

## Bergman and Bloch spaces of vector-valued functions

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**Abstract.** We investigate Bergman and Bloch spaces of analytic vector-valued functions in the unit disc. We show how the Bergman projection from the Bochner-Lebesgue space  $L_p(\mathbb{D}, X)$  onto the Bergman space  $B_p(X)$  extends boundedly to the space of vector-valued measures of bounded  $p$ -variation  $V_p(X)$ , using this fact to prove that the dual of  $B_p(X)$  is  $B_p(X^*)$  for any complex Banach space  $X$  and  $1 < p < \infty$ . As for  $p = 1$  the dual is the Bloch space  $\mathcal{B}(X^*)$ . Furthermore we relate these spaces (via the Bergman kernel) with the classes of  $p$ -summing and positive  $p$ -summing operators, and we show in the same framework that  $B_p(X)$  is always complemented in  $\ell_p(X)$ .

### 1. Introduction.

Throughout the paper  $X$  will be a complex Banach space,  $1 \leq p < \infty$ ,  $\mathcal{H}(\mathbb{D}, X)$  (respect.  $\mathcal{P}(X)$ ) denotes the space of analytic functions (respect. polynomials) on the unit disc  $\mathbb{D}$  taking values in  $X$  and  $L_p(m, X)$  stands for the Bochner-Lebesgue  $p$ -integrable functions on  $\mathbb{D}$  where  $m$  is the normalized Borel-Lebesgue measure on  $\mathbb{D}$ . We write  $H_p(X)$  and  $B_p(X)$  for the Hardy and Bergman spaces of vector-valued analytic functions respectively, which, using the notation  $M_p(f, r) = (\frac{1}{2\pi} \int_{-\pi}^{\pi} \|f(re^{it})\|^p dt)^{1/p}$ , consist of those functions in  $\mathcal{H}(\mathbb{D}, X)$  where  $\sup_{0 < r < 1} M_p(f, r) = \|f\|_{H_p(X)} < \infty$  and  $(\int_0^1 M_p^p(f, r) r dr)^{1/p} = \|f\|_{B_p(X)} < \infty$ .

A limiting case in the scale of Bergman spaces, which is useful for many purposes, is the Bloch space  $\mathcal{B}(X)$ , its elements being all functions in  $\mathcal{H}(\mathbb{D}, X)$  such that  $\sup_{|z| < 1} (1 - |z|^2) \|f'(z)\| < \infty$ .

For  $X = \mathbb{C}$  the reader is referred to [3], [15] and [22] for the scalar-valued theory on these spaces, to [8], [9] or [10] for several properties of Bloch functions and their connection with multipliers between  $H_1$  and  $BMOA$  in the vector-valued setting, and finally to the paper [4] for properties on Taylor coefficient of functions in  $B_p(X)$  and different results on multipliers between vector-valued Bergman spaces.

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In this paper we shall study questions such as the boundedness of Bergman projection, the duality or the atomic decomposition in the vector-valued setting. The relationship between vector-valued analytic functions, vector measures and operators is also considered.

The paper is divided into four sections. In the first one we prove some elementary facts on the spaces  $B_p(X)$  and  $\mathcal{B}(X)$ , showing that the norm of a function in  $B_p(X)$  can be described in terms of its derivatives, in particular that  $f \in B_p(X)$  if and only if  $(1 - |z|)\|f'(z)\| \in L_p(m)$ , which makes natural to introduce  $\mathcal{B}(X)$  in the scale as a limiting case.

The second section is devoted to analyze the Bergman projection in the vector-valued setting. We see that the Bergman projection is bounded not only on  $L_p(m, X)$  but even on the space of vector measures of bounded  $p$ -variation  $V_p(m, X)$ . This allows us, as in the scalar-valued case, to get the duality  $(B_p(X))^* = B_{p'}(X^*)$  without conditions on the space  $X$ . It is also shown that the Bergman projection is bounded from  $V_\infty(m, X)$  onto  $\mathcal{B}(X)$ , and a projection from the space of vector-valued measures of bounded variation  $M(X)$  onto  $B_1(X)$  is also presented. Then we prove that  $B_1(X)$  coincides with the projective tensor product  $B_1 \hat{\otimes} X$  and that  $\mathcal{B}(X)$  can be identified with  $\mathcal{L}(B_1, X)$ . As a consequence the duality  $(B_1(X))^* = \mathcal{B}(X^*)$  is obtained.

