REMARKS ON VECTOR–VALUED BMOA AND VECTOR–VALUED MULTIPLIERS.

Oscar Blasco

Abstract. In this paper we consider the vector-valued interpretation of the space $BMOA$ defined in terms of Carleson measures and analyze the relationship with the one defined in terms of oscillation. We study the space of multipliers between H^p and $BMOA$ in the vector-valued setting. This leads us to the consideration of some geometric properties depending upon the validity of certain inequalities due to Littlewood and Paley on the g-function for vector-valued functions.

Introduction.

In [B1, B2] the author considered the vector-valued situation of the result by M. Mateljevich and M. Pavlovic ([MP]) which establishes that the space of multipliers between H^1 and $BMOA$ can be identified with the space of Bloch functions, i.e. $(H^1, BMOA) = Bloch$. For such a purpose it was introduced the notion of pairs (X, Y) having the (H^1, BMO)-property for those where the space of multipliers $(H^1(X), BMOA(Y))$, with its natural definition (see Section 3), coincides with $Bloch(L(X, Y))$.

It was observed there that the validity of $(H^1(X), BMOA(Y)) = Bloch(L(X, Y))$ depends on the fact that X and Y satisfy the vector-valued formulation of some inequalites due to Hardy and Littlewood (see [HL]) in the scalar-valued case.

In this paper we consider the vector-valued interpretation of the space $BMOA$ defined in terms of Carleson measures (see Definition 1.2 below) instead of the one considered in [B1] and analyze the relationship with the previous one, studying the result on vector-valued multipliers for this formulation of $BMOA$.

This leads us to the consideration of some other geometric properties coming from other inequalities due to Littlewood and Paley on the g-function which have been already considered in [B3] and more recently in [Bl1, Bl2, X].

Throughout the paper all spaces are assumed to be complex Banach spaces, D stands for the unit disc and $\mathbb{T}$ for its boundary. Given $1 \leq p < \infty$, we shall denote by $L^p(X)$ the space of X-valued Bochner p-integrable functions on the circle $\mathbb{T}$ and write $||f||_{p,X} = \left(\int_0^{2\pi} ||f(e^{it})||^p \frac{dt}{2\pi}\right)^{\frac{1}{p}}$ and $M_{p,X}(F, r) = ||F_r||_{p,X} =$

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$\left(\int_0^{2\pi}||F(re^{it})||^p\frac{dt}{2\pi}\right)^{\frac{1}{p}}$ for an X-valued analytic function F on D. We shall write $H^p(X)$ (respec. $H^p_0(X)$) for the vector-valued Hardy spaces, i.e. space of functions in $L^p(X)$ whose negative (respec. non positive) Fourier coefficients vanish. Of course Hardy spaces $H^p(X)$ (respec. $H^p_0(X)$) can be regarded as spaces of analytic functions on the disc. Actually they coincide with the closure of the X-valued polynomials, denoted by $\mathcal{P}(X)$ (respec. those which vanish at $z=0$, denoted by $\mathcal{P}_0(X)$) under the norm given by $\sup_{0<r<1} M_{p,X}(f,r)$.

The paper is divided into three sections. In the first one we consider the vector valued version of $BMOA$ in terms of Carleson measures, giving the connection with the standard notion considered in [B1]. It is shown that both notions only coincide for Hilbert spaces and also a proof of the extension of Kahane's inequalities to vector-valued BMO is provided. Section 2 is devoted to the consideration of vector-valued multipliers between H^1 and $BMOA$ and some properties that will play an important role in this setting. Finally in section 3 we mention some elementary facts on vector valued Bloch functions and apply the previous theorems to get some applications.

As usual p' is the conjugate exponent of p when $1\le p\le\infty$, i.e. $\frac{1}{p}+\frac{1}{p'}=1$ and C stands for a constant that may vary from line to line.

1.- Vector-valued BMOA

Definition 1.1. *Let X be a complex Banach space. $BMOA(X)$ stands for the space of functions $f\in L^1(X)$ with $\hat{f}(n)=0$ for $n<0$ such that*

$$||f||_{*,X}=\sup_I\frac{1}{|I|}\int_I||f(e^{it})-f_I||\frac{dt}{2\pi}<\infty,$$

where the supremum is taken over all intervals $I\in[0,2\pi)$, $|I|$ stands for the normalized Lebesgue measure of I and $f_I=\frac{1}{|I|}\int_I f(e^{it})\frac{dt}{2\pi}$.

The norm in the space is given by

$$||f||_{BMO(X)}=||\int_{-\pi}^{\pi}f(e^{it})\frac{dt}{2\pi}||+||f||_{*,X}.$$

The same technique as in the scalar-valued case allows us to replace the average over intervals by convolution with the Poisson kernel. According to this and the previous formulation one has that

$$||f||_{*,X}\approx\sup_{|z|<1}\int_0^{2\pi}||f(e^{it})-f(z)||P_z(e^{-it})\frac{dt}{2\pi}$$

where P_z is the Poisson Kernel $P_z(w)=\frac{1-|z|^2}{|1-zw|^2}$ and $f(z)=\int_0^{2\pi}f(e^{it})P_z(e^{-it})\frac{dt}{2\pi}$.

