(p,q)-SUMMING SEQUENCES OF OPERATORS.

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Abstract. A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $\mathcal{L}(X,Y)$ is a (p,q)-*summing* multiplier (or (p,q)-summing sequence of operators), in short $(u_j)\in\ell_{\pi_{p,q}}(X,Y)$, if there exists a constant $C>0$ such that, for any finite collection of vectors $x_1,x_2,\dots x_n$ in X, it holds that

$$\Big(\sum_{j=1}^{n}\|u_jx_j\|^p\Big)^{1/p}\le C\sup\Big\{\Big(\sum_{j=1}^{n}|x^*x_j|^q\Big)^{1/q}\ ;\ x^*\in B_{X^*}\Big\}.$$

Some examples of these operators, inclussions between the spaces and connections with spaces of multipliers are presented.

§1 Introduction.

Let X and Y be two real or complex Banach spaces and let $E(X)$ and $F(Y)$ be two Banach spaces whose elements are defined by sequences of vectors in X and Y (containing any eventually null sequence in X or Y). A sequence of operators $(u_n)\in\mathcal{L}(X,Y)$ is called a *multiplier sequence* from $E(X)$ to $F(Y)$ if there exists a constant $C>0$ such that

$$\big\|(u_jx_j)_{j=1}^n\big\|_{F(Y)}\le C\big\|(x_j)_{j=1}^n\big\|_{E(X)}$$

for all finite families $x_1,\dots,x_n$ in X.

The set of all of multiplier sequences is denoted by $(E(X),F(Y))$.

For the study of such multipliers for the cases of $E(X)$ and $F(Y)$ corresponding to vector-valued Hardy spaces, vector-valued Bergman spaces, vector-valued $BMOA$ or spaces of vector valued Bloch functions the reader is referred to [AB1, Bl1, Bl2, Bl3, Bl4].

Given a real or complex Banach space X and $1\le p\le\infty$, we denote by $\ell_p(X)$, $\ell_p^w(X)$ and $\ell_p\langle X\rangle$ the Banach spaces of sequences in X with norms $\|(x_j)\|_{\ell_p(X)}=\|(\|x_j\|)\|_{\ell_p}$, $\|(x_j)\|_{\ell_p^w(X)}=\sup_{x^*\in B_{X^*}}\|(x^*x_j)\|_{\ell_p}$ and $\|(x_j)\|_{\ell_p\langle X\rangle}=\sup\{\|(x_j^*x_j)\|_{\ell_1}:\|(x_j)^*\|_{\ell^w_{p'}(X^*)}=1\}$ respectively. The space $\ell_p\langle X\rangle$ was first introduced in [C] and recently it has been described in different ways (see [AB] for a description as the space of integral operators from $\ell_{p'}$ into X or [BD] and [FR] for the identification with the projective tensor product $\ell_p\hat{\otimes}X$).

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The aim of this paper is to consider the cases where $E(X)$ or $F(X)$ correspond to the spaces $\ell_p(X)$, $\ell_p^w(X)$ or $\ell_p\langle X\rangle$. The study of such multipliers was iniciated in [B] for the case $E(X) = \ell_p^w(X)$, and $F(Y) = \ell_p(Y)$ where several examples and results were achieved. The particular case $E(X) = \ell_p^w(X)$ and $F(Y) = \ell_q(\mathbb{K})$ corresponds to the notion of (p,q)-summing sequences studied in [AB], and the case $u_j = \lambda_j I$, where I stands for the identity operator on a Banach space, was considered in [AF] and [FR].

If $1 \le p \le q < \infty$, the space $\Pi_{p,q}(X,Y)$ of *(p,q)-summing operators* is formed by those operators $u : X \to Y$ mapping sequences in $\ell_q^w(X)$ into sequences in $\ell_p(Y)$, in other words $u \in \Pi_{p,q}$ if there exists C such that

$$\|(ux_j)\|_{\ell_p(Y)} \le C\|(x_j)\|_{\ell_q^w(X)}$$

for any finite family of vectors x_j in X, and the least of such C is the (p,q)-summing norm of u, denoted by $\pi_{p,q}(u)$. This, in our terminology, means that (u_j) belongs to $(\ell_q^w(X), \ell_p(Y))$ if $u_j = u$ for all n.

If we set $u_j = \lambda_j u$ then $(u_j) \in (\ell_q^w(X), \ell_1(Y))$ for all $(\lambda_j) \in \ell_{p'}$, where $(1/p) + (1/p') = 1$, if and only if $u \in \Pi_{p,q}(X,Y)$. These facts suggest the use of the notation $\ell_{\pi_{p,q}}(X,Y)$ instead of $(\ell_q^w(X), \ell_p(Y))$ and $\ell_{\pi_p}(X,Y)$ for $q = p$.

