(p, q)-summing sequences

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Abstract

A sequence (x_j) in a Banach space X is (p, q)-summing if for any weakly q-summable sequence (x_j^*) in the dual space we get a p-summable sequence of scalars $(x_j^*(x_j))$. We consider the spaces formed by these sequences, relating them to the theory of (p, q)-summing operators. We give a characterization of the case $p = 1$ in terms of integral operators, and show how these spaces are relevant for a general question on Banach spaces and their duals, in connection with Grothendieck theorem.

Key words: Sequences in Banach spaces, bounded, integral and (p, q)-summing operators, type and cotype, Grothendieck theorem.

1 Definitions and basic results

In all that follows X is a Banach space over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We shall use the usual terms X^* for the dual space of X, $\mathcal{L}(X, Y)$ for the space of bounded linear operators between two Banach spaces, and B_X and S_X for the unit ball and sphere in X; $X \simeq Y$ means that X and Y are isometrically isomorphic. We write the action of an operator or functional on x merely as ux and x^*x, though we prefer to use $x^*(x)$ or $\langle x^*, x\rangle$ if we think it helps, and we use the tensor form for expressing finite rank operators: $(x^* \otimes y)x = x^*(x)y$. Finally, (e_j) is the canonical basis of the sequence spaces ℓ_p and c_0, p' denotes the

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conjugate exponent of p, $\alpha^+ = \max\{\alpha, 0\}$ for any real α, and $\|\cdot\|_p$ stands for the usual p-norm of a sequence or function.

Definition 1 *Let $p, q \in [1, \infty)$. A sequence (x_j) in X is called a (p,q)-summing sequence if there exists a constant $C \geq 0$ for which*

$$\Big(\sum_{j=1}^{n} |x_j^* x_j|^p\Big)^{1/p} \leq C \sup\Big\{\Big(\sum_{j=1}^{n} |x_j^* x|^q\Big)^{1/q} \ : \ x \in B_X\Big\}$$

for any finite collection of vectors $x_1^*, \dots, x_n^*$ in X^*.

The least such C is the (p,q)-summing norm of (x_j), denoted by $\pi_{p,q}[x_j]$ or (in case of ambiguity) $\pi_{p,q}[x_j; X]$, and $\ell_{\pi_{p,q}}(X)$ is the space of all (p,q)-summing sequences in X. If $p = q$ we simply write $\pi_p[x_j]$ and $\ell_{\pi_p}(X)$, the space of p-summing sequences in X.

We believe our notations are justified as long as these sequences in $X \subseteq X^{**}$ are multiplier sequences from $\ell_q^w(X^*)$ to ℓ_p, special instances of the more general class of (p,q)-summing sequences of operators (u_j) in $\mathcal{L}(X, Y)$ for two Banach spaces X and Y: those such that $\|(u_j x_j)\|_{\ell_p(Y)} \leq C \|(x_j)\|_{\ell_p^w(X)}$ for a constant C. Note that a constant sequence $(u_j = u)$ satisfies this if and only if $u \in \Pi_{p,q}(X, Y)$, i.e. u is a (p,q)-summing operator, and the least C equals $\pi_{p,q}(u)$, the (p,q)-summing norm of u (the p-summing norm $\pi_p(u)$ if $p = q$).

We refer the reader to the forthcoming paper [1] for further results on this more general setting; see also [2] for the particular case $p = q$, $X = Y$ and $u_j = \alpha_j \, \mathrm{id}_X$. A quite recent and very good source book on p-summing norms and related topics is [3]. Some other good references are [4], [5] and [6].

Remark 1 *$(\ell_{\pi_{p,q}}(X), \pi_{p,q})$ is a Banach space. This follows readily once we note that it is closed as a subset of $\mathcal{L}(\ell_q^w(X^*), \ell_p)$.*

Remark 2 *The obvious modifications in the definition for $p = \infty$ or $q = \infty$ make sense, but then clearly $\ell_{\pi_{p,\infty}}(X) = \ell_p(X)$ and $\ell_{\pi_{\infty,q}}(X) = \ell_\infty(X)$.*

Remark 3 *A standard use of the weak Principle of Local Reflexivity (see [6], p. 73) shows that $(x_j^*) \subset X^*$ is (p,q)-summing if and only if*

$$\Big(\sum_{j=1}^{n} |x_j^* x_j|^p\Big)^{1/p} \leq C \sup\Big\{\Big(\sum_{j=1}^{n} |x^* x_j|^q\Big)^{1/q} \ : \ x^* \in B_{X^*}\Big\},$$

where C is a constant independent from n and $x_1, \dots, x_n \in X$.

In particular $\ell_{\pi_{p,q}}(X) = \ell_{\pi_{p,q}}(X^{**}) \cap \ell_\infty(X)$.

Let us omit as well the simple proofs of the following facts:

Lemma 1 *Let $1 \leq p, q < \infty$, $(\alpha_j) \subseteq \mathbb{K}$ and $x \in X$: Then*

$$\pi_{p,q}[\alpha_j x] = \|(\alpha_j)\|_r \|x\| ,$$

where $1/r = ((1/p) - (1/q))^+$.

Proposition 1 *Given $1 \leq p, q$, let r such that $(1/r) = ((1/p) - (1/q))^+$. Then*

$$\ell_p(X) \subseteq \ell_{\pi_{p,q}}(X) \subseteq \ell_r(X),$$

with continuous inclusions of norm 1.

Actually, if X is finite dimensional then $\ell_{\pi_{p,q}}(X) = \ell_r(X)$.

To verify the last claim, recall that X is finite dimensional if and only if $\ell_q^w(X) = \ell_q(X)$ for any $q \in [1, \infty)$.

Remark 4 *Note that $\ell_{\pi_{p,q}}(X) \subset c_0(X)$ if and only if $p < q$.*

Furthermore, any non trivial constant sequence is in $\ell_{\pi_{p,q}}(X)$ if and only if $p \geq q$; this corresponds to the fact that the notion of (p, q)-summing operator only makes sense for $p \geq q$, since otherwise $\pi_{p,q}(u) < \infty$ only if $u = 0$; in contrast with that, any finite sequence is obviously a (p, q)-summing sequence for any p and q.