Recall now that in the vector valued setting, although Khintchine's inequalites do not generally remain valid, at least one still has the so called Kahane's inequalities, i.e. for any $0<p<\infty$ there exist constants $C_1,C_2>0$ such that for any $n\in\mathbb{N}$

$$C_1\left(\int_0^{2\pi}||\sum_{k=0}^n x_ke^{i2^kt}||^p\frac{dt}{2\pi}\right)^{\frac{1}{p}}\le\int_0^{2\pi}||\sum_{k=0}^n x_ke^{i2^kt}||\frac{dt}{2\pi}\le C_2\left(\int_0^{2\pi}||\sum_{k=0}^n x_ke^{i2^kt}||^p\frac{dt}{2\pi}\right)^{\frac{1}{p}}.$$

There exists an extension of Kahane-Khintchine inequalities to vector valued BMO which is part of the folklore. Let us present a proof based upon the following lemma.

Lemma A. *(see* [Pe, Pi1]*) Let X be a Banach space. Let $\lambda_k \in \mathbb{R}^+$ such that $\frac{\lambda_{k+1}}{\lambda_k} \geq C > 1$ and $\inf_{k\in\mathbb{Z}} \lambda_{k+1} - \lambda_k = d > 0$. Then there exist constants $K_1, K_2 > 0$, depending only on C and d, such that for any $x_0, x_1, x_2, ..., x_n \in X$*

$$K_1 \int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}|| \frac{dt}{2\pi} \leq \int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i\lambda_k t}|| \frac{dt}{2\pi} \leq K_2 \int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}|| \frac{dt}{2\pi}$$

Theorem 1.1. *Let X be a Banach space. Then there exist constants $C_1, C_2 > 0$ such that for any $x_0, x_1, x_2, ..., x_n \in X$*

$$C_1 \int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}|| \frac{dt}{2\pi} \leq ||\sum_{k=0}^{n} x_k e^{i2^k t}||_{*,X} \leq C_2 \int_0^{2\pi} ||\sum_{k=0}^{n} x_k e^{i2^k t}|| \frac{dt}{2\pi}.$$

Proof. Let us write $f(e^{it}) = \sum_{k=0}^{n} x_k e^{i2^k t}$. Given an interval, say $J = \{e^{it} : |t - t_J| < 2\pi|J|\}$, then consider $n(J) \in \mathbb{N}$ such that $|J|2^{n(J)} \leq 1 < |J|2^{n(J)+1}$.

Now, assuming $n \geq n(J)$, we split $f = g + h$ where $g(e^{it}) = \sum_{k=0}^{n(J)} x_k e^{i2^k t}$.

Note that

$$(g - g_J)(e^{it}) = \frac{1}{2} \sum_{k=0}^{n(J)} x_k \frac{1}{|J|} \int_{t_J - 2\pi|J|}^{t_J + 2\pi|J|} (e^{i2^k t} - e^{i2^k s}) \frac{ds}{2\pi}.$$

Hence

$$\|(g - g_J)(e^{it})\| \leq \frac{1}{2} \sum_{k=0}^{n(J)} \|x_k\| \frac{1}{|J|} \int_{t_J - 2\pi|J|}^{t_J + 2\pi|J|} 2^k |t - s| \frac{ds}{2\pi}.$$

Now if $e^{it} \in J$ then

$$\begin{aligned} \|(g - g_J)(e^{it})\| &\leq C \sum_{k=0}^{n(J)} \|x_k\| 2^k |J| \\ &\leq C\|f\|_1 (\sum_{k=0}^{n(J)} 2^k) |J| \\ &\leq C\|f\|_1 2^{n(J)} |J| \\ &\leq C\|f\|_{1,X}. \end{aligned}$$

For the function h we have that

$$\frac{1}{|J|} \int_J ||h(e^{it}) - h_J|| \frac{dt}{2\pi} \leq \frac{2}{|J|} \int_J ||h(e^{it})|| \frac{dt}{2\pi} = 2 \int_0^{2\pi} || \sum_{k=n(J)+1}^{n} x_k e^{i2^k (|J|t)} || \frac{dt}{2\pi}.$$

Now applying Lemma A for $\lambda_k = 2^k|J| \geq 1$ we get

$$\frac{1}{|J|}\int_J ||h(e^{it}) - h_J||\frac{dt}{2\pi} \leq 2K_2\int_0^{2\pi} ||\sum_{k=n(J)+1}^{n} x_k e^{i2^k t}||\frac{dt}{2\pi}.$$

Now, making use of the contraction principle, we can say that

$$\int_0^{2\pi} ||\sum_{k=n(J)+1}^{n} x_k e^{i2^k t}||\frac{dt}{2\pi} \leq \|f\|_{1,X}.$$

Adding both inequalities and taking now the supremum over J we get the direct inequality.

The converse inequality is trivial and the proof is finished. $\square$

Let us now recall the formulation of functions in $BMOA$ in terms of Carleson measures (see [G, Z]) that we shall use later on.

Definition 1.2. *Given an analytic function $f(z) = \sum_{k=0}^{\infty} x_k z^k$ we define*

$$||f||_{\mathcal{C},X} = \sup_{|z|<1}\Big(\int_D (1-|w|^2)||f'(w)||^2 P_z(\bar{w})dA(w)\Big)^{\frac{1}{2}}$$

where P_z is the Poisson Kernel $P_z(w) = \frac{1-|z|^2}{|1-zw|^2}$.