Next section is devoted to relate vector-valued analytic functions and operators. For any  $\mathcal{L}(X, Y)$ -valued analytic function  $F(z) = \sum_{n=0}^{\infty} T_n z^n$  we can associate two linear operators,  $T_F(x) = F_x$  where  $F_x(z) = \sum_{n=0}^{\infty} T_n(x) z^n$  which maps elements in  $X$  into  $Y$ -valued analytic functions and  $S_F(g) = \sum_{n=0}^{\infty} \frac{T_n(x_n)}{n+1}$  for any  $g \in \mathcal{P}(X)$  such that  $g(z) = \sum_{n \geq 0} x_n z^n$ , which maps  $X$ -valued polynomials into vectors in  $Y$ . Under these identifications it is shown that  $\mathcal{B}(\mathcal{L}(X, Y))$  can be regarded either as  $\mathcal{L}(X, \mathcal{B}(Y))$  or as  $\mathcal{L}(B_1(X), Y)$ . Of course if  $F \in B_p(\mathcal{L}(X, Y))$  then  $T_F \in \mathcal{L}(X, B_p(Y))$  and  $S_F \in \mathcal{L}(B_{p'}(X), Y)$  but the converse does not hold true in general. Some connections with the theory of  $p$ -summing and positive  $p$ -summing operators are provided. It is observed that  $B_p(\Pi_p(X, Y))$  is continuously embedded into  $\Pi_p(X, B_p(Y))$  but again the converse is false. As a final result of our considerations we see that if  $T \in \mathcal{L}(B_{p'}, X)$  and  $f_T(z) = T(K_z)$ , where  $K_z$  stands for the Bergman kernel, then  $f_T$  belongs to  $B_p(X)$  if and only if the composition with the Bergman projection  $TP$  gives a positive  $p$ -summing operator from  $L_{p'}(m)$  into  $X$ .

Finally, in the last section we show that  $B_p(X)$  is always isomorphic to a complemented subspace of  $\ell_p(X)$ .

We write  $\mathcal{L}(X, Y)$  (resp.  $\mathcal{K}(X, Y)$ ) for the space of bounded (resp. compact) linear operators between the spaces  $X$  and  $Y$ , we denote  $x^*x$  the duality pairing in  $(X^*, X)$ ,  $u_n(z) = z^n$  for  $n \geq 0$  and any  $f \in \mathcal{P}(X)$  is written  $f = \sum_{n=0}^N u_n \otimes x_n$  for some  $N \in \mathbb{N}$  where  $(\phi \otimes x)(z) = \phi(z)x$  for  $\phi \in \mathcal{H}(\mathbb{D}, \mathbb{C})$  and  $x \in X$ . As usual we use  $p'$  for the conjugate exponent, i.e.  $1/p + 1/p' = 1$ , and  $C$  denotes a constant that may vary from line to line.

## 2. Preliminaries

**Definition 2.1.** Let  $1 \leq p < \infty$  and let  $X$  be a complex Banach space.  $B_p(X)$  is

defined as the space of  $X$ -valued analytic functions on the unit disc  $\mathbb{D}$  such that

$$\|f\|_{B_p(X)} = \left( \int_{\mathbb{D}} \|f(z)\|^p dm(z) \right)^{1/p} < \infty.$$

As in the scalar-valued case one gets the following facts, whose proofs are left to the reader.

**Proposition 2.2.** *Let  $1 \leq p < \infty$  and let  $X$  be a complex Banach space.*

(i)  $B_p(X)$  is a Banach space.

(ii) If  $f \in B_p(X)$  then  $\lim_{r \rightarrow 1} \|f - f_r\|_{B_p(X)} = 0$ , where  $f_r(z) = f(rz)$ .

(iii) The space of  $X$ -valued analytic polynomials  $\mathcal{P}(X)$  is dense in  $B_p(X)$ .

**Remark 2.3.** If  $H$  is a complex Hilbert space, then  $B_2(H)$  is also a Hilbert space under the scalar product given by

$$\langle\langle f, g \rangle\rangle = \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dm(z) \quad (f, g \in B_2(H))$$

where  $\langle \cdot, \cdot \rangle$  denotes the scalar product on  $H$ .