Lemma 2 *Given $1 \leq t \leq s < \infty$, let r such that $1/r = (1/t) - (1/s)$. Then we have, for any $x_1^*, x_2^*, \ldots, x_n^* \in X^*$,*

$$(\sum_{j=1}^{n} \|x_j^*\|^s)^{1/s} = \sup\{(\sum_{j=1}^{n} |x_j^* x_j|^t)^{1/t} : \|(x_j)\|_{\ell_r(X)} = 1\}.$$

PROOF. For $t = 1$ this is just the duality $\ell_s(X^*) = (\ell_{s'}(X))^*$.

The general case follows from

$$(\sum_{j=1}^{n} |x_j^* x_j|^t)^{1/t} = \sup\{\sum_{j=1}^{n} |\alpha_j x_j^* x_j| : \sum_{j=1}^{n} |\alpha_j|^{t'} = 1\}.$$

Note that $\ell_{s'}(X) = \ell_{t'} \ell_r(X)$, and then

$$\sup\{\sum_{j=1}^{n} |\alpha_j x_j^* x_j| : \sum_{j=1}^{n} |\alpha_j|^{t'} = 1, \|(x_j)\|_{\ell_r(X)} = 1\}$$
$$= \sup\{\sum_{j=1}^{n} |x_j^* y_j| : \sum_{j=1}^{n} \|y_j\|^{s'} = 1\} = (\sum_{j=1}^{n} \|x_j^*\|^s)^{1/s}.$$

Theorem 1 *If $1 \leq p \leq q < \infty$, the following are equivalent:*

(a) *X is finite dimensional.*

(b) $\ell_{\pi_{p,q}}(X) = \ell_r(X)$ *for* $1/r = (1/p) - (1/q)$*.*

PROOF. We only have to show that (b) implies (a). By the previous lemma

$$(\sum_{k=1}^{n} \|x_k^*\|^q)^{1/q} = \sup\{(\sum_{k=1}^{n} |x_k^* x_k|^p)^{1/p} : \sum_{k=1}^{n} \|x_k\|^r = 1\}.$$

Therefore $\ell_r(X) \subseteq \ell_{\pi_{p,q}}(X)$ implies $\ell_q^w(X^*) = \ell_q(X^*)$.

We'll see later on that there are infinite dimensional spaces X such that $\ell_{\pi_{p,q}}(X) = \ell_\infty(X)$ for certain $p > q$.

Let us remark now another difference between the cases $p < q$ and $p \geq q$: note first that, in general, the $\pi_{p,q}$-norm of any sequence is independent from any reordering of its terms:

Proposition 2 *Let (x_j) a bounded sequence in X, and let $1 \leq p, q$. Then*

$$\pi_{p,q}[x_{\sigma(j)}] = \pi_{p,q}[x_j]$$

for any bijection $\sigma \colon \mathbb{N} \to \mathbb{N}$.

The proof follows from the definition and the fact that the p-norm and the weak q-norm are reordering invariant.

When $p \geq q$ we can say more:

Proposition 3 *Let (x_j) a bounded sequence in X, and let $1 \leq q \leq p < \infty$. Then*

$$\pi_{p,q}[x_{\sigma(j)}] \leq \pi_{p,q}[x_j]$$

for any map $\sigma \colon \mathbb{N} \to \mathbb{N}$.

PROOF. Given $x_1^*, x_2^*, \dots, x_n^* \in X^*$ we have

$$\begin{aligned}
\Big(\sum_j |x_j^* x_{\sigma(j)}|^p\Big)^{1/p} &= \Big(\sum_k \big(\sum_{\sigma(j)=k} |x_j^* x_k|^p\big)\Big)^{1/p} \le \Big(\sum_k \big(\sum_{\sigma(j)=k} |x_j^* x_k|^q\big)^{p/q}\Big)^{1/p} \\
&= \Big(\sum_k \Big|\big(\sum_{\sigma(j)=k} \alpha_j x_j^*\big) x_k\Big|^p\Big)\Big)^{1/p} \qquad (\text{where } (\alpha_j)_{\sigma(j)=k} \in B_{\ell_{q'}}) \\
&= \Big(\sum_k |y_k^* x_k|^p\Big)^{1/p} \qquad (\text{with } y_k^* = \sum_{\sigma(j)=k} \alpha_j x_j^* \in X^*) \\
&\le \pi_{p,q}[x_j]\, \|(y_k^*)\|_{\ell_q^w(X^*)} = \pi_{p,q}[x_j] \sup_{\|(\beta_k)\|_{q'} \le 1} \Big\|\sum_k \beta_k y_k^*\Big\| \\
&= \pi_{p,q}[x_j] \sup_{\|(\beta_k)\|_{q'} \le 1} \Big\|\sum_j \alpha_j \beta_{\sigma(j)} x_j^*\Big\| \\
&\le \pi_{p,q}[x_j] \sup_{\|(\gamma_j)\|_{q'} \le 1} \Big\|\sum_k \gamma_j x_j^*\Big\| = \pi_{p,q}[x_j]\, \|(x_j^*)\|_{\ell_q^w(X^*)}\,.
\end{aligned}$$

The result does not hold if $1 \le p < q$: take σ a constant map.

Proposition 3 implies that all (p,q)-sequences satisfy something apparently stronger than the condition in Definition 1:

Corollary 1 *For any $p \ge q \ge 1$, a sequence $(x_j) \subset X$ is (p,q)-summing if and only if there exists a constant C such that*

$$\Big(\sum_{k=1}^n \sup_j |x_k^* x_j|^p\Big)^{1/p} \le C \sup_{x \in B_X} \Big(\sum_{k=1}^n |x_k^* x|^q\Big)^{1/q}$$

for any $x_1^*, \dots, x_n^* \in X^*$, and the least such C is $\pi_{p,q}[x_j]$.

2 $(1,q)$-summing sequences as integral operators

Recall that $u \in \mathcal{L}(X,Y)$ is *p-integral* if the composition $X \xrightarrow{u} Y \xrightarrow{j_Y} Y^{**}$ equals $X \xrightarrow{\beta} L_\infty(\mu) \xrightarrow{i_p} L_p(\mu) \xrightarrow{\alpha} Y^{**}$ for some positive measure μ and bounded operators α and β (i_p and j_Y are the respective inclusions).

The p-integral norm of u is the infimum of all the possible values of $\|\alpha\|\|\beta\|$ in the previous expression. The set of p-integral operators (a Banach operator ideal) is denoted by $I_p(X,Y)$. For $p = 1$ it is denoted simply by $I(X,Y)$, the space of *integral* operators.

Any p-integral operator u is also p-summing, and $\pi_p(u)$ is not greater than the p-integral norm, but the converse is not true in general. Basic results on p-integral operators can be seen in [3].