We shall denote $BMOA_{\mathcal{C}}(X)$ the space of functions such that $||f||_{\mathcal{C},X} < \infty$.

$BMOA_{\mathcal{C}}(X)$ becomes a Banach space endowed with the norm

$$||f||_{BMOA_{\mathcal{C}}(X)} = ||f(0)|| + ||f||_{\mathcal{C},X}$$

Let us now recall the notions of type and cotype of a Banach space. Although they are usually defined in terms of the Rademacher functions we shall replace them by lacunary sequences $e^{i2^n t}$, which gives an equivalent definition ([MPi, Pi]).

Given $1 \leq p \leq 2 \leq q \leq \infty$. A Banach space has cotype q (respectively type p) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and for all $x_0, x_1, x_2, ...x_N \in X$ one has

$$\left(\sum_{k=0}^{N} ||x_k||^q\right)^{\frac{1}{q}} \leq ||\sum_{k=0}^{N} x_k e^{2^k it}||_{1,X},$$

(respectively

$$||\sum_{k=0}^{N} x_k e^{2^k it}||_{1,X} \leq C\left(\sum_{k=0}^{N} ||x_k||^p\right)^{\frac{1}{p}}.)$$

Recall also the well-known result by S. Kwapien ([Kw]), which establishes that X has type 2 and cotype 2 if and only if X is isomorphic to a Hilbert space.

First of all let us establish the connection between $BMOA(X)$ and $BMOA_{\mathcal{C}}(X)$.

Theorem 1.2. *Let X be a complex Banach space.*

(i) If there exists a constant $C > 0$ such that

$$||f||_{\mathcal{C},X} \leq C||f||_{*,X}$$

for any $f \in \mathcal{P}_0(X)$ then X has cotype 2.

(ii) If there exists a constant $C > 0$ such that

$$||f||_{*,X} \leq C||f||_{\mathcal{C},X}$$

for any $f \in \mathcal{P}_0(X)$ then X has type 2.

Proof.

(i) Let us take $f(z) = \sum_{k=0}^{n} x_k z^{2^k}$. Assume first that $||f||_{\mathcal{C},X} \leq C||f||_{*,X}$. Note, that choosing $z = 0$, we have

$$\int_0^1 (1-s) M_{2,X}^2(f', s) ds \leq ||f||_{\mathcal{C},X} \leq C||f||_{1,X}.$$

Since $2^n||x_n||r^{2^n-1} \leq M_{2,X}(f', r)$ for $n \in \mathbb{N}$ then we can write

$$\left(\int_0^1 (1-r) M_{2,X}^2(f', r) dr\right)^{\frac{1}{2}} \geq \left(\sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} (1-r) 2^{2k} ||x_k||^2 r^{2(2^k-1)} dr\right)^{\frac{1}{2}}$$
$$\geq C \left(\sum_{k=0}^{n} ||x_{2^k}||^2 (1-2^{-k})^{2(2^k-1)}\right)^{\frac{1}{2}}.$$

Using now the fact that $(1-2^{-k})^{2^k} \geq Ce^{-1}$ one gets the cotype 2 condition

$$\left(\sum_{k=0}^{n} ||x_{2^k}||^2\right)^{\frac{1}{2}} \leq C||f||_{1,X}.$$

(ii) Assume now that $||f||_{*,X} \leq C||f||_{\mathcal{C},X}$. Therefore, if $f(z) = \sum_{k=0}^{n} x_k z^{2^k}$ then

$$||f||_{1,X}^2 \leq C||f||_{*,X}^2 \leq C \sup_{z\in D} \int_D (1-|w|^2) (\sum_{k=0}^{n} 2^k ||x_k|| |w|^{2^k-1})^2 P_z(\bar{w}) dA(w).$$

From the Cauchy-Schwarz inequality

$$(\sum_{k=0}^{n} 2^k ||x_k|| |w|^{2^k-1})^2 \leq (\sum_{k=0}^{n} 2^k ||x_k||^2 |w|^{2^k-1})(\sum_{k=0}^{n} 2^k |w|^{2^k-1})$$
$$\leq (\sum_{k=0}^{n} 2^k ||x_k||^2 |w|^{2^k-1})(\frac{C}{1-|w|^2}).$$

This gives that

$$||f||_{1,X}^2 \leq C \int_D \sum_{k=0}^{n} 2^k ||x_k||^2 |w|^{2^k-1} P_z(\bar{w}) dA(w)$$
$$= C \int_0^1 \sum_{k=0}^{n} 2^k ||x_k||^2 r^{2^k-1} dr = C \sum_{k=0}^{n} ||x_{2^k}||^2. \quad \square$$

As a consequence we get the following characterization of Hilbert spaces which is part of the folklore.

Corollary 1.1. *Let X be a complex Banach space. $BMOA(X) = BMOA_{\mathcal{C}}(X)$ (with equivalent norms) if and only if X is isomorphic to a Hilbert space.*

Proof. Recall that the classical proof ([G, Theorem 3.4]) can be reproduced in the case of Hilbert spaces because it merely relies upon Plancherel's theorem.