For any  $f \in B_2(H)$  such that  $f(z) = \sum_{n=0}^{\infty} x_n z^n$  we have

$$(2.1) \quad \|f\|_{B_2(H)} = \left( \sum_{n=0}^{\infty} \frac{\|x_n\|^2}{n+1} \right)^{1/2}.$$

This shows that  $B_2(H)$  is isometrically isomorphic to  $\ell_2(H)$ . Actually, if  $H$  is separable with an orthonormal basis  $(e_n)_{n \geq 0}$  then  $(\sqrt{n+1}u_n \otimes e_k)_{n,k \geq 0}$  is an orthonormal basis of  $B_2(H)$ .

Let us mention that (2.1) is no longer true for Banach spaces, as follows from the next easy example.

**Example 2.4.** Let  $2 < p < \infty$  and let  $(e_n)$  be the canonical basis of  $\ell_p$ . If  $f(z) = \sum_{n=0}^{\infty} e_n z^n = (z^n)_{n=0}^{\infty}$  then  $f \in B_2(\ell_p)$  but  $\sum_{n=0}^{\infty} \frac{\|e_n\|^2}{n+1} = \infty$ .

The reader is referred to [4] for further results on Taylor coefficients of functions in vector valued Bergman spaces and for connections with geometry of Banach spaces.

Let us point out that, as in the scalar-valued case, we have that for  $f \in \mathcal{H}(\mathbb{D}, X)$ ,  $0 < r < 1$  and  $1 \leq q \leq \infty$ , the following inequalities hold true:

$$(2.2) \quad r^2 M_q(f', r^2) \leq \frac{M_q(f, r)}{1-r}.$$

$$(2.3) \quad M_q(f, r) \leq \|f(0)\| + \int_0^r M_q(f', s) ds.$$

These facts can be used to get an equivalent norm in  $B_p(X)$  by looking at the derivatives of the function rather than the function itself.

**Theorem 2.5.** (see [22]) *Let  $f \in \mathcal{H}(\mathbb{D}, X)$ ,  $n \in \mathbb{N}$ ,  $1 \leq p < \infty$ . Then  $f \in B_p(X)$  if and only if the function  $z \mapsto (1 - |z|^2)^n f^{(n)}(z) \in L_p(m, X)$ .*

*Proof.* Let us show that for any  $g \in \mathcal{H}(\mathbb{D}, X)$  and  $k \geq 0$ , the function  $(1 - |z|^2)^k g(z)$  belongs to  $L_p(m, X)$  if and only if  $(1 - |z|^2)^{k+1} g'(z)$  also does. Then a recurrence argument gives the statement.

Note that  $(1 - |z|^2)^{k+1} g'(z) \in L_p(m, X)$  if and only if

$$\int_{\mathbb{D}} (1 - |z|^2)^{pk+p} \|zg'(z)\|^p dm(z) < \infty.$$

Let us denote  $h(z) = zg'(z) = \sum_{n=0}^{\infty} nx_n z^n$ , and observe that for each  $r < 1$  one has that  $h_{r^2} = g_r * \lambda_r$ , where  $\lambda_r(e^{i\theta}) = re^{i\theta}(1 - re^{i\theta})^{-2}$ .

Since  $M_1(\lambda, r) = \frac{r}{1-r^2}$  and  $M_p(h, r^2) \leq M_1(\lambda, r)M_p(g, r)$ , one gets that

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{pk+p} \|zg'(z)\|^p dm(z) &= \int_0^1 4r^3 (1 - r^4)^{pk+p} M_p^p(h, r^2) dr \\ &\leq \int_0^1 8r (1 - r^2)^{pk} M_p^p(g, r) dr \\ &= C \int_{\mathbb{D}} (1 - |z|^2)^{pk} \|g(z)\|^p dm(z). \end{aligned}$$

Conversely, let us take  $g$  such that  $(1 - r^2)^{k+1} M_p(g', r) \in L_p((0, 1), dr)$ . We may assume that  $\int_0^1 (1 - r)^{(k+1)p} M_p^p(g', r) dr = 1$  and also that  $g(0) = 0$ .