We'll make use of the following fact: $u\colon X \to Y$ is integral if and only if there exists a constant $C > 0$ such that

$$|\operatorname{tr}(uv)| \leq C\|v\|$$

for any finite rank linear operator $v\colon Y \to X$, and the least such C is the integral norm of u.

This makes easy to characterize the $(1,q)$-sequences in terms of integral operators:

Theorem 2 *For any $1 \leq q < \infty$, a sequence $(x_j) \subset X$ is $(1,q)$-summing if and only if it defines an integral operator $u\colon \ell_q \to X$ by $ue_j = x_j$, and the integral norm of u is then $\pi_{1,q}[x_j]$.*

PROOF. Let u an integral operator $\ell_q \to X$ with $ue_j = x_j$ for all j, and let C its integral norm. Given $x_1^*, \ldots x_n^* \in X^*$, let $v = \sum_{j=1}^n x_j^* \otimes \lambda_j e_j$, where $\lambda_j = \operatorname{sgn}(x_j^* x_j)$. Then

$$\sum_{j=1}^n |x_j^* x_j| = \sum_{j=1}^n \lambda_j x_j^* x_j = \operatorname{tr}(uv)\,,$$

so $\sum_{j=1}^n |x_j^* x_j| \leq C\|v\|$, and $\|v\|$ is just $\|(x_j^*)\|_{\ell_q^w(X^*)}$. Then $\pi_{1,q}[x_j] \leq C$.

Conversely, let $(x_j) \in \ell_{\pi_{1,q}}(X)$. Then $(x_j) \in \ell_{q'}(X)$, so $u\colon e_j \mapsto x_j$ defines a bounded operator in $\mathcal{L}(\ell_q, X)$. Now, if $v = \sum_{j=1}^n x_j^* \otimes \xi_j$ with $\xi_j = (\xi_{jk})_k \in \ell_q$ then, for $v_k^* = \sum_j \xi_{jk} x_j^* \in X^*$, it turns out that $|\operatorname{tr}(uv)| = \sum_k |v_k^* x_k| \leq \pi_{1,q}[x_k]\|(v_k^*)\|_{\ell_q^w(X^*)}$ and $\|(v_k^*)\|_{\ell_q^w(X^*)} = \|v\|$, giving that the integral norm of u is bounded by $\pi_{1,q}[x_j]$.

As an application of Theorem 2, we can identify the sequences in $\ell_{\pi_{1,q}}(L_1(\mu))$, for any σ-finite space μ:

For any Banach lattice X, an operator $u\colon X \to L_1(\mu)$ is integral if and only if $\int \Big(\sup_{x \in B_X} |ux|\Big) d\mu < \infty$, its value being the integral norm of u (see Th. 5.19 in [3]). If applied to $X = \ell_q$, Theorem 2 gives the following:

Theorem 3 *Let $1 \leq q < \infty$, and let μ a σ-finite measure. Then $(f_j) \in \ell_{\pi_{1,q}}(L_1(\mu))$ if and only if*

$$\int \|(f_j(w))\|_{\ell_{q'}} d\mu(w) < \infty\,,$$

and then the integral equals $\pi_{1,q}[f_j]$.

PROOF. Just note that $\sup_{\|(\lambda_j)\|_q=1} \left|\sum_j \lambda_j f_j(w)\right| = \|(f_j(w))\|_{q'}$ for any w in the measure space.

When $1 < q < \infty$ it results that $\ell_{\pi_{1,q}}(L_1(\mu)) \simeq L_1(\mu, \ell_{q'})$. This is true for $q = \infty$, since $\ell_{\pi_{1,\infty}}(L_1(\mu)) = \ell_1(L_1(\mu)) \simeq \mathcal{L}_1(\mu, \ell_1)$.

As for $q = 1$, recall that we can have $\int \sup_j |f_j(w)| d\mu(w) < \infty$ with $w \mapsto (f_j(w))$ not being a measurable function. For example, for the Rademacher functions r_j in $\big([0,1], dt\big)$ we have that $\{(r_j(t)) : \ t \in [0,1]\} = \{-1,1\}^{\mathbb{N}}$ is not esentially separable and then the sequence does not define a function in $L_1(dt, \ell_\infty)$. Anyway $(r_j) \in \ell_{\pi_1}\big(L_1[0,1]\big)$, as Theorem 3 gives the following for $q = 1$:

Corollary 2 *Let μ a σ-finite measure. Then $(f_j) \in \ell_{\pi_1}(L_1(\mu))$ if and only if there exists another function $f \in L_1(\mu)$ such that, for every j, $|f_j| \leq f$ μ-a.e.*

Another consequence of the interpretation of π_1-sequences as integral operators is the following:

Corollary 3 *Let (x_j) be a bounded sequence in X. Then $(x_j) \in \ell_{\pi_1}(X)$ if and only if there exist a Banach space Y, a sequence $(y_j^*) \in \ell_\infty(Y^*)$ and $u \in \Pi_1(X^*, Y)$ such that $x_j = y_j^* \circ u \in X^{**}$ for each j.*

PROOF. Let us assume that such u and (y_j^*) do exist. The constant sequence $(u_j = u)$ is a multiplier from $\ell_1^w(X^*)$ to $\ell_1(Y)$, and so it is (y_j^*) from $\ell_1(Y)$ to ℓ_1, so the composition $(x_j) = (y_j^* \circ u)$ belongs to $\ell_{\pi_1}(X^{**})$.

Conversely, if $(x_j) \in \ell_{\pi_1}(X)$ then Theorem 2 says that $v \colon \ell_1 \to X$ given by $ve_j = x_j$ is an integral operator, and in particular v^* is absolutely summing (v^* is integral if v is so, and integral operators with values in ℓ_∞ are absolutely summing). Then we can take $Y = \ell_\infty$, $u = v^*$ and $(y_j^*) = (e_j)$ in $\ell_1 \subset (\ell_\infty)^*$. Since $e_j(v^*x^*) = x^*(ve_j) = x^*x_j$ for any $x^* \in X^*$ and each j, the result follows.

3 Inclusions among the spaces $\ell_{\pi_{p,q}}(X)$

Let us point out first some elementary embeddings among these spaces.