A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $\mathcal{L}(X,Y)$ is a (p,q)-*summing* multiplier, in short $(u_j) \in \ell_{\pi_{p,q}}(X,Y)$, if there exists a constant $C > 0$ such that, for any finite collection of vectors $x_1, x_2, \dots x_n$ in X, it holds that

$$\Big(\sum_{j=1}^n \|u_j x_j\|^p\Big)^{1/p} \le C \sup\Big\{\Big(\sum_{j=1}^n |x^* x_j|^q\Big)^{1/q}\ ;\ x^* \in B_{X^*}\Big\}.$$

The basic theory of p-summing and (p,q)-summing operators can be found, for example, in the books [DJT], [DF], [J], [TJ], [Pi] or [W].

The reader is referred to [AF] for the particular case $p = q$, $X = Y$ and $u_j = \alpha_j\, I$. A scalar sequence (α_j) is there defined to be a p-summing multiplier if $u_j = \alpha_j I$ belongs to $\ell_{\pi_p}(X,Y)$.

In [AB] it was considered the case $Y = \mathbb{K}$, what lead to define a new family of spaces of vector valued sequences, not only for dual spaces, that were called spaces of (p,q)-*summing sequences* in X.

For any Banach space X, it was defined the space $\ell_{\pi_{p,q}}(X)$ as the set of all sequences (x_j) in X such that there exists a constant $C > 0$ for which

$$\Big(\sum_{j=1}^n |x_j^* x_j|^p\Big)^{1/p} \le C \sup\Big\{\Big(\sum_{j=1}^n |x_j^* x|^q\Big)^{1/q}\ ;\ x \in B_X\Big\}$$

for any finite collection of vectors $x_1^*, \dots, x_n^*$ in X^*. The reader should notice that $\ell_p\langle X\rangle = \ell_{\pi_{1,p'}}(X)$ and that a sequence $(x_j) \in \ell_{\pi_{p,q}}(X)$, considered as operators in $\mathcal{L}(X^*, \mathbb{K})$, corresponds to a (p,q)-summing multiplier. The main objective of such a notion was to describe some classical aspects of the theory of geometry of Banach spaces and operator ideals, from the point of view of those sequence spaces.

The aim of the paper is to get some descriptions for particular cases of multipliers, to deal with the special case of (p,q)-summing multipliers and to get inclusions between them. These objectives are done in sections 2, 3 and 4 respectively.

Notation is fairly standard. We follow the usual terms $\mathcal{L}(X,Y)$ for the space of bounded linear operators between Banach spaces, B_X and S_X for the unit ball and sphere in X, $X \sim Y$ if two Banach spaces are isomorphic and $X \simeq Y$ if they are isometrically isomorphic. We write the action of an operator or functional on x merely as ux and x^*x, though we prefer to use $x^*(x)$ or $\langle x^*, x\rangle$ sometimes; p' denotes the conjugate exponent of p, $x^+ = \max\{x, 0\}$ and $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$ if no difference is relevant.

§2 Identifications of some spaces of multipliers.

In [BS] it was considered another intermediate space of sequences of operators, by using the strong operator topology.

Let us define for $1 \le p < \infty$ the space $\ell_p^s(\mathcal{L}(X,Y))$ as

$$\{(u_j) : u_j\colon X \to Y \text{ linear and bounded}, \sum_j \|u_j(x)\|^p < \infty \text{ for } x \in X\}.$$

We endow it with the norm $\|(u_j)\|_{\ell_p^s(\mathcal{L}(X,Y))} = \sup_{\|x\|=1}(\sum_j \|u_j(x)\|^p)^{1/p}$.

Of course we have

$$\ell_p(\mathcal{L}(X,Y)) \subset \ell_p^s(\mathcal{L}(X,Y)) \subset \ell_p^w(\mathcal{L}(X,Y)).$$

We shall see that these spaces of operators actually correspond to certain spaces of multipliers.

Proposition 2.1. *Let X and Y be Banach spaces, and $1 \le p, q \le \infty$. For $1/r = ((1/p) - (1/q))^+$ we have that*

$$(\ell_q(X), \ell_p(Y)) = \ell_r(\mathcal{L}(X,Y)).$$

Proof. Any multiplier sequence (u_j) must be in $\ell_\infty(\mathcal{L}(X,Y))$, as we see by taking sequences in X of the form $(0, \dots, 0, x_j, 0, 0, \dots)$. If $q \le p$ it is plain that the converse is true.