Thanks to (2.3) we have

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{kp} \|g(z)\|^p dm(z) &= \int_0^1 2r (1 - r^2)^{kp} M_p^p(g, r) dr \\ &\leq \int_0^1 2r (1 - r^2)^{kp} \left( \int_0^r M_p(g', s) ds \right)^p dr \\ &\leq C \int_0^1 (1 - r)^{kp} \left( \int_0^r M_p(g', s) ds \right)^p dr. \end{aligned}$$

For  $p = 1$  we get

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^k \|g(z)\| dm(z) &\leq C \int_0^1 (1 - r)^k \left( \int_0^r M_1(g', s) ds \right) dr \\ &= C \int_0^1 (1 - s)^{k+1} M_1(g', s) ds = C. \end{aligned}$$

For  $p > 1$ , we write for each  $t \in (0, 1)$

$$I_t = \int_0^t (1 - r)^{kp} \left( \int_0^r M_p(g', s) ds \right)^p dr.$$

Let  $u(r) = -\frac{1}{pk+1}(1 - r)^{pk+1}$  and  $v(r) = \left( \int_0^r M_p(g', s) ds \right)^p$ .

Since  $u(t)v(t) < 0$  and  $v(0) = 0$ , we have

$$I_t = \int_0^t u'(r)v(r)dr \leq - \int_0^t u(r)v'(r)dr$$

That is

$$\begin{aligned} I_t &\leq \frac{p}{pk+1} \int_0^t (1-r)^{pk+1} M_p(g', r) \left( \int_0^r M_p(g', s) ds \right)^{p-1} dr \\ &= \frac{p}{pk+1} \int_0^t (1-r)^{k+1} M_p(g', r) (1-r)^{(p-1)k} \left( \int_0^r M_p(g', s) ds \right)^{p-1} dr. \end{aligned}$$

Then the assumption and Hölder's inequality show that  $I_t \leq CI_t^{1/p'}$ . Hence  $I_t \leq C$  for all  $t$  and the proof is finished.  $\square$

Taking the formulation in terms of the first derivative, it makes sense to look at the extreme case  $p = \infty$  of Bergman spaces as functions in  $\mathcal{H}(\mathbb{D}, X)$  such that the function  $(1 - |z|)^2 f'(z)$  belongs to  $L_\infty(m, X)$ .

**Definition 2.6.** The *Bloch space*  $\mathcal{B}(X)$  is defined as the set of all functions in  $\mathcal{H}(\mathbb{D}, X)$  for which  $\sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\| < \infty$ . Under the norm

$$\|f\|_{\mathcal{B}(X)} = \|f(0)\| + \sup_{z \in \mathbb{D}} (1 - |z|^2) \|f'(z)\|$$

it becomes a Banach space.

The *little Bloch space*  $\mathcal{B}_0(X)$  is the subspace of  $\mathcal{B}(X)$  given by those functions for which

$$\lim_{r \rightarrow 1} (1 - r^2) M_\infty(f', r) = 0.$$

**Remark 2.7.**  $f \in \mathcal{B}(X)$  if and only if  $x^* f \in \mathcal{B}$  for all  $x^* \in X^*$ .

And, interchanging the suprema, we have that

$$(2.4) \quad \|f\|_{\mathcal{B}(X)} \approx \sup_{\|x^*\|=1} \|x^* f\|_{\mathcal{B}}$$

where  $x^* f(z) = \langle f(z), x^* \rangle$ .

**Proposition 2.8.** If  $f \in \mathcal{B}(X)$  then  $\|f\|_{\mathcal{B}(X)} = \lim_{r \rightarrow 1} \|f_r\|_{\mathcal{B}(X)}$ .

Proof. Note that

$$(1 - |z|^2) \|f'_r(z)\| = r(1 - |z|^2) \|f'(rz)\| \leq (1 - |rz|^2) \|f'(rz)\|$$

what implies that  $\|f_r\|_{\mathcal{B}(X)} \leq \|f\|_{\mathcal{B}(X)}$  for all  $0 < r < 1$ .

Now, given  $\varepsilon > 0$  take  $z_0 \in \mathbb{D}$  such that  $(1 - |z_0|^2) \|f'(z_0)\| > \|f\|_{\mathcal{B}(X)} - \varepsilon/2$  and take  $r_0$  verifying that  $r(1 - |z_0|^2) \|f'(rz_0)\| > (1 - |z_0|^2) \|f'(z_0)\| - \varepsilon/2$  for any  $r > r_0$ . Hence

$$\|f_r\|_{\mathcal{B}(X)} > \|f\|_{\mathcal{B}(X)} - \varepsilon.$$

$\square$

**Theorem 2.9.** *Let  $f \in \mathcal{B}(X)$ . The following are equivalent.*

- (i)  $f \in \mathcal{B}_0(X)$ .
- (ii)  $\lim_{r \rightarrow 1} \|f - f_r\|_{\mathcal{B}(X)} = 0$ .
- (iii)  $f$  belongs to the closure of  $\mathcal{P}(X)$ .