Proposition 4 *Let $1 \leq r, s < \infty$, $1 \leq p_1 \leq p_2$, $1 \leq q_1 \leq q_2$ and $1 \leq p \leq q$.*

Then

$$\ell_{\pi_{p_1,s}}(X) \subseteq \ell_{\pi_{p_2,s}}(X)\,,$$
$$\ell_{\pi_{r,q_2}}(X) \subseteq \ell_{\pi_{r,q_1}}(X) \;\; \textit{and}$$
$$\ell_{\pi_p}(X) \subseteq \ell_{\pi_q}(X)\,,$$

with continuous inclusions of norm 1.

In particular, for $1 \le p, q < \infty$

$$\ell_{\pi_{1,q}}(X) \subseteq \ell_{\pi_1}(X) \subseteq \ell_{\pi_p}(X) \subseteq \ell_{\pi_{p,1}}(X).$$

We can actually show the following more general result:

Theorem 4 *Let* p, q, r *and* s *such that* $1 \le p \le r$, $1 \le q, s$ *and* $(1/q)+(1/r) \le (1/p)+(1/s)$. *Then* $\ell_{\pi_{p,q}}(X) \subseteq \ell_{\pi_{r,s}}(X)$, *with continuous inclusion of norm* 1.

PROOF. The case $s \le q$ follows from the norm 1 inclusions $\ell_s^w(X^*) \subseteq \ell_q^w(X^*)$ and $\ell_p(X) \subseteq \ell_r(X)$. If $q < s$, then for $r = \infty$ or $s = \infty$ the result is true by Remark 2 and Proposition 1. So we assume that $q < s$ and $r, s < \infty$. Then $1 < r/p, s/q < \infty$; let a and b their conjugate numbers, that is $1 = (1/a) + (p/r) = (1/b) + (q/s)$.

If $\pi_{p,q}[x_j] \le C$, for any finite set of vectors x_j^* in X^* we have, for appropriate scalars $\alpha_j \ge 0$ such that $\sum \alpha_j^a = 1$, that

$$\Big(\sum_j |x_j^* x_j|^r\Big)^{1/r} = \Big(\sum_j |x_j^*(\alpha_j^{1/p} x_j)|^p\Big)^{1/p} \le C \sup_{\|x^*\|\le 1} \Big(\sum_j \alpha_j^{q/p} |x^* x_j|^q\Big)^{1/q}.$$

From our assumptions we have that $ap \le bq$, so that $\sum_j \alpha_j^{\frac{q}{p}b} \le 1$, and for any x^* Hölder inequality gives $\Big(\sum_j \alpha_j^{q/p} |x^* x_j|^q\Big)^{1/q} \le \Big(\sum_j |x^* x_j|^s\Big)^{1/s}$. This shows that $\pi_{r,s}[x_j] \le C$.

3.1 The role of type and cotype

Recall that $\mathrm{Rad}_p(X)$ is the closure in $L_p([0,1], X)$ of the set of functions of the form $\sum_{j=1}^n r_j x_j$, where $x_j \in X$ and $(r_j)_{j\in\mathbb{N}}$ are the Rademacher functions on $[0,1]$. By Kahane–Khintchine inequalities (see [3], page 211) it follows that $\mathrm{Rad}_p(X)$ coincide up to equivalent norms for all $p < \infty$. The space is denoted then $\mathrm{Rad}(X)$. Given $1 \le p \le 2$ (respect. $q \ge 2$), a Banach space X is said to have (Rademacher) type p (respect. (Rademacher) cotype q) if $\ell_p(X) \subseteq \mathrm{Rad}(X)$ (respect. $\mathrm{Rad}(X) \subseteq \ell_q(X)$).

We know by Proposition 1 that, for finite dimensional X, if $(1/p) - (1/q) = (1/r) - (1/s)$ then $\ell_{\pi_{p,q}}(X) = \ell_{\pi_{r,s}}(X)$. In order to find conditions that ensure $\ell_{\pi_{p,q}}(X) = \ell_{\pi_{r,s}}(X)$ if $(1/q) + (1/r) = (1/p) + (1/s)$ we give the following lemma:

Lemma 3 *Let $1 < r < \infty$. Then $\ell_1^w(X) = \ell_r \ell_{r'}^w(X)$ if and only if $\mathcal{L}(c_0, X) = \Pi_r(c_0, X)$.*

PROOF. Assume $\ell_1^w(X) = \ell_r \ell_{r'}^w(X)$ and take $u \in \mathcal{L}(c_0, X)$. If $x_j = u(e_j)$ then $(x_j) \in \ell_1^w(X)$, so we write $x_j = u(e_j) = \alpha_j x'_j$ where $(\alpha_j) \in \ell_r$ and $(x'_j) \in \ell_{r'}^w(X)$. This allows to factorize $u = wv$, where $v \in \mathcal{L}(c_0, \ell_r)$ is given by $v(e_j) = \alpha_j e_j$ and $w \in \mathcal{L}(\ell_r, X)$ is given by $w(e_j) = x'_j$. It is not difficult to show (see [3], page 41) that $v \in \Pi_r(c_0, \ell_r)$, and then $u \in \Pi_r(c_0, X)$.

Conversely, assume $\mathcal{L}(c_0, X) = \Pi_r(c_0, X)$ and let us take $(x_j) \in \ell_1^w(X)$. Consider now the operator $u : c_0 \to X$ defined by $u(e_j) = x_j$. From the assumption $u \in \Pi_r(c_0, X)$. Now, since $(e_j) \in \ell_1^w(c_0)$ and $u \in \Pi_r(c_0, X)$, then (see [3], page 53) $u(e_j) = \alpha_j x'_j$ with $(\alpha_j) \in \ell_r$ and $(x'_j) \in \ell_{r'}^w(X)$.

Proposition 5 *Assume that $\mathcal{L}(c_0, X^*) = \Pi_{s'}(c_0, X^*)$ for some $1 < s < \infty$. Then $\ell_{\pi_{r,s}}(X) \subseteq \ell_{\pi_{p,q}}(X)$ for any $1 \le p, q, r < \infty$ such that $(1/p) - (1/q) = (1/r) - (1/s)$.*

PROOF. Let us take $(x_j) \in \ell_{\pi_{r,s}}(X)$ and $(x_j^*) \in \ell_q^w(X^*)$. To show that $(x_j^* x_j) \in \ell_p$, it suffices to see that for any $(\alpha_j) \in \ell_{q'}$ we get $(\alpha_j x_j^* x_j) \in \ell_u$ where $(1/p) + (1/q') = 1/u$. Given now a sequence $(\alpha_j) \in \ell_{q'}$ we have that $(\alpha_j x_j^*) \in \ell_1^w(X^*)$. Using Lemma 3 we have that there exist $(\beta_j) \in \ell_{s'}$ and $(y_j^*) \in \ell_s^w(X^*)$ such that $\alpha_j x_j^* = \beta_j y_j^*$. Therefore $(\alpha_j x_j^*) = (\beta_j y_j^* x_j) \in \ell_{s'} \ell_r = \ell_u$ since $1/u = (1/p) + (1/q') = (1/s') + (1/r)$.