Let $q > p$ and $1/p = (1/r) + (1/q)$. By Hölder's inequality,

$$(\sum_j \|u_j x_j\|^p)^{1/p} \le (\sum_j \|u_j\|^p \|x_j\|^p)^{1/p} \le (\sum_j \|u_j\|^r)^{1/r} (\sum_j \|x_j\|^q)^{1/q}.$$

Conversely, given n we note that the $\ell_{r/p}$-norm of $(\|u_j\|^p)_{j=1}^n$ equals, by duality, the norm of $(\lambda_j \|u_j\|^p)$ in ℓ_1 for some $0 \le \lambda_j$ such that $\sum_j \lambda_j^{q/p} = 1$. Let $\beta_j = \lambda_j^{1/p}$ and $x_j \in S_X$ such that $\|u_j x_j\|$ is arbitrarily close to $\|u_j\|$; then $(\sum_j \|u_j(\beta_j x_j)\|^p)^{1/p}$ approximates $(\sum_j \|u_j\|^p \beta_j^p)^{1/p}$, and hence $\left(\sum_{j=1}^n \|u_j\|^r\right)^{1/r}$ is bounded by a constant independent of n. $\square$

We recall the following crucial description of $\ell_p\langle X\rangle$ to be used in the sequel.

Lemma 2.1. *(see* [BD],[FR]*) Let X be a Banach space, and $1 \le p < \infty$. Then $\ell_p\langle X\rangle = \ell_{\pi_{1,p'}}(X) = \ell_p \hat{\otimes} X$.*

Proposition 2.2. *Let X and Y be Banach spaces, and $1 \leq p, q \leq \infty$. For $1/r = ((1/p) - (1/q))^+$ we have that*

$$(\ell_q\langle X\rangle, \ell_p^w(Y)) = \ell_r^w(\mathcal{L}(X,Y)).$$

Proof. Only the case $p < q$ needs a proof. Let $1/p = (1/r) + (1/q)$. Observe first that $(u_j) \in \ell_r^w(\mathcal{L}(X,Y))$ if and only if

$$\sup_{\|x\|=1, \|y^*\|=1} \sum_{j=1}^{\infty} |\langle u_j(x), y^*\rangle|^r < \infty.$$

Let $(u_j) \in \ell_r^w(\mathcal{L}(X,Y))$ and let $x_j = \lambda_j x$ where $(\lambda_j) \in \ell_q$ and $x \in X$. To show that $u_j(x_j) \in \ell_p^w(Y)$ let us take $y^* \in Y^*$. By Hölder's inequality

$$(\sum_j |\langle u_j x_j, y^*\rangle|^p)^{1/p} \leq (\sum_j |\langle u_j x, y^*\rangle|^p |\lambda_j|^p)^{1/p} \leq (\sum_j |\langle u_j x, y^*\rangle|^r)^{1/r} \|(\lambda_j)\|_q.$$

Now use Lemma 2.1 to extend to $\ell_q\langle X\rangle$ by continuity.

For the converse, assume $(u_j) \in (\ell_q\langle X\rangle, \ell_p^w(Y))$. Since for any $\|x\| = 1$ and $\|y^*\| = 1$

$$(\sum_{j=1}^{\infty} |\langle u_j(x), y^*\rangle|^r)^{1/r} = \sup\{(\sum_j |\langle u_j x, y^*\rangle|^p |\lambda_j|^p)^{1/p} : \|(\lambda_j)\|_q = 1\},$$

we obtain, by writing $x_j = \lambda_j x$, where $(\lambda_j) \in \ell_q$ and $x \in X$, which belongs to $\ell_q\langle X\rangle$ by Lemma 2.1, that

$$(\sum_{j=1}^{\infty} |\langle u_j(x), y^*\rangle|^r)^{1/r} \leq \|(u_j)\|_{(\ell_q\langle X\rangle, \ell_p^w(Y))} \sup\{\|(x_j)\|_{\ell_q\langle X\rangle} : \|(\lambda_j)\|_q = 1\} \leq C.$$