*Proof.* (i) $\Rightarrow$ (ii) Assume that  $\lim_{s \rightarrow 1} (1 - s^2)M_\infty(f', s) = 0$ . Note that for all  $0 < s < 1$  we have

$$\sup_{|z| < 1} (1 - |z|^2) \|f'(z) - rf'(rz)\| \leq 2 \sup_{|z| > s} (1 - |z|^2) M_\infty(f', |z|) + \sup_{|z| \leq s} \|f'(z) - f'_r(z)\|.$$

Hence, given  $\varepsilon > 0$  choose  $s_0 < 1$  such that  $\sup_{|z| > s_0} (1 - |z|^2) M_\infty(f', |z|) < \frac{\varepsilon}{4}$  and then use that  $f'_r$  converges uniformly on compact sets to get  $r_0 < 1$  such that  $\sup_{|z| \leq s_0} \|f'(z) - f'_r(z)\| < \frac{\varepsilon}{2}$  for  $r > r_0$ . Then  $\|f - f_r\|_{\mathcal{B}(X)} < \varepsilon$  for  $r > r_0$ .

(ii) $\Rightarrow$ (iii) Assume now that, for each  $\varepsilon > 0$ , there exists  $r_0 < 1$  such that  $\|f - f_{r_0}\|_{\mathcal{B}(X)} < \varepsilon/2$ . Now we can take a Taylor polynomial of  $f_{r_0}$   $P_N = P_N(f_{r_0})$  such that  $\|f_{r_0} - P_N\|_{H_\infty(X)} < \varepsilon/2$ . Therefore

$$\|f - P_N(f_{r_0})\|_{\mathcal{B}(X)} \leq \|f - f_{r_0}\|_{\mathcal{B}(X)} + \|f_{r_0} - P_N\|_{H_\infty(X)} < \varepsilon.$$

(iii) $\Rightarrow$ (i) Note that  $\mathcal{P}(X) \subset \mathcal{B}_0(X)$ , because if  $P \in \mathcal{P}(X)$  then

$$(1 - r^2)M_\infty(P', r) \leq 2(1 - r) \max_{|z| \leq 1} \|P'(z)\|.$$

Since  $\mathcal{B}_0(X)$  is closed the result is proved.  $\square$

### 3. Bergman kernels and projections.

Let us write  $K(z, w) = \frac{1}{(1-zw)^2}$  and  $K_z(w) = K(z, w)$  for  $z, w \in \mathbb{D}$ . That is

$$K_z = \sum_{n=0}^{\infty} (n+1)u_n z^n.$$

Since  $\|u_n\|_{H_p} = 1$ ,  $\|u_n\|_{B_p} \sim n^{-1/p}$  and  $\|u_n\|_{\mathcal{B}} \sim e^{-1}$  we have that, for each  $|z| < 1$ , the series  $\sum_{n=0}^{\infty} (n+1)u_n z^n$  is absolutely convergent considered as a  $\mathcal{B}$ ,  $H_p$  or  $B_p$ -valued function. This allows us to consider  $K : \mathbb{D} \rightarrow X$  given by  $K(z) = K_z$  as an  $X$ -valued analytic function where  $X$  is either  $\mathcal{B}$ ,  $H_p$  or  $B_p$  for  $1 \leq p \leq \infty$ .

We will call  $K(z, w)$  the *Bergman kernel*, and the map  $K : \mathbb{D} \rightarrow X$  the *Bergman function*. Of course  $(n+1)u_n$  are its Taylor coefficients, and its derivative is given by  $K'(z) = \sum_{n=1}^{\infty} (n+1)nu_n z^{n-1}$ , with  $K'(z)(w) = \frac{2w}{(1-zw)^3}$ .

In order to estimate the norms of  $K$  in different spaces we simply need the following lemmas.

**Lemma 3.1.** (see [15], page 65) *Let  $J_\alpha(r) = \int_0^1 \frac{dt}{|1-re^{it}|^\alpha}$  for  $\alpha > 0$ . Then*

- (i)  $J_\alpha(r)$  is bounded in  $(0, 1)$  for  $\alpha < 1$ ,

- (ii)  $J_\alpha(r) \sim \log \frac{1}{1-r}$  as  $r \rightarrow 1$  for  $\alpha = 1$ , and  
 (iii)  $J_\alpha(r) \sim \frac{1}{(1-r)^{\alpha-1}}$  as  $r \rightarrow 1$  for  $\alpha > 1$ .

