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A NOTE ON VECTOR-VALUED HARDY AND PALEY INEQUALITIES

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ABSTRACT. The values of p and q for $L_p(L_q)$ that satisfy the extension of Paley and Hardy inequalities for vector-valued H^1 functions are characterized. In particular, it is shown that $L_2(L_1)$ is a Paley space that fails Hardy inequality.

INTRODUCTION

In [BP] the vector-valued analogue of two classical inequalities in the theory of Hardy spaces were investigated. A complex Banach space X is said to be a Paley space if

$$\text{(P)} \qquad \left(\sum_{k=0}^{\infty} \|\hat{f}(2^k)\|^2\right)^{1/2} \leq C\|f\|_1 \quad \text{for all } f \in H^1(X).$$

A complex Banach space X is said to verify vector-valued Hardy inequality (for short X is a (HI)-space) if

$$\text{(H)} \qquad \sum_{n=0}^{\infty} \frac{\|\hat{f}(n)\|}{n+1} \leq C\|f\|_1 \quad \text{for all } f \in H^1(X),$$

where $H^1(X) = \{f \in L^1(\mathbb{T}, X) \colon \hat{f}(n) = 0 \textit{ for } n < 0\}$.

Both inequalities can be regarded in the framework of vector-valued extensions of multipliers from H^1 to l^1. Recall that a sequence (m_n) is a $(H^1 - l^1)$-multiplier, to be denoted by $m_n \in (H^1 - l^1)$, if $T_{m_n}(f) = (\hat{f}(n)m_n)$ defines a bounded operator from H^1 into l^1.

The $(H^1 - l^1)$-multipliers were characterized by C. Fefferman in the following way (see [SW] for a proof):

$$(*) \qquad (H^1 - l^1) = \left\{ m_n \colon \sup_{s\geq 1} \left(\sum_{k\geq 1} \left(\sum_{j=ks+1}^{(k+1)s} |m_j| \right)^2 \right)^{1/2} < \infty \right\}.$$

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A complex Banach space is said to have $(H^1 - l^1)$-Fourier type if

$$\text{(F)} \quad \sum_{n=0}^{\infty} \|\hat{f}(n)\||m_n| \leq C\|f\|_1 \quad \text{for all } f \in H^1(X) \text{ and for all } m_n \in (H^1 - l^1).$$

The reader is referred to [BP] for examples of spaces having and failing these properties and for their connection with the notions of Rademacher type and Fourier type.

Using $(*)$ it is easy to see that any space of $(H^1 - l^1)$-Fourier type must be a Paley and a (HI)-space. Unfortunately the only examples of spaces without $(H^1 - l^1)$-Fourier type that we had at our disposal also behave badly with respect to the other two properties. The problem of finding a Paley space failing Hardy inequality or without $(H^1 - l^1)$-Fourier type was left open (see [BP, Remark 4.1]).

Surprisingly it is enough to deal with Lebesgue spaces of mixed norm, namely, $L_p(L_q)$, to produce a simple example of Paley space failing Hardy inequality. In fact we shall see that $L_2(L_1)$ is such an example.

Given $1 \leq p \leq \infty$, (Ω, Σ, μ) a σ-finite measure space, and a Banach space Y we denote by $L_p(\mu, Y)$ the space of Y-valued strongly measurable functions such that $\|f\| \in L_p(\mu)$.

Throughout the paper $1 \leq p, q \leq \infty$ and we shall use the notation $L_p(L_q) = L_p(\mathbb{T}, L_q(\mathbb{T}))$.

Paley spaces

For self-containedness of the paper, we provide here simple direct proofs of special cases of Corollary 3.2 and Theorem 3.2 of [BP] that show how the Paley property behaves with respect to the vector-valued extension.

Lemma 1. *Let $1 \leq p \leq 2$, (Ω, Σ, μ) be a σ-finite measure space and Y a Paley space. Then $L_p(\mu, Y)$ is a Paley space.*

Proof. The case $p = 2$ is a simple consequence of Fubini's theorem. Let us assume $1 \leq p < 2$ and $q = (2/p)' = 2/(2-p)$. Let us take $f(t) = \sum_{n\geq 0} x_n e^{int}$ where $x_n \in L_p(\mu, Y)$.

$$\left(\sum_{k\geq 0} \|x_{2^k}\|^2_{L_p(\mu, Y)}\right)^{1/2} = \left(\sum_{k\geq 0} \left(\int_\Omega \|x_{2^k}(w)\|_Y^p d\mu(w)\right)^{2/p}\right)^{1/2}$$

$$= \sup_{\sum \alpha_k^q = 1} \left(\sum_{k\geq 0} \int_\Omega \|x_{2^k}(w)\|_Y^p \alpha_k d\mu(w)\right)^{1/p}$$

$$\leq \left(\int_\Omega \left(\sum_{k\geq 0} \|x_{2^k}(w)\|_Y^2\right)^{p/2} d\mu(w)\right)^{1/p}$$

$$\leq C \left(\int_\Omega \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \|\sum_{n\geq 0} x_n(w)e^{int}\|_Y dt\right)^p d\mu(w)\right)^{1/p}$$

Now since Φ is uniformly bounded then $\sup_{0\le r<1} M_1(\|\Phi\|, r) < \infty$. Using the fact that for $1 \le p < \infty$ the Banach space $l_p(H^1)$ is a separable dual by a routine argument, we show that the radial limit $F(z)$ exists almost everywhere and that $F \in H^1(l_p(H^1))$ for $1 < p < \infty$.

On the other hand

$$\|x_n\|_{l_p(H^1)} = a_n \left(\sum_{k=1}^{\infty} 2^{kp(1-p)}(1-\frac{1}{2^k})^{np}\right)^{1/p} \ge a_n \left(\sum_{k\ge \log_2 n}^{\infty} 2^{kp(1-p)}(1-\frac{1}{n})^{np}\right)^{1/p}$$

Since $(1-1/n)^{np}$ converges to e^{-p}, for n big enough we have

$$\|x_n\|_{l_p(H^1)} \ge C_p a_n \left(\sum_{k\ge \log_2 n}^{\infty} 2^{kp(1-p)}\right)^{1/p} \ge C_p a_n n^{1-p}.$$

Now using estimate (1) we get $\sum_{n=1}^{\infty} \|x_n\|_{l_p(H^1)}/(n+1) = \infty$. □

Remark. If $1 < p < \infty$ then $l_p(H_1)$ is a Paley space but is not a (HI)-space. (Hence it does not have $(H^1 - l^q)$-Fourier type.)

Theorem 3. *$L_p(L_q)$ is a (HI)-space if and only if either $1 < p, q < \infty$ or $p = 1$ and $1 \le q < \infty$.*

Proof. Let us first show that under such assumptions on p, q we get (HI)-spaces. It is an application of Fubini's theorem that if Y is a (HI)-space then $L_1(\mu, Y)$ is a (HI)-space. Combining this with the result that every B-convex space (Rademacher type bigger than 1) is a (HI)-space (see [BP, Bo]) we get this implication.

For the other implication observe that the cases $p = \infty$ or $q = \infty$ must be excluded because then $L_p(L_q)$ would contain c_0. The case $q = 1$ follows from Theorem 1, since l_p embbeds into $L_p(\mathbb{T})$ and H^1 into $L_1(\mathbb{T})$. □

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