□

Proposition 2.3. *Let X, Y be Banach spaces, $1 \leq p, q < \infty$ and $1/r = ((1/p) - (1/q))^+$. Then*

$$\big(\ell_q\langle X\rangle, \ell_p(Y)\big) = \ell_r^s(\mathcal{L}(X,Y)).$$

Proof. Observe that $\big(\ell_q(X), \ell_p(Y)\big) \subset \big(\ell_q\langle X\rangle, \ell_p(Y)\big)$. Hence we may assume again $p < q$ and $(u_j) \in (\ell_q\langle X\rangle, \ell_p(Y))$. For each $x \in X$

$$(\sum_{j=1}^{\infty} \|u_j(x)\|^r)^{1/r} = \sup\{(\sum_j \|u_j(x)\|^p |\lambda_j|^p)^{1/p} : \|(\lambda_j)\|_q = 1\},$$

and then we obtain, writing $x_j = \lambda_j x$, where $(\lambda_j) \in \ell_q$, which belongs to $\ell_q\langle X\rangle$ by Lemma 2.1,

$$(\sum_{j=1}^{\infty} \|u_j(x)\|^r)^{1/r} \leq \|(u_j)\|_{(\ell_q\langle X\rangle, \ell_p(Y))} \sup\{\|(x_j)\|_{\ell_q\langle X\rangle} : \|(\lambda_j)\|_q = 1\} \leq C\|x\|.$$

Conversely, if $(u_j) \in \ell_r^s(\mathcal{L}(X,Y))$ and $x_j = \lambda_j x$ with $(\lambda_j) \in \ell_q$ and $x \in X$, then

$$(\sum_j \|u_j(x_j)\|^p)^{1/p} \leq (\sum_j \|u_j(x)\|^p |\lambda_j|^p)^{1/p} \leq (\sum_j \|u_j x\|^r)^{1/r} \|(\lambda_j)\|_q.$$

Now use Lemma 2.1 to extend to $\ell_q\langle X\rangle$ by continuity. □

There is still another case that is rather simple to describe.

Proposition 2.4. *Let X and Y be Banach spaces, $u_j \in \mathcal{L}(X,Y)$ for $j \in \mathbb{N}$, $1 \le p, q \le \infty$ and $1/r = ((1/p) - (1/q))^+$.*

Then $(u_j) \in (\ell_q(X), \ell_p^w(Y))$ if and only if $(u_j^*) \in \ell_r^s(\mathcal{L}(Y^*, X^*))$.

Proof. The case $p \ge q$ is rather direct. Let us assume $p < q$, $(u_j) \in (\ell_q(X), \ell_p^w(Y))$ and $y^* \in Y^*$. Then

$$(\sum_j \|u_j^*(y^*)\|^r)^{1/r} = \sup\{(\sum_j |\langle x_j, u_j^*(y^*)\rangle|^p)^{1/p} : (\sum_j \|x_j\|^q)^{1/q} = 1\}$$

$$= \sup\{(\sum_j |\langle u_j(x_j), y^*\rangle|^p)^{1/p} : (\sum_j \|x_j\|^q)^{1/q} = 1\} \le C\|y^*\|.$$

Assume $(u_j^*) \in \ell_r^s(\mathcal{L}(Y^*, X^*))$. By Hölder's inequality

$$(\sum_j |\langle u_j x_j, y^*\rangle|^p)^{1/p} = (\sum_j |\langle x_j, u_j^*(y^*)\rangle|^p)^{1/p} \le (\sum_j \|u_j^*(y^*)\|^r)^{1/r} (\sum_j \|x_j\|^q)^{1/q}.$$

□

§3 (p,q)-summing sequences of operators.

The study of multiplier sequences between $\ell_q^w(X)$ and $\ell_p(Y)$ is far more complicated. For the reason explained in the introduction, we find more convenient to change the notation from $(\ell_q^w(X), \ell_p(Y))$ to the following:

Definition 3.1. *Let X and Y be Banach spaces, and let p, $q \ge 1$. A sequence $(u_j)_{j\in\mathbb{N}}$ of operators in $\mathcal{L}(X,Y)$ is a (p,q)-summing multiplier if there exists a constant $C > 0$ such that, for any finite collection of vectors $x_1, x_2, \dots x_n$ in X, it holds that*

$$\Big(\sum_{j=1}^n \|u_j x_j\|^p\Big)^{1/p} \le C \sup\Big\{\Big(\sum_{j=1}^n |x^* x_j|^q\Big)^{1/q} ;\ x^* \in B_{X^*}\Big\}.$$

We use $\ell_{\pi_{p,q}}(X,Y)$ to denote the set of (p,q)-summing multipliers, and $\pi_{p,q}[u_j]$ is the least constant C for which (u_j) verifies the inequality in the definition. In order to avoid ambiguities, sometimes we shall use $\pi_{p,q}[u_j; X, Y]$. Of course if $p = q$ we simply say that the sequence (u_j) is a p-summing multiplier and write $\ell_{\pi_p}(X,Y)$, $\pi_p[u_j; X, Y]$ (see [Bl]).