Combining Theorem 4 and Proposition 5 we get the following:

Theorem 5 *Let X such that $\mathcal{L}(c_0, X^*) = \Pi_{s'}(c_0, X^*)$ for some $1 < s < \infty$. Then $\ell_{\pi_{r,s}}(X) = \ell_{\pi_{p,q}}(X)$ whenever $1 \le p, q, r, s < \infty$ are such that $1 \le p \le r$ and $(1/p) - (1/q) = (1/r) - (1/s)$.*

Proposition 6

(a) *If X has cotype 2 then $\ell_1^w(X) = \ell_2 \ell_2^w(X)$.*

(b) *If X has cotype $q > 2$ then $\ell_1^w(X) = \ell_r \ell_{r'}^w(X)$ for any $r > q$.*

PROOF. Use Lemma 3 and the fact that $\mathcal{L}(c_0, Y) = \Pi_2(c_0, Y)$ for any Y of cotype 2 and $\mathcal{L}(c_0, Y) = \Pi_r(c_0, Y)$ for any Y of cotype $q > 2$ and $r > q$ (see Theorem 11.14 in [3]).

Remark 5 *Let X be any space with GL-property (see Page 350, [3] for definition and results). Then X has cotype 2 if and only if $\ell_1^w(X) = \ell_2 \ell_2^w(X)$. Actually it holds that $\mathcal{L}(c_0, X) = \Pi_2(c_0, X)$ if an only if X is of cotype 2, (see page 352, [3]).*

Remark 6 *Recall that X is a G.T. space if $\mathcal{L}(X, \ell_2) = \Pi_1(X, \ell_2)$ (the term comes after Grothendieck theorem, that asserts that this is the case for $X = L_1(\mu)$). Then $\ell_1^w(X) = \ell_2 \ell_2^w(X)$.*

Indeed, if $u \in \mathcal{L}(c_0, X)$ then $u^* \in \mathcal{L}(X^*, \ell_1)$. Now GT property on X gives that $u^* \in \Pi_2(X^*, \ell_1)$ (see [4], page 71) which implies that u^* factors through a Hilbert space, and so u does. Therefore $u \in \Pi_2(c_0, X)$.

Corollary 4 *If X^* has cotype 2 then $\ell_{\pi_{p,q}}(X) = \ell_{\pi_{r,2}}(X)$ for any $p \leq r$ and $1/q = (1/p) - (1/r) + (1/2)$.*

In particular $\ell_{\pi_1}(X) = \ell_{\pi_2}(X)$ and $\ell_{\pi_{1,q}}(X) = \ell_{\pi_{r,2}}(X)$ for $1/r = (1/q') + (1/2)$.

Corollary 5 *If X^* has cotype $q_0 > 2$ then $\ell_{\pi_{p,q}}(X) = \ell_{\pi_{r,s}}(X)$ for any $p \leq r$, $s < q_0'$ and $(1/p) - (1/q) = (1/r) - (1/s)$.*

In particular $\ell_{\pi_p}(X) = \ell_{\pi_1}(X)$ for any $1 \leq p < q_0'$ and $\ell_{\pi_{1,q}}(X) = \ell_{\pi_{r,s}}(X)$ for $s < q_0'$ and $1/r = (1/q') + (1/s)$.

Proposition 7 *Let $1 \leq q \leq p < \infty$ and $r \geq p'$. Then the following are equivalent:*

(a) id_{X^*} *is (p, q)-summing.*

(b) *$\ell_r(X) \subseteq \ell_{\pi_{s,q}}(X)$ for any $1 \leq s \leq r$ such that $1/s = (1/r) + (1/p)$.*

Moreover, $\pi_{p,q}(\mathrm{id}_{X^*}) = \sup\{\pi_{s,q}[x_j] : \|(x_j)\|_{\ell_r(X)} = 1\}$.

PROOF. Assume first that the identity in X^* is (p, q)-summing. If r and s are as stated, $(x_j) \in B_{\ell_r(X)}$ and $x_1^*, \ldots, x_n^* \in X^*$ we see that

$$\Big(\sum_j |x_j^* x_j|^s\Big)^{1/s} \leq \Big(\sum_j \|x_j^*\|^p\Big)^{1/p} \leq \pi_{p,q}(\mathrm{id}_{X^*})\, \|(x_j^*)\|_{\ell_q^w(X^*)}.$$

Conversely, we assume now that $\ell_r(X) \subseteq \ell_{\pi_{s,q}}(X)$ and take $x_1^*, \ldots, x_n^*$ in X^*. From Lemma 2 we have

$$\Big(\sum_j \|x_j^*\|^p\Big)^{1/p} = \sup\{(\sum_{k=1}^n |x_k^* x_k|^s)^{1/s} : \sum_{k=1}^n \|x_k\|^r = 1\}.$$

Then (x_j) is of norm 1 in $\ell_r(X)$, and if C is the norm of the inclusion of $\ell_r(X)$ in $\ell_{\pi_{s,q}}(X)$ we have $\Big(\sum_j |x_j^* x_j|^s\Big)^{1/s} \leq C\|(x_j^*)\|_{\ell_q^w(X^*)}$. This yields $\Big(\sum_j \|x_j^*\|^p\Big)^{1/p} \leq C\|(x_j^*)\|_{\ell_q^w(X^*)}$.

Some particularly interesting cases are given in the following corollaries.

Corollary 6 *For any X and $1 \leq p$ the following are equivalent:*

(a) *id_{X^*} is $(p,1)$-summing.*

(b) $\ell_\infty(X) = \ell_{\pi_{p,1}}(X)$.

(c) $\ell_{p'}(X) \subseteq \ell_{\pi_1}(X)$.

Moreover, if $p \geq 2$ they hold if and only if X^* has cotype p.

PROOF. Only the last claim deserves a proof. It is due to the deep result, due to M. Talagrand (see [7]), that asserts that for $2 < q < \infty$ the identity in any Banach space Y is $(q,1)$-summing if and only if Y has cotype q.