The converse follows by combining Theorem 1.1 with Kwapien's theorem. □

Let us give some easy sufficient conditions to get functions in $BMOA_{\mathcal{C}}(X)$. We need the following

Lemma B.
Let $0 < p \leq q \leq \infty$ and g an X-valued analytic function. Then

$$M_{q,X}(g, r^2) \leq C(1-r)^{\frac{1}{q}-\frac{1}{p}} M_{p,X}(g, r) \qquad \text{(see [D, page 84])} \tag{1.1}$$

Let $\gamma > 1$ then

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{\gamma}} = O((1-|z|)^{1-\gamma}) \quad \text{(see [D, page 65])} \tag{1.2}$$

Let $\gamma < \beta$ then

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{(1-rs)^{\beta}} dr = O\left((1-s)^{\gamma-\beta}\right) \qquad \text{(see [SW, Lemma 6])} \tag{1.3}$$

Next theorem, with $BMOA_{\mathcal{C}}(X)$ replaced by $BMOA(X)$, corresponds to Theorem 2.1 in [B1].

Theorem 1.3. *Let f be a X-valued analytic function. If there exists $0 < p < \infty$ such that*

$$M_{p,X}(f', r) = O\left((1-r)^{-1/p'}\right)$$

then $f \in BMOA_{\mathcal{C}}(X)$.

Proof. Notice that (1.1) implies that if there exists $0 < p_0 < \infty$ such that $M_{p_0,X}(f', r) = O\left((1-r)^{-1/p_0'}\right)$ then the same property holds for any $p \geq p_0$. Therefore it suffices to prove the result assuming $2 < p < \infty$.

Set then $q = \frac{p}{2}$ and take $z \in D$. Then using Hölder's inequality and (1.2) we have

$$\int_0^1 \int_0^{2\pi} \frac{(1-s^2)(1-|z|^2)||f'(se^{it})||^2}{|1-zse^{-it}|^2} \frac{dt}{2\pi} ds$$
$$\leq \int_0^1 (1-s^2)(1-|z|^2) M_{p,X}^2(f', s) \left(\int_0^{2\pi} \frac{1}{|1-zse^{-it}|^{2q'}} \frac{dt}{2\pi} \right)^{\frac{1}{q'}} ds$$
$$\leq C \int_0^1 \frac{(1-s)^{1-\frac{2}{p'}}(1-|z|^2)}{(1-|z|s)^{2-\frac{1}{q'}}} ds.$$

Applying now (1.3) for $\gamma = \frac{2}{p}$ and $\beta = 1 + \frac{2}{p}$ one gets

$$\int_0^1 \frac{(1-s)^{1-\frac{2}{p'}}}{(1-|z|s)^{2-\frac{1}{q'}}} ds \leq \frac{C}{1-|z|}.$$

This gives then that f belongs $BMOA_{\mathcal{C}}(X)$. $\square$

EXAMPLE 1.1. Let $(\alpha_n) \geq 0$ such that $\sum_{n=1}^{\infty} \alpha_n^p < \infty$ for some $1 < p < \infty$ and let s_n be an increasing sequence in $(0,1)$ with $\lim_{n\to\infty} s_n = 1$. If $f_n(z) = log(\frac{1}{(1-s_n z)^{\alpha_n}})$ and $f(z) = \big(f_n(z)\big)_{n\in\mathbb{N}}$ then $f \in BMOA(l^p) \cap BMOA_{\mathcal{C}}(l^p)$.

It suffices to see that $M_{p,l^p}(f',r) = O((1-r)^{-\frac{1}{p'}})$. Now using (1.2) we get

$$\begin{aligned} M^p_{p,l^p}(f',r) &= \sum_{n=1}^{\infty} M_p^p(f_n',r) \\ &= \sum_{n=1}^{\infty} \alpha_n^p \int_0^{2\pi} \frac{s_n}{|1-s_n re^{-it})|^p} \frac{dt}{2\pi} \\ &\leq C \sum_{n=1}^{\infty} \alpha_n^p (1-s_n r)^{1-p} \leq C(1-r)^{1-p}. \end{aligned}$$

A simple and useful sufficient condition for a function to belong to $BMOA_{\mathcal{C}}(X)$ is given in the following proposition.

Proposition 1.1. *Let f be a X-valued analytic function. If*

$$\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||^2 dr < \infty$$

then $f \in BMOA_{\mathcal{C}}(X)$.

Proof. For any $z \in D$ one has

$$\begin{aligned} &\int_D \frac{(1-|z|^2)(1-|w|^2)|||f'(w)||_X^2}{|1-\bar{w}z|^2} dA(w) \\ &\leq 2\int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 \Big(\int_0^{2\pi} \frac{1-r^2|z|^2}{|1-re^{-it}z|^2} \frac{dt}{2\pi} \Big) dr \\ &= 2\int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 dr. \end{aligned}$$

Therefore

$$||f||_{\mathcal{C},X} \leq C \left(\int_0^1 (1-r) \sup_{|w|=r} ||f'(w)||_X^2 dr \right)^{\frac{1}{2}} < \infty. \quad \square$$

It was proved in [B1] Example 3.1 that if $X = l^1$ and $f(z) = \big(\frac{1}{nlog(n+1)} z^n\big)_{n=0}^{\infty}$, then

$$\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||_{l^1}^2 dr < \infty$$

but $f \notin H^1(l^1)$.