Remarks 3.1.

1. The obvious modifications for $p = \infty$ or $q = \infty$ make sense, but then

$$\ell_{\pi_{p,\infty}}(X,Y) = \ell_p(\mathcal{L}(X,Y)) \text{ and } \ell_{\pi_{\infty,q}}(X,Y) = \ell_\infty(\mathcal{L}(X,Y)).$$

2. Let $u \ne 0$ be a bounded linear operator between two Banach spaces X and Y. If u maps sequences $(x_j) \in \ell_q(X)$ into sequences $(ux_j) \in \ell_p(Y)$ then necessarily $q \le p$ (for $q > p$ one can take $x_j = (1/j)^{1/p} x$, where $x \notin \mathrm{Ker}(u)$, to get a contradiction).

This example shows that $\Pi_{p,q}(X,Y) = \{0\}$ if $p < q$, but in our setting if $c_{00}(\mathcal{L}(X,Y))$ stands for all sequences of operators with a finite number of non-zero elements, then, for any $1 \le p,q \le \infty$, one gets

$$c_{00}(\mathcal{L}(X,Y)) \subset \ell_{\pi_{p,q}}(X,Y).$$

Actually, if $u_j = 0$ for all $j > N$ then $\pi_{p,q}[u_j] \le N^{1/p} \max_{j \le N} ||u_j||$.

3. For any Banach space X, and the usual identification between X and $\mathcal{L}(\mathbb{K}, X)$, it follows from Proposition 2.1 that if $1/r = ((1/p) - (1/q))^+$ then

$$\ell_{\pi_{p,q}}(\mathbb{K}, X) = \ell_r(X).$$

4. For any couple of Banach spaces X and Y, $1 \le p,q < \infty$ and $u_j \in \mathcal{L}(X,Y)$, we clearly have

$$(u_j) \in \ell_{\pi_{p,q}}(X,Y) \text{ if and only if } (\lambda_j u_j) \in \ell_{\pi_{1,q}}(X,Y) \text{ for all } (\lambda_j) \in \ell_{p'}.$$

Moreover

$$\pi_{p,q}[u_j; X, Y] = \sup\{\pi_{1,q}[\lambda_j u_j; X, Y] : ||\lambda_j||_{\ell_{p'}} = 1\}.$$

5. Note that if X, Y are Banach spaces and $u_j \in \mathcal{L}(X,Y)$ then

$$\pi_{p,q}[u_j] = \sup_{\|y_j^*\|=1} \pi_{p,q}[y_j^* u_j; X^*]\,.$$

Let us mention that the characterization of the absolutely summing operators in terms of unconditional series can be generalized as follows, with the same standard proof (see [DJT]):

Proposition 3.2. *A sequence (u_j) is in $\ell_{\pi_{p,1}}(X,Y)$ if and only if it holds that for any unconditionally convergent series $\sum x_j$ in X we have $(u_j x_j)_j \in \ell_p(Y)$.*

The subspace of $\mathcal{L}(\ell_q^w(X), \ell_p(Y))$ formed by (p,q)-summing sequences of operators is closed, and then the summing norm $\pi_{p,q}$ is complete:

Proposition 3.3. *For any X and Y Banach spaces, and for any $1 \le p,q < \infty$, $(\ell_{\pi_{p,q}}(X,Y), \pi_{p,q})$ is a Banach space.*

Easy examples can be constructed by tensoring some elements in classical spaces.

Examples 3.1. *Let X and Y be Banach spaces, and $1 \le p,q \le \infty$.*
(1) $\ell_{\pi_{r,q}}(X,\mathbb{K}) \hat{\otimes} \ell_s(Y) \subset \ell_{\pi_{p,q}}(X,Y)$ for $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.
(2) $\ell_s \hat{\otimes} \Pi_{r,q}(X,Y) \subset \ell_{\pi_{p,q}}(X,Y)$ for $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$.
In particular $\ell_p \hat{\otimes} X \subset \ell_{\pi_{1,p'}}(X) = \ell_p\langle X \rangle$.
(3) $\ell_s(Y) \hat{\otimes} X^* \subset \ell_{\pi_{p,q}}(X,Y)$ for $p < q$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{s}$.

Proof. (1) Take $u_j = x_j^* \otimes y_j$ where $(x_j^*) \in \ell_{\pi_{r,q}}(X,\mathbb{K})$ and $(y_j) \in \ell_s(Y)$. If $(x_j) \in \ell_q^w(X)$ then $(\langle x_j^*, x_j \rangle) \in \ell_r(\mathbb{K})$. Hence $(u_j(x_j)) = (\langle x_j^*, x_j \rangle y_j) \in \ell_p(Y)$.

