_I U}\|^{1/2}_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \|S_{P_I V}\|^{1/2}_{L^1(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\
\leq \; & 2(\log(n+1))^2 \|P_I U\|_{L^2(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \|P_I U\|_{L^2(\mathbb{T},\mathrm{Mat}(\mathbb{C},n\times n))} \\
\leq \; & 2(\log(n+1))^2 |I| \|U\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}} \|V\|_{\mathrm{BMO}^{\mathrm{d}}_{\mathrm{norm}}},
\end{aligned}
$$

where we obtain the third inequality from (41) and the fourth inequality from the proof of (33). This finishes the proof of (40). □

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