This example shows that the condition in Proposition 1.1 is not enough to get functions in $BMOA(X)$ for general Banach spaces and gives sense to the following definition.

Definition 1.3. *(see* [B1]*) A complex Banach space X is said to have the $(HL)^*$-property if there exists a constant $C > 0$ such that*

$$||f||_{*,X} \leq C \left(\int_0^1 (1-r) \sup_{|z|=r} ||f'(z)||^2 dr \right)^{\frac{1}{2}}.$$

for any $f \in \mathcal{P}(X)$.

The reader is referred to [B1] to find spaces having and failing such a property.

2.- Vector valued Multipliers and some geometric properties

Let us mention first some notions introduced in other papers.

Definition 2.1. *(see* [B1]*) Let X, Y be complex Banach spaces. If $F(z) = \sum_{n=0}^\infty T_n z^n$ is an $L(X,Y)$-valued analytic function and $f(z) = \sum_{n=0}^\infty x_n z^n$ is an X-valued analytic function then we can define the Y-valued analytic function*

$$F * f(z) = \sum_{n=0}^\infty T_n(x_n) z^n = \int_0^{2\pi} F(ze^{it}) \left(f(e^{-it})\right) \frac{dt}{2\pi}.$$

Definition 2.2. *(see* [AB]*) Let $1 \leq p < \infty$. A complex Banach space X is said to have property $(H)_p$, to be denoted $X \in (H)_p$, if there exists a constant $C > 0$ such that*

$$(\int_0^1 (1-r)^{max\{2,p\}-1} M_{p,X}^{max\{2,p\}}(f',r) dr)^{\frac{1}{max\{2,p\}}} \leq C ||f||_{p,X}$$

for any polynomial $f \in \mathcal{P}(X)$.

Remark 2.1. The property $(H)_1$ was already defined and studied in [B1], denoted there by (HL) and then again in [AB].

Remark 2.2. The property $(H)_\infty$ would mean

$$M_{\infty,X}(f',r) \leq C \frac{M_{\infty,X}(f,r)}{1-r},$$

which holds true for any Banach space.

Remark 2.3. Observe that

$$\int_0^1 (1-r)^{max\{p,2\}-1} M_{p,X}^{max\{p,2\}}(f',r) dr = \sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} (1-r)^{max\{p,2\}-1} M_{p,X}^{max\{p,2\}}(f',r) dr,$$

for $r_k = 1 - 2^{-k}$ and then, since $M_{p,X}(f,r)$ is increasing, the properties $(HL)^*$ and $(H)_p$ can be replaced by

$$||f||_{*,X} \leq C \left(\sum_{k=0}^\infty 2^{-2k} \sup_{|z|=r_k} ||f'(z)||^2 \right)^{\frac{1}{2}} \tag{2.1}$$

and

$$(2.2)\qquad \left(\sum_{k=0}^{\infty} 2^{-max\{p,2\}k} M_{p,X}^{max\{p,2\}}(f', r_k)\right)^{\frac{1}{max\{p,2\}}} \leq C||f||_{p,X}.$$

Hence X has the $(H)_p$-property if and only if the operator $f \to \big(2^{-k} f'(r_k e^{it})\big)_k$ is bounded from $H^p_0(X)$ into $l^{max\{p,2\}}\big(L^p(X)\big)$.

EXAMPLE 2.1. Let $X = c_0$ fails to have $(H)_p$-property for any $1 \leq p < \infty$.

Indeed, take $f_N(z) = \sum_{n=1}^{N} e_n z^n$. On the one hand $sup_{N\in\mathbb{N}}||f_N||_{p,c_0} = 1$ and on the other hand $M_{p,c_0}(f'_N, r_k) = \sup\limits_{1\leq n\leq N} n r_k^{n-1}$. Hence $M_{p,c_0}(f'_N, r_k) \geq C2^k$ for $N \geq 2^k$. Therefore

$$\sum_{k=0}^{log_2(N)} 2^{-max\{p,2\}k} M_{p,c_o}^{max\{p,2\}}(f'_N, r_k) \geq Clog(N).$$

This completes the proof, using (2.2). $\square$

Regarding properties $(H)_p$ the reader is referred to [AB, B4] for different results and examples.

Let us introduce other property which appears from the consideration of Hardy spaces in terms of the g-function. The reader is referred to [B3] for some related properties and to [X] for similar formulations on the Lusin area function for vector-valued Lebesgue spaces.

Definition 2.3. *A complex Banach space X is said to have property (g), in short $X \in (g)$, if there exists a constant $C > 0$ such that*

$$\int_0^{2\pi} (\int_0^1 (1-r)||f'(re^{i\theta})||^2 dr)^{\frac{1}{2}} d\theta \leq C||f||_{1,X}$$

for any $f \in \mathcal{P}(X)$.