(2) Take $u_j = \lambda_j u$ where $u \in \Pi_{r,q}(X,Y)$ and $(\lambda_j) \in \ell_s(\mathbb{K})$. If $(x_j) \in \ell_q^w(X)$ then $(u(x_j)) \in \ell_r(Y)$. Hence $(u_j(x_j)) = (\lambda_j u(x_j)) \in \ell_p(Y)$.

(3) Take $u_j = x^* \otimes y_j$ where $x^* \in X^*$ and $(y_j) \in \ell_s(Y)$. If $(x_j) \in \ell_q^w(X)$ then $(\langle x^*, x_j \rangle) \in \ell_q(\mathbb{K})$. Hence $(u_j(x_j)) = (\langle x^*, x_j \rangle y_j) \in \ell_p(Y)$. □

Theorem 3.1. *Let X, Y be Banach spaces and $1 < p$. Then*

$$\ell_p^s(\mathcal{L}(X,Y)) \subset \ell_{\pi_{p,1}}(X,Y).$$

Proof. Let $u_1, ..., u_n \in \mathcal{L}(X,Y)$ and $x_1, ..., x_n \in \ell_1^w(X)$. Then

$$\sum_{j=1}^n \|u_j(x_j)\|^p = \sup\{\big|\sum_{j=1}^n \langle u_j(x_j), y_j^*\rangle\big| : \sum_{j=1}^n \|y_j^*\|^{p'} = 1\}$$

$$= \sup\{\big|\int_0^1 \langle \sum_{j=1}^n x_j r_j(t), \sum_{j=1}^n u_j^*(y_j^*) r_j(t)\rangle dt\big| : \sum_{j=1}^n \|y_j^*\|^{p'} = 1\}$$

$$\leq \|(x_j)\|_{\ell_1^w(X)} \sup\{\int_0^1 \|\sum_{j=1}^n u_j^*(y_j^*) r_j(t)\| dt : \sum_{j=1}^n \|y_j^*\|^{p'} = 1\}$$

$$\leq \|(x_j)\|_{\ell_1^w(X)} \sup\{\big|\langle \sum_{j=1}^n u_j^*(y_j^*) r_j(t), x\rangle\big| : \sum_{j=1}^n \|y_j^*\|^{p'} = 1, \|x\| = 1, t \in [0,1]\}$$

$$\leq \|(x_j)\|_{\ell_1^w(X)} \sup\{\big|\sum_{j=1}^n \langle u_j(x) r_j(t), y_j^*\rangle\big| : \sum_{j=1}^n \|y_j^*\|^{p'} = 1, \|x\| = 1, t \in [0,1]\}$$

$$\leq \|(x_j)\|_{\ell_1^w(X)} \sup\{(\sum_{j=1}^n \|u_j(x)\|^p)^{1/p} : \|x\| = 1\}.$$

□

We finish this section with a result on multipliers in $\big(\ell_{\pi_{p,q}}(X), \ell_p(Y)\big)$, showing that these spaces coincide for any $1 \leq p \leq q$:

Theorem 3.2. *Let X, Y be Banach spaces and $1 \leq p \leq q$. Then*

$$\big(\ell_{\pi_{p,q}}(X), \ell_p(Y)\big) = \ell_q^s(\mathcal{L}(X,Y)).$$

Proof. Assume first that $(u_j) \in \big(\ell_{\pi_{p,q}}(X), \ell_p(Y)\big)$; let r such that $1/p = (1/r) + (1/q)$. Given $x \in X$, we may write

$$\|(u_j x)\|_q = \|(u_j \alpha_j x)\|_p$$

for some numbers (α_j) such that $\|(\alpha_j)\|_r = 1$. Now the assumption and (2) in Examples 3.1 give

$$\|(u_j x)\|_q \leq \|(u_j)\|_{(\ell_{\pi_{p,q}}(X), \ell_p(Y))} \pi_{p,q}[\alpha_j x] = \|(u_j)\|_{(\ell_{\pi_{p,q}}(X), \ell_p(Y))} \|x\|.$$