Remark 7 *As for $p = 2$, we get that $\ell_2(X) \subseteq \ell_{\pi_1}(X)$ if and only if $\ell_\infty(X) = \ell_{\pi_{2,1}}(X)$, if and only if X^* has the so-caled* Orlicz property*, i. e. id_{X^*} is $(2,1)$-summing. However, although cotype 2 is a sufficient condition to have the Orlicz property it is not necessary (see [8]).*

These inclusions are the best possible when dealing with infinite dimensional spaces:

Corollary 7 *For any Banach space X the following are equivalent:*

(a) *X is finite dimensional.*

(b) *$\ell_{\pi_{p,q}}(X) = \ell_\infty(X)$ for some $p \geq q$ with $(1/q) - (1/p) < 1/2$.*

(c) *$\ell_s(X) \subseteq \ell_{\pi_{p,q}}(X)$ for some $1 \leq p \leq q$ and $p < s < r$ with $(1/s) - (1/r) < 1/2$.*

(d) *$\ell_{\pi_{p,1}}(X) = \ell_\infty(X)$ for some (or for all) $1 \leq p < 2$.*

(e) $\ell_{p'}(X) \subseteq \ell_{\pi_1}(X)$ *for some (or for all)* $1 \leq p < 2$.

PROOF. To see that (b) implies (a) use the fact that $id_{X^*} \in \Pi_{p,q}(X^*, X^*)$ for $(1/q) - (1/p) < 1/2$. This gives that X^* is finite dimensional (see [3], page 199).

If (c) is true then Proposition 7 says that $id_{X^*} \in \Pi_{q_1,q}(X^*, X^*)$ for $(1/s) + (1/q_1) = (1/p)$, what again gives (a) because $(1/q) - (1/q_1) < 1/2$.

(d) is the particular case of (b) for $q = 1$.

(e) is equivalent to (d) by Corollary 6.

Remark 8 *For* $p > 1$ *and* $1 \leq q < \infty$*, in general* $\ell_{\pi_{p,q}}(X) \neq I_p(\ell_q, X)$.

Indeed, recalling that $I_2(X,Y) = \Pi_2(X,Y)$ *for every couple of spaces* X *and* Y *(see Corollary 5.9 in [3]), we conclude that* $\ell_{\pi_{2,1}}(\ell_\infty) \neq I_2(\ell_1, \ell_\infty)$*: By Corollary 6 we have that* $\ell_{\pi_{2,1}}(\ell_\infty) = \ell_\infty(\ell_\infty) \simeq \mathcal{L}(\ell_1, \ell_\infty)$*, but* $\mathcal{L}(\ell_1, \ell_\infty)$ *does not coincide with* $\Pi_2(\ell_1, \ell_\infty)$ *because* $\Pi_2(\ell_1, \ell_\infty) = \Pi_1(\ell_1, \ell_\infty)$ *(for* ℓ_1 *is of cotype 2, and Corollary 11.16 in [3] applies), and on the other hand* $\Pi_1(\ell_1, \ell_\infty) \neq \mathcal{L}(\ell_1, \ell_\infty)$*: the operator given by* $x \in \ell_1 \mapsto (\sum_{j=1}^n x_j)_n \in \ell_\infty$ *is not absolutely summing (see [5], exercise III.F.3).*

Proposition 8 *Let* E *a Banach subspace of* X*. Then we have that* $\ell_{\pi_{p,q}}(E) \subseteq \ell_\infty(E) \cap \ell_{\pi_{p,q}}(X)$*, but equality does not hold in general.*

PROOF. The embedding is straightforward.

Let us show that for $p = q = 1$ there exists E such that $\ell_{\pi_1}(E) \neq \ell_\infty(E) \cap \ell_{\pi_1}(X)$:

Take E such that $\ell_2(E) \not\subseteq \ell_{\pi_1}(E)$ (for instance $E = \ell_1$). Since E is a subspace of $X = \ell_\infty(\Gamma)$ for $\Gamma = B_{E^*}$ and $(\ell_\infty(\Gamma))^* = (\ell_1(\Gamma))^{**}$ is of cotype 2 , then $\ell_2(E) \subseteq \ell_\infty(E) \cap \ell_{\pi_1}(X)$. Therefore $\ell_\infty(E) \cap \ell_{\pi_1}(X)$ does not coincide with $\ell_{\pi_1}(E)$.

3.2 The (p,q)*-summing norm of the canonical basis in* ℓ_r

Theorem 6 *Let* $p > q$ *and* $1/s' = (1/q) - (1/p)$*. Then* $\ell_s^w(X) \subseteq \ell_{\pi_{p,q}}(X)$ *with inclusion of norm* 1.

PROOF. For any finite family of vectors $(x_j)_{1\le j\le n}$ in X and $(x_j^*)_{1\le j\le n}$ in X^*, since $1/p' > 1/q'$ and $1/p' = (1/s') + (1/q')$ we can write

$$\begin{aligned}(\sum_j |x_j^* x_j|^p)^{1/p} &= \sup_{\|(\alpha_j)\|_{p'}=1} |\sum_j \alpha_j x_j^* x_j| \\ &= \sup_{\|(\beta_j)\|_{s'}=1} \sup_{\|(\lambda_j)\|_{q'}=1} |\sum_j \beta_j \lambda_j x_j^* x_j| \\ &= \sup_{\|(\beta_j)\|_{s'}=1} \sup_{\|(\lambda_j)\|_{q'}=1} |\int_0^1 \langle \sum_j r_j(t)\lambda_j x_j^*, \sum_k r_k(t)\beta_k x_k\rangle dt| \\ &\le \sup_{\|(\beta_j)\|_{s'}=1} \sup_{\|(\lambda_j)\|_{q'}=1} \sup_{t\in[0,1]} \|\sum_j r_j(t)\lambda_j x_j^*\|_{X^*} \|\sum_k r_k(t)\beta_k x_k\|_X \\ &\le \|(x_j^*)\|_{\ell_q^w(X^*)} \|(x_j)\|_{\ell_s^w(X)}.\end{aligned}$$

Corollary 8 *For any $p \ge 1$, $\ell_p^w(X) \subset \ell_{\pi_{p,1}}(X)$ with inclusion of norm 1.*

As an application, we see next whether the sequence given by the canonical basis (e_j) belongs to $\ell_{\pi_{p,q}}(\ell_r)$, depending on the values of p, q and r.