Theorem 2.1. *Let X, Y be Banach spaces and $X \in (g)$. If $f \in H^1(X)$ and $F : \mathbb{D} \to L(X, Y)$ is an analytic function satisfying that*

$$\int_0^1 \int_0^1 (1-r)(1-s) M^2_{\infty,L(X,Y)}(g'', rs) dr ds < \infty$$

then

$$\int_0^1 (1-r) \sup_{|z|=r} ||(F * f)'(z)||^2 dr < \infty.$$

In particular $F * f \in BMOA_{\mathcal{C}}(Y)$.

Proof. Let us write $F(z) = \sum_{n=0}^{\infty} T_n z^n$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

We have that

$$(F*f)'(z) = \sum_{n=1}^{\infty} nT_n(x_n)z^{n-1}$$
$$= 2\int_0^1 (1-s^2)\sum_{n=1}^{\infty} n^2(n-1)T_n(x_n)z^{n-1}s^{2n-3}ds$$
$$= 2\int_0^1\int_0^{2\pi}(1-s^2)(\sum_{n=1}^{\infty} n(n-1)T_n z^{n-2}s^{n-2}e^{i(n-2)t})\big(\sum_{n=1}^{\infty} nx_n s^{n-1}e^{-i(n-1)t}\big)\frac{dt}{2\pi}ze^{it}ds$$
$$= 2\int_0^1\int_0^{2\pi}(1-s^2)F''(zse^{it})\big(f'(se^{-it})\big)ze^{it}\frac{dt}{2\pi}ds.$$

Therefore, using that $X \in (g)$, we have

$$||(F*f)'(z)|| \leq 2|z|\int_0^{2\pi}(\int_0^1(1-s^2)||f'(se^{i\theta})||^2ds)^{\frac{1}{2}}(\int_0^1(1-s^2)||F''(sze^{i\theta})||^2ds)^{\frac{1}{2}}d\theta$$
$$\leq 2|z|(\int_0^1(1-s^2)M_\infty^2(F'',s|z|)ds)^{\frac{1}{2}}\int_0^{2\pi}(\int_0^1(1-s^2)||f'(se^{i\theta})||^2ds)^{\frac{1}{2}}d\theta$$
$$\leq C|z|(\int_0^1(1-s^2)M_\infty^2(F'',s|z|)ds)^{\frac{1}{2}}\|f\|_{1,X}.$$

Hence

$$\sup_{|z|=r}||(F*f)'(z)||^2 \leq C(\int_0^1(1-s)M_\infty^2(F'',s|z|)ds)\|f\|_{1,X}^2.$$

Now

$$\int_0^1(1-r)\sup_{|z|=r}||(F*f)'(z)||^2dr \leq C\int_0^1\int_0^1(1-s)(1-r)M_{\infty,X}^2(F'',sr)dsdr\|f\|_{1,X}^2. \quad \Box$$

Let us now give a result which improves the previous theorem as well as Theorem 3.2 in [B1].

Theorem 2.2. *Let $1 \leq p < \infty$ and X, Y be Banach spaces with $X \in (H)_p$. If $f \in H^p(X)$ and $F : \mathbb{D} \to L(X,Y)$ is an analytic function such that*

$$M_{p',L(X,Y)}(F',r) = O(\frac{1}{1-r}) \qquad (r \to 1)$$

then

$$\int_0^1(1-r)^{max\{p,2\}-1}\sup_{|z|=r}||(F*f)'(z)||^{max\{p,2\}}dr < \infty.$$

In particular $F*f \in BMOA_{\mathcal{C}}(Y)$ provided $1 \leq p \leq 2$.

Proof. Let us write $F(z) = \sum_{n=0}^{\infty} T_n z^n$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

Now let us observe that

$$\begin{aligned} z(F*f)'(z^2) &= \sum_{n=1}^{\infty} nT_n(x_n)z^{2n-1}s^{2n-1}ds \\ &= 2\int_0^1 \sum_{n=1}^{\infty} n^2T_n(x_n)z^{2n-1} \\ &= 2\int_0^1\int_0^{2\pi} (\sum_{n=1}^{\infty} nT_nz^{n-1}s^{n-1}e^{i(n-1)t})(\sum_{n=1}^{\infty} nx_nz^{n-1}s^{n-1}e^{-i(n-1)t})\frac{dt}{2\pi}sds \\ &= 2\int_0^1\int_0^{2\pi} F'(zse^{it})\big(f'(zse^{-it})\big)se^{it}\frac{dt}{2\pi}ds. \end{aligned}$$

Therefore if $q = max\{p,2\}$ then we have

$$\begin{aligned} ||z(F*f)'(z^2)|| &\leq 2\int_0^1 M_{p,X}(f',s|z|)M_{p',L(X,Y)}(F',s|z|)ds \\ &\leq C\left(\int_0^1 \frac{ds}{(1-s|z|)^{q'}}\right)^{\frac{1}{q'}}\left(\int_0^{|z|} M^q_{p,X}(f',s)ds\right)^{\frac{1}{q}} \\ &\leq C\frac{\left(\int_0^{|z|} M^q_{p,X}(f',s)ds\right)^{\frac{1}{q}}}{(1-|z|)^{\frac{1}{q}}}. \end{aligned}$$