Conversely, let $u_j \in \ell_q^s(\mathcal{L}(X,Y))$ and (x_j) such that $\pi_{p,q}[x_j] \leq 1$. Then

$$\|(u_j x_j)\|_p = \|(y_j^*(u_j x_j))\|_p = \|((u_j^* y_j^*) x_j)\|_p$$

for some $y_j^* \in Y^*$ with $\|y_j^*\| = 1$, so

$$\|(u_j x_j)\|_p \leq \pi_{p,q}[x_j] \|(u_j^* y_j^*)\|_{\ell_q^w(X^*)} \leq \sup_{\|x\| \leq 1} \|((u_j^* y_j^*) x)\|_q$$

$$= \sup_{\|x\| \leq 1} \|(y_j^*(u_j x))\|_q \leq \sup_{\|x\| \leq 1} \|(u_j x)\|_q. \quad \square$$

Remark 3.2. *It is known and easy (see* [BS], *Proposition 2.5) that* $\ell_q^s(\mathcal{L}(X,Y) \simeq \mathcal{L}(X,\ell_q(Y))$. *If* $1 \leq p \leq q$ *then*

$$\big(\ell_{\pi_{p,q}}(X), \ell_p(Y)\big) \simeq \mathcal{L}(X, \ell_q(Y)).$$

The isometry is given by mapping $(u_j) \in \big(\ell_{\pi_{p,q}}(X), \ell_p(Y)\big)$ *to the bounded linear operator* $U : X \to \ell_q(Y)$ *defined by* $U(x) = (u_j x)$.

Remark 3.3. *Let* X, Y *be Banach spaces and* $1 \leq q$. *If* $UC(Y)$ *stands for the space of unconditionally convergent series, also identified with the space of compact operators* $\mathcal{K}(c_0, Y)$ *then (see* [FR], *Theorem 3.13)*

$$\big(\ell_q(X), UC(Y)\big) \simeq \mathcal{L}(\ell_q(X), Y).$$

The isometry is given by mapping $(u_j) \in \big(\ell_q\langle X\rangle, UC(Y)\big)$ *to the bounded linear operator defined by* $T_{(u_j)}(x_j) = \sum_j u_j(x_j)$.

§4 Inclusions among the spaces $\ell_{\pi_{p,q}}(X)$.

Let us point out first some elementary embeddings among these spaces.

Proposition 4.1. *Let* $1 \leq r, s < \infty$, $1 \leq p_1 \leq p_2$, $1 \leq q_1 \leq q_2$ *and* $1 \leq p \leq q$. *Then*

$$\ell_{\pi_{p_1,s}}(X,Y) \subseteq \ell_{\pi_{p_2,s}}(X,Y)\,,$$

$$\ell_{\pi_{r,q_2}}(X,Y) \subseteq \ell_{\pi_{r,q_1}}(X,Y)\,,$$

$$\ell_{\pi_p}(X,Y) \subseteq \ell_{\pi_q}(X,Y)$$

with continuous inclusions of norm 1.

In particular, for $1 \leq p, q < \infty$,

$$\ell_{\pi_{1,q}}(X,Y) \subset \ell_{\pi_1}(X,Y) \subset \ell_{\pi_p}(X,Y) \subset \ell_{\pi_{p,1}}(X,Y).$$

Proof. The proofs of the two first embeddings are straighforward.

To see the last one, take $(u_j) \in \ell_{\pi_p}(X,Y)$, $(x_j) \in \ell_q^w(X)$ and $(\lambda_j) \in \ell_r$ where $(1/r) + (1/q) = (1/p)$. Then

$$(\sum_{j=1}^n |\lambda_j u_j(x_j)|^p)^{1/p} \leq \pi_p[u_j] \|(\lambda_j x_j)\|_{\ell_p^w(X)} \leq \pi_p[u_j] \|(x_j)\|_{\ell_q^w(X)} \|(\lambda_j)\|_{\ell_r}.$$

Taking the supremum over the unit ball of ℓ_r we get the result. □

We can actually get a general formulation which cover all the cases above and many more ones. Similar proof was given in [AB], but we include here the modificiation for sequences of operators for the sake of completeness.

Theorem 4.1. *Let X and Y be Banach spaces, $1 \le p \le r$, $1 \le q, s$ and $(1/q) + (1/r) \le (1/p) + (1/s)$. Then $\ell_{\pi_{p,q}}(X,Y) \subseteq \ell_{\pi_{r,s}}(X,Y)$, with continuous inclusion of norm 1.*

Proof. The case $s \le q$ follows from the norm 1 inclusions $\ell_s^w(X^*) \subseteq \ell_q^w(X^*)$ and $\ell_p(X) \subseteq \ell_r(X)$. For $q < s$ and either $r = \infty$ or $s = \infty$ Proposition 2.1 and Remarks 3.1 give the result. So we assume that $q < s$ and that both $r, s < \infty$. Then $1 < r/p, s/q < \infty$; let a and b their conjugate numbers, that is $1 = (1/a) + (p/r) = (1/b) + (q/s)$.