Proposition 9 *For any $p \ge 1$ we have $(e_j) \in \ell_{\pi_{p,1}}(\ell_{p'})$, with $\pi_{p,1}[e_j; \ell_{p'}] = 1$.*

PROOF. Note that for $p \ge 2$ this follows from Corollary 6, because $(\ell_{p'})^* = \ell_p$ has cotype p.

For $1 \le p < 2$, apply Corollary 8 to $(e_j) \in \ell_p^w(\ell_{p'})$.

Theorem 7 *$(e_j) \in \ell_{\pi_{p,q}}(\ell_r)$ if and only if it holds that $p = \infty$ or $1/r \le (1/q) - (1/p)$. Moreover, in these cases $\pi_{p,q}[e_j] = 1$.*

PROOF. For $p < q$ we have that $\ell_{\pi_{p,q}}(\ell_r) \subset \ell_{(\frac{1}{p}-\frac{1}{q})^{-1}}(\ell_r)$. Hence $(e_j) \in \ell_{\pi_{p,q}}(\ell_r)$ is only possible for $q \le p$. As the norm of the inclusion $\ell_{q'}^n \to \ell_{r'}^n$ is $n^{(\frac{1}{q}-\frac{1}{r})^+}$, we see that

$$\Big(\sum_{j=1}^n |\langle e_j, e_j\rangle|^p\Big)^{1/p} = n^{\frac{1}{p}} \le \pi_{p,q}[e_j] n^{(\frac{1}{q}-\frac{1}{r})^+},$$

which leads to $p = \infty$ or $q < r$ with $0 \le 1/q - 1/p - 1/r$.

Conversely, if $p = \infty$ then $(e_j) \in \ell_\infty(\ell_r) = \ell_{\pi_{\infty,q}}(\ell_r)$. And if $1/q - 1/p - 1/r \ge 0$ then, by Proposition 9 and Theorem 4, we obtain

$$(e_j) \in \ell_{\pi_{r',1}}(\ell_r) \subseteq \ell_{\pi_{p,q}}(\ell_r).$$

The inclusion above is of norm 1, so $\pi_{p,q}[e_j] = 1$ when bounded.

This gives a new proof of the well-known fact that id: $\ell_p \hookrightarrow \ell_q$ is integral if and only if $p = 1$ and $q = \infty$, according to Theorem 2.

4 (p, q)-summing sequences and Grothendieck theorem

Theorem 8 *Let X be a Banach space. Then*

$$\ell_{\pi_{1,2}}(X) \subseteq \mathrm{Rad}(X) \subseteq \ell_{\pi_1}(X).$$

PROOF. Let us take a finite family of vectors $(x_j)_{1\leq j\leq n}$ in X. Using that $L_1([0,1], X)$ isometrically embedds into the dual of $L_\infty([0,1], X^*)$, we have

$$\begin{aligned}
\int_0^1 \Big\| \sum_{k=1}^n x_k r_k(t) \Big\| dt &= \sup_{\|g\|_{L^\infty([0,1],X^*)}=1} \Big| \sum_{k=1}^n \langle x_k \int_0^1 g(t) r_k(t) dt \rangle \Big| \\
&\leq \pi_{1,2}[x_j] \sup_{\|g\|_{L^\infty([0,1],X^*)}=1} \sup_{\|(\alpha_k)\|_2=1} \Big\| \sum_{k=1}^n \alpha_k \int_0^1 g(t) r_k(t) dt \Big\| \\
&= \pi_{1,2}[x_j] \sup_{\|g\|_{L^\infty([0,1],X^*)}=1} \sup_{\|(\alpha_k)\|_2=1} \Big\| \int_0^1 (\sum_{k=1}^n \alpha_k r_k(t)) g(t) dt \Big\| \\
&= \pi_{1,2}[x_j] \sup_{\|(\alpha_k)\|_2=1} \int_0^1 \Big| \sum_{k=1}^n \alpha_k r_k(t) \Big| dt \\
&\leq \pi_{1,2}[x_j].
\end{aligned}$$

On the other hand, for any finite family of vectors $(x_j)_{1\leq j\leq n}$ in X and $(x_j^*)_{1\leq j\leq n}$ in X^* we can write

$$\begin{aligned}
\sum_{j=1}^n |x_j^* x_j| &\sim \sup_{\varepsilon_k=\pm1} \Big| \sum_{j=1}^n \langle x_j^*, \varepsilon_j x_j \rangle \Big| \\
&= \sup_{\varepsilon_k=\pm1} \Big| \int_0^1 \langle \sum_{j=1}^n \varepsilon_j x_j^* r_j(t), \sum_{j=1}^n x_j r_j(t) \rangle dt \Big| \\
&\leq \Big\| \sum_{j=1}^n x_j r_j \Big\|_{\mathrm{Rad}(X)} \|(x_j^*)\|_{\ell_1^w(X^*)}
\end{aligned}$$

This gives the other inclusion.

By Khintchine inequalities one sees that $L_1(\mu, \ell_2) = \mathrm{Rad}(L_1(\mu))$, and Theorem 3 gives that $\ell_{\pi_{1,2}}(L_1(\mu)) = \mathrm{Rad}(L_1(\mu))$. Actually, combining Theorem 8 with Pisier's results on G.T. spaces (see Theorem 6.6 and Corollary 6.7 in [4]) it is easy to prove the following:

Theorem 9 $\mathrm{Rad}(X) = \ell_{\pi_{1,2}}(X)$ *if and only if* X *is a G.T. space of cotype* 2.

Grothendieck theorem has been stated in a lot of equivalent ways. We shall give yet another formulation of it in terms of the $\ell_{\pi_{p,q}}$ spaces. It gives a partial answer to a general question about the way that bounded sequences in X^* interact with bounded sequences in X.

For any Banach space X, let us consider the bilinear map

$$V_X \colon \ell_\infty(X^*) \times \ell_\infty(X) \to \ell_\infty(\ell_\infty)$$

given by $V_X\Big((x_j^*), (x_k)\Big) = \Big((x_j^* x_k)_k\Big)_j$. It is obvious that V_X is bounded.