Hence

$$\sup_{|z|=r} ||z(F*f)'(z^2)|| \leq \frac{C}{(1-r)^{\frac{1}{q}}}\left(\int_0^r M^q_{p,X}(f',s)ds\right)^{\frac{1}{q}}.$$

Now, using the $(H)_p$-property on X, we can estimate

$$\begin{aligned} \int_0^1 (1-r^2)^{q-1}\sup_{|z|=r^2} ||(F*f)'(z)||^q rdr &\leq C_q\int_0^1 (1-r)^{q-2}\Big(\int_0^r M^q_{1,X}(f',s)ds\Big)dr \\ &= C\int_0^1 (1-s)^{q-1}M^q_{p,X}(f',s)ds \leq C||f||^q_{p,X}. \quad \square \end{aligned}$$

Since $Bloch(L(X,Y))$ corresponds to $M_{\infty,L(X,Y)}(g',r) = O(\frac{1}{1-r})$ then we recover the following

Corollary 2.1. ([B1]) *Let X,Y be a Banach spaces such that $X \in (H)_1$ and $Y \in (HL)^*$.*
If $f \in H^1(X)$ and $F \in Bloch(L(X,Y))$ then $F*f \in BMOA(Y)$.

3.- Vector valued Bloch functions and applications.

Let us now recall some results on vector valued Bloch functions.

Definition 3.1. *Given a complex Banach space E we shall use the notation $Bloch(E)$ for the space of E- valued analytic functions on D, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, such that*

$$\sup_{|z|<1} (1 - |z|)||f'(z)|| < \infty.$$

We endow the space with the following norm

$$||f||_{Bloch(E)} = max\{||f(0)||, \sup_{|z|<1} (1 - |z|)||f'(z)||\}.$$

Remark 3.1. It follows clearly from the definition that, for any Banach space E and $F(z) = \sum_{n=0}^{\infty} x_n z^n$, one has that $F \in Bloch\,(E)$ if and only if

$$F_{x^*}(z) = \sum_{n=0}^{\infty} < x^*, x_n > z^n \in Bloch$$

for any $x^* \in E^*$. Moreover

$$||F||_{Bloch(E)} = \sup_{||x^*||\leq 1} ||F_{x^*}||_B loch.$$

Remark 3.2. Let $E = L(X, Y)$, the space of bounded linear operators from X into Y and $(T_n) \subset L(X, Y)$. It is elementary to see that $F(z) = \sum_{n=0}^{\infty} T_n z^n \in Bloch\,(L(X, Y))$ if and only if the functions $F_{x,y^*}(z) = \sum_{n=0}^{\infty} < T_n(x), y^* > z^n \in Bloch$ for any $x \in X, y^* \in Y^*$. Moreover

$$||F||_{Bloch(L(X,Y))} = \sup_{||x||\leq 1, ||y^*||\leq 1} ||F_{x,y^*}||_B loch.$$

Remark 3.3. In the case $E = l^\infty$ one can identify $Bloch(l^\infty) = l^\infty(Bloch)$. Moreover if $f = (f_n)$

$$\sup_{n\in\mathbb{N}} ||f_n||_{Bloch} = ||f||_{Bloch(l^\infty)}.$$

EXAMPLE 3.1. Let $1 \leq p \leq \infty$ and

$$f_p(z) = \sum_{n=1}^{\infty} n^{\frac{-1}{p}} e_n z^n, \qquad f_\infty(z) = \sum_{n=1}^{\infty} \frac{a_n}{n} z^n$$

where e_n stands for the canonical basis in l^p and $a_n = \sum_{k=1}^{n} e_k$. Then $f_p \in Bloch(l^p)$.

EXAMPLE 3.2. Let $1 \leq p \leq \infty$ and

$$g_p(z) = \frac{1}{(1 - z)^{\frac{1}{p}}}, \qquad g_\infty(z) = log\frac{1}{1 - z}.$$

Then $F_p(z) = (g_p)_z \in Bloch(H^p)$.

There are also other procedures to get X-valued Bloch functions that we state in the following propositions, already pointed out in [B1].

Proposition 3.1. *(see* [B1], *Prop. 1.2). Let X be a Banach space and $T \in L(L^1(D), X)$ where $L^1(D)$ stands for the Lebesgue space on the disc with the area measure. Then $f(z) = T(K_z)$ is a X-valued Bloch function, where K_z denotes the Bergman Kernel $K_z(w) = \frac{1}{(1-zw)^2}$.*

Proposition 3.2. *(see* [B1] *Prop. 1.1) Let E be a Banach space and $x_n \in E$.*

(i) If $\sup_{||x^*||\leq 1} \sup_{n\geq 0} \sum_{k=2^n}^{2^{n+1}} | < x^*, x_k > | < \infty$ *then* $\sum_{n=0}^{\infty} x_n z^n \in Bloch(E)$.

(ii) $|| \sum_{n=0}^{\infty} x_n z^{2^n} ||_{Bloch(E)} \approx \sup_{n\geq 0} ||x_n||$.