If $\pi_{p,q}[u_j] \le C$, for any finite set of vectors x_j in X we have, for appropiate scalars $\alpha_j \ge 0$ such that $\sum \alpha_j^a = 1$, that

$$\Big(\sum_j \|u_j x_j\|^r\Big)^{1/r} = \Big(\sum_j \|u_j(\alpha_j^{1/p} x_j)\|^p\Big)^{1/p} \le C \sup_{\|x^*\|\le 1} \Big(\sum_j \alpha_j^{q/p} |x^* x_j|^q\Big)^{1/q}.$$

From our assumptions we have that $ap \le bq$, so that $\displaystyle\sum_j \alpha_j^{\frac{q}{p}b} \le 1$, and for any x^* we get, by Hölder's inequality, $\big(\sum_j \alpha_j^{q/p}|x^* x_j|^q\big)^{1/q} \le \big(\sum_j |x^* x_j|^s\big)^{1/s}$. This shows that $\pi_{r,s}[u_j] \le C$. $\square$

Note that, in the scalar-valued case, for $(1/p) - (1/q) = (1/r) - (1/s)$ we have

$$(\ell_p, \ell_q) = (\ell_r, \ell_s).$$

To find cases where $\ell_{\pi_{p,q}}(X,Y) = \ell_{\pi_{r,s}}(X,Y)$ for $(1/q) + (1/r) = (1/p) + (1/s)$ we need the following lemma:

Lemma 4.1. *(see Lemma 3,* [AB]*) Let X be a Banach space and $1 < r < \infty$. Then $\ell_1^w(X) = \ell_r \ell_{r'}^w(X)$ if and only if $\mathcal{L}(c_0, X) = \Pi_r(c_0, X)$.*

Proposition 4.2. *Let X be a Banach space such that $\mathcal{L}(c_0, X) = \Pi_{s'}(c_0, X)$ for some $1 < s < \infty$. Then $\ell_{\pi_{r,s}}(X,Y) \subseteq \ell_{\pi_{p,q}}(X,Y)$ for $1 \le p, q, r, s < \infty$ such that $(1/p) - (1/q) = (1/r) - (1/s)$ and for any Banach space Y.*

Proof. Let us take $(u_j) \in \ell_{\pi_{r,s}}(X,Y)$ and $(x_j) \in \ell_q^w(X)$. To show that $(u_j(x_j)) \in \ell_p$, it suffices to see that for any $(\alpha_j) \in \ell_{q'}$ we get $(\alpha_j u_j(x_j)) \in \ell_u$ where $(1/p) + (1/q') = 1/u$. Given now a sequence $(\alpha_j) \in \ell_{q'}$ we have that $(\alpha_j x_j) \in \ell_1^w(X)$. Using Lemma 4.1 we have that there exists $(\beta_j) \in \ell_{s'}$ and $(y_j) \in \ell_s^w(X)$ so that $\alpha_j x_j = \beta_j y_j$. Therefore $(\alpha_j u_j(x_j)) = (\beta_j u_j(y_j)) \in \ell_{s'} \ell_r = \ell_u$ because $1/u = (1/p) + (1/q') = (1/s') + (1/r)$. $\square$

Combining Theorem 4.1 and Proposition 4.2 we get the main result of this section:

Theorem 4.2. *Let X be a Banach space such that $\mathcal{L}(c_0, X) = \Pi_{s'}(c_0, X)$ for some $1 < s < \infty$ and let Y be any Banach space. Then $\ell_{\pi_{r,s}}(X,Y) = \ell_{\pi_{p,q}}(X,Y)$ for $1 \le p, q, r, s < \infty$ such that $1 \le p \le r$ and $(1/p) - (1/q) = (1/r) - (1/s)$.*

Corollary 4.1. *(see* [Bl]*, Theorem 3.8) If X has cotype 2 and Y is any Banach space then $\ell_{\pi_{p,q}}(X,Y) = \ell_{\pi_{r,2}}(X,Y)$ for any $p \le r$ and $1/q = (1/p) - (1/r) + (1/2)$.*

In particular $\ell_{\pi_1}(X,Y) = \ell_{\pi_2}(X,Y)$ and $\ell_{\pi_{1,q}}(X,Y) = \ell_{\pi_{r,2}}(X,Y)$ for $1/r = (1/q') + (1/2)$.

Proof. Use Lemma 4.1 and the fact that $\mathcal{L}(c_0, Y) = \Pi_2(c_0, Y)$ for any Y of cotype 2. $\square$

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