Note that, for the restricted map $V_{n,X} \colon \ell_\infty^n(X^*) \times \ell_\infty^n(X) \to M_n(\mathbb{K})$ (defined in the same way), it always holds that the linear span of the image is $M_n(\mathbb{K})$. Actually, for $X = \mathbb{K}$,

$$(\alpha_{j,k}) = \sum_{j=1}^n \sum_{k=1}^n V_n(\alpha_{j,k} e_j, e_k).$$

It is also easy to observe that $V_{\ell_1}(\ell_\infty(\ell_\infty) \times \ell_\infty(\ell_1)\Big) = \ell_\infty(\ell_\infty)$: for any uniformly bounded infinite matrix $(\alpha_{j,k})$, if we set $x_j^* = (\alpha_{j,k})_k \in \ell_\infty$ then

$$(\alpha_{j,k}) = V_{\ell_1}\Big((x_j^*)_j, (e_k)_k\Big).$$

However, for other Banach spaces the bilinear map is actually bounded not only into $\ell_\infty(\ell_\infty)$, but into a smaller space. This is the case for ℓ_p if $1 < p < \infty$:

Theorem 10 *Given* $1 \le q \le p$, $\Pi_{p,q}(\ell_1, X) = \mathcal{L}(\ell_1, X)$ *if and only if* V_X *defines a bounded bilinear map* $V_X \colon \ell_\infty(X^*) \times \ell_\infty(X) \to \ell_{\pi_{p,q}}(\ell_\infty)$.

PROOF. Let $(x_j) \subset X$ and $(x_j^*) \subset X^*$ be such that $\|x_j\|, \|x_j^*\| \le 1$ for all j. Let $u \colon \ell_1 \to X$ the continuous operator such that $ue_j = x_j$ for all j; clearly $\|u\| \le 1$.

By hypothesis we can take C (independently of (x_j)) such that $\pi_{p,q}(u) \le C\|u\| \le C$. That is,

$$\|(uy_j)\|_{\ell_p(X)} \le C\|(y_j)\|_{\ell_q^w(\ell_1)}$$

for any finite family $(y_j) \subset \ell_1$. Therefore if $\xi_j = x_j^* \circ u$ for each j then

$$((\langle \xi_j, e_k \rangle)_k)_j = ((x_j^*(ue_k))_k)_j = ((x_j^* x_k)_k)_j = V_X((x_j^*), (x_j)).$$

Consequently

$$\|(\langle \xi_j, y_j \rangle)\|_{\ell_p} = \|(\langle x_j^*, uy_j \rangle)\|_{\ell_p} \le \|(uy_j)\|_{\ell_p(X)},$$

and then

$$\|(\langle \xi_j, y_j \rangle)\|_{\ell_p} \leq C\|(y_j)\|_{\ell_q^w(\ell_1)},$$

showing that $\pi_{p,q}[\xi_j; \ell_\infty] \leq C$.

Let us assume now that $V_X \colon \ell_\infty(X^*) \times \ell_\infty(X) \to \ell_{\pi_{p,q}}(\ell_\infty)$ is bounded with norm C. Given $u \in \mathcal{L}(\ell_1, X)$, for every finite family $(y_j) \in \ell_1$ we have that

$$\begin{aligned}\|(uy_j)\|_{\ell_p(X)} &= \sup\{\|(\langle x_j^*, uy_j \rangle)\|_{\ell_p} : \ (x_j^*) \subset B_{X^*}\} \\ &\leq \sup\{\pi_{p,q}[V_X((x_j^*), (ue_j)); \ell_\infty] : \ (x_j^*) \subset B_{X^*}\} \, \|(y_j)\|_{\ell_q^w(\ell_1)} \\ &\leq C\|u\| \, \|(y_j)\|_{\ell_q^w(\ell_1)},\end{aligned}$$

and then $\pi_{p,q}(u) \leq C\|u\|$.

In view of this, Grothendieck theorem is equivalent to the following result:

Corollary 9 *If H is a Hilbert space, the bilinear form*

$$V_H \colon \ell_\infty(H) \times \ell_\infty(H) \to \ell_{\pi_1}(\ell_\infty)$$

is bounded, and its norm is Grothendieck constant K_G.

Taking $H = \ell_2$ (with no loss of generality), this is a particular case of the following result:

Corollary 10 *If $1 \leq p \leq \infty$ and $1/r = 1 - |(1/p) - (1/2)|$, then the bilinear form*

$$V_{\ell_p} \colon \ell_\infty(\ell_{p'}) \times \ell_\infty(\ell_p) \to \ell_{\pi_{r,1}}(\ell_\infty)$$

is bounded, with $\|V_{\ell_p}\| \leq 2^{\frac{a}{2}} K_G^{1-a}$, where $a = |1 - 2/p|$.

PROOF. Equivalently

$$\pi_{r,1}(u) \leq 2^{\frac{a}{2}} K_G^{1-a} \|u\|$$

for every operator $u \in \mathcal{L}(\ell_1, \ell_p)$, which is an extension, due to Kwapień, of Grothendieck theorem (see [9], and also 34.11 in [6]).

Remark 9 *Note in the previous result that $1 \leq r \leq 2$. The case $r = 2$ is for $p = 1$ (or $p = \infty$). By Corollary 6 we know that $\ell_{\pi_{2,1}}(\ell_\infty) = \ell_\infty(\ell_\infty)$, so the statement is trivial in this case. However, Corollary 6 tells us that for $r < 2$ the inclusion $\ell_{\pi_{r,1}}(\ell_\infty) \subseteq \ell_\infty(\ell_\infty)$ is proper.*

References

[1] J. L. Arregui and O. Blasco, On summing sequences of operators (to appear)

[2] S. Aywa and J. H. Fourie, On summing multipliers and applications *J. Math. Anal. Appl.* 253 (2001), 166–186, doi:10.1006/jmaa.2000.7081.

[3] J. Diestel, H. Jarchow and A. Tonge, "Absolutely Summing Operators" Cambridge University Press, 1995.

[4] G. Pisier, "Factorization of Operators and Geometry of Banach Spaces", CBM 60, Amer. Math. Soc., Providence (R.I.), 1985.

[5] P. Wojtaszczyk, "Banach spaces for analysts" Cambridge University Press, 1991.

[6] A. Defant and K. Floret, "Tensor Norms and Operator Ideals", North-Holland, 1993.

[7] M. Talagrand, Cotype of operators from $C(K)$, *Inventiones Math.* 107, (1992), 1–40.

[8] M. Talagrand, Cotype and $(q,1)$-summing norm in a Banach space, *Inventiones Math.* 110 (1992), 545–556.

[9] S. Kwapień, Some remarks on (p,q)-summing operators in ℓ_p-spaces *Studia Math.*, 29 (1968), 327–337.