It is well known (see [D, page 103]) that the space of multipliers (H^1, H^2) can be identified with the space of sequences (λ_n) such that

$$\sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} |\lambda_k|^2 < \infty.$$

Therefore one has the following:

If $f(z) = \sum_{n=0}^{\infty} x_n z^n \in Bloch(X)$ then $< f(z), x^* >\in (H^1, BMOA)$. In particular, since $BMOA \subset H^2$, we have that $< f(z), x^* >\in (H^1, H^2)$ and then

$$\sup_{||x^*||=1} \sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} | < x^*, x_k > |^2 < \infty.$$

We shall see that we can get better information assuming some conditions on X.

Let us recall the notion of Fourier-type introduced by J. Peetre ([Pee]). Given $1 \leq p \leq 2$, a Banach space X is said to have Fourier type p if there exists a constant $C > 0$ such that

$$\left(\sum_{n=-\infty}^{\infty} ||\hat{f}(n)||^{p'} \right)^{\frac{1}{p'}} \leq C||f||_{L^p(X)}.$$

Typical examples of spaces of Fourier type p are the Lebesgue spaces $L^r(\mu)$ for $p \leq r \leq p'$ or those obtained by interpolation $[X, H]_\theta$ between any Banach space X and a Hilbert space H for $1/p = 1 - \theta/2$.

Proposition 3.3. *Let X be a Banach space with $(HL)^*$-property and Fourier type p.*

If $f(z) = \sum_{n\in\mathbb{N}} x_n z^n \in Bloch(X)$ then $\|x_n\| \in (H^1, l^{p'})$.

In particular

$$\sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} ||x_k||^{p'} < \infty. \tag{3.1}$$

Proof. Using that $\|f * \phi\|_{p,X} \leq C\|f * \phi\|_{BMOA(X)}$ and Corollary 2.1 we have

$$\|f * \phi\|_{p,X} \leq C\|f\|_{Bloch(X)}\|\phi\|_1$$

for any function $\phi \in H^1$.

Applying now the Fourier type condition

$$\left(\sum_{n\in\mathbb{N}} \alpha_n^{p'} ||x_n||^{p'}\right)^{\frac{1}{p'}} \leq C\|f\|_{Bloch(X)}\|\phi\|_1$$

for any $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^1$.

This means that $\|x_n\| \in (H^1, l^{p'})$.

Choosing $\phi_r(z) = \frac{1}{(1-rz)^2}$ we shall have

$$\left(\sum_{n\in\mathbb{N}} n^{p'} ||x_n||^{p'} r^{np'}\right)^{\frac{1}{p'}} \leq C\|f\|_{Bloch(X)} \frac{1}{(1-r)}.$$

This implies

$$\sum_{n=1}^{N} n^{p'} ||x_n||^{p'} \leq CN^{p'} ||f||^{p'}_{Bloch(X)}.$$

Which obviously gives (3.1) □

Let me point out now another applications.

Proposition 3.4. *Let X be a Banach space with the $(HL)^*$-property and $f \in Bloch(X)$. Then*

$$||f_r||_{BMOA(X)} \leq Clog\frac{1}{1-r}||f||_{Bloch(X)}$$

where $f_r(z) = f(rz)$.

Proof. It is a simple consequence of Corollary 2.1 and the fact

$$\int_0^{2\pi} \frac{1}{|1-re^{it}|}\frac{dt}{2\pi} \approx log\frac{1}{1-r}. \quad \square$$

Proposition 3.5. *Let X be a Banach space with the $(H)_1$-property.*

If $\sum_{n\in\mathbb{N}} x_n z^n \in H^1(X)$ and $(x_n^*) \subset X^*$ satisfies $\sup_{||x||=1} \sup_{n\in\mathbb{N}} \sum_{k=2^n}^{2^{n+1}} | < x_k^*, x > | < \infty$ then

$$\sum_{n\in\mathbb{N}} | < x_n^*, x_n > |^2 < \infty.$$

Proof. It follows from (i) in Proposition 3.2 that for any sequence $\varepsilon_n \in \{0,1\}$ we have $\sum_{n\in\mathbb{N}} \varepsilon_n x_n^* z^n \in Bloch(X^*)$ with norm bounded by a constant independent of the choice of ε_n. Then, from Corollary 2.1, since $f(z) = \sum_{n\in\mathbb{N}} x_n z^n \in H^1(X)$ we have

$$||\sum_{n\in\mathbb{N}} \varepsilon_n < x_n^*, x_n > z^n||_{BMOA} \leq C||\sum_{n\in\mathbb{N}} \varepsilon_n x_n^* z^n||_{Bloch(X^*)}||f||_{1,X}$$

This shows that for any $t \in [0,1]$

$$||\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > z^n||_{BMOA} \leq C||f||_{1,X}.$$

Therefore

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}} | < x_n^*, x_n > |^2\right)^{\frac{1}{2}} &\approx \int_0^1 \int_0^{2\pi} |\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > e^{in\theta}| dt \frac{d\theta}{2\pi} \\ &= \int_0^1 ||\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > z^n||_{H^1} dt \\ &\leq \int_0^1 ||\sum_{n\in\mathbb{N}} r_n(t) < x_n^*, x_n > z^n||_{BMOA} dt \\ &\leq C||f||_{1,X} < \infty. \quad \square\end{aligned}$$

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Oscar Blasco. Departamento de Análisis Matemático, Universidad de Valencia, 46100 Burjassot (Valencia), Spain.

E-mail: Oscar.Blasco@uv.es