**Lemma 3.2.** (see [22] 4.2.2.) Let  $I_{\alpha,\beta}(r) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-rw|^\beta} dm(w)$  for  $\beta > 0$  and  $\alpha > -1$ . Then

- (i)  $I_{\alpha,\beta}(r)$  is bounded in  $(0, 1)$  for  $\beta - \alpha < 2$ ,  
 (ii)  $I_{\alpha,\beta}(r) \sim \log \frac{1}{1-r}$  as  $r \rightarrow 1$  for  $\beta - \alpha = 2$ , and  
 (iii)  $I_{\alpha,\beta}(r) \sim \frac{1}{(1-r)^{\beta-\alpha-2}}$  as  $r \rightarrow 1$  for  $\beta - \alpha > 2$ .

From these lemmas we get the next estimates as  $|z| \rightarrow 1$ :

$$(3.1) \quad \|K(z)\|_{B_1} \sim \log \frac{1}{1-|z|} \text{ and } \|K(z)\|_{B_p} \sim \frac{1}{(1-|z|)^{2/p'}}, \text{ for } p > 1,$$

$$(3.2) \quad \|K(z)\|_{H_p} \sim \frac{1}{(1-|z|)^{2-1/p}} \text{ for } p \geq 1,$$

$$(3.3) \quad \|K'(z)\|_{B_p} \sim \frac{1}{(1-|z|)^{3-2/p}} \text{ and } \|K'(z)\|_{H_p} \sim \frac{1}{(1-|z|)^{3-1/p}},$$

$$(3.4) \quad \|K(z)\|_{\mathcal{B}} \sim \frac{1}{(1-|z|)^2} \text{ and } \|K'(z)\|_{\mathcal{B}} \sim \frac{1}{(1-|z|)^3}.$$

**Proposition 3.3.** Let  $1 \leq p, q < \infty$ . Let  $X \in \{B_q, H_q, \mathcal{B}, 1 \leq q \leq \infty\}$ .

- (i) The Bergman function  $K \in B_p(X)$  if and only if  $X = B_q$  and  $2p < q'$ .  
 (ii) The Bergman function  $K \in \mathcal{B}(X)$  if and only if  $X = B_1$ .  
 (iii) The Bergman function  $K \notin H_p(X)$ .

**Definition 3.4.** (see [14] or [12]) For any Banach space  $X$ , we denote by  $M(X)$  the Banach space of vector ( $X$ -valued) measures of bounded variation defined on the Borel subsets of  $\mathbb{D}$ , with norm given by  $\|G\|_1 = |G|(\mathbb{D})$ .

For  $1 < p < \infty$ . A measure  $G$  is said to have bounded  $p$ -variation,  $G \in V_p(m, X)$ , if

$$\|G\|_p = \sup_{\pi} \left( \sum_{A \in \pi} \frac{\|G(A)\|^p}{m(A)^{p-1}} \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all finite partitions  $\pi$  of  $\mathbb{D}$  into Borel sets of positive measure.

For  $p = \infty$  we have that  $G \in V_\infty(m, X)$  if there exists a constant  $C > 0$  such that  $\|G(A)\| \leq Cm(A)$  for any Borel set  $A$ , and its norm is given by

$$\|G\|_\infty = \sup \left\{ \frac{\|G(A)\|}{m(A)} : m(A) > 0 \right\}.$$

**Remark 3.5.** Given an  $X$ -valued simple measurable function  $f = \sum_{k=1}^n x_k \chi_{A_k}$  and a  $X^*$ -valued measure  $G$  we denote by

$$\langle f, G \rangle = \sum_{k=1}^n x_k^* x_k$$

where  $x_k^* = G(\bar{A}_k)$  and  $\bar{A}_k = \{z \in \mathbb{D} : \bar{z} \in A_k\}$ .

It is not difficult to see that if  $G \in V_p(m, X^*)$  this extends to a linear functional in  $L_{p'}(m, X)$  and actually we have the duality  $(L_{p'}(m, X))^* = V_p(m, X^*)$  under this pairing (see [14]).

If  $G$  is an  $X$ -valued measure of bounded variation, and  $\phi = \sum_{k=1}^n \alpha_k \chi_{E_k}$  is a simple function then we define

$$\int_{\mathbb{D}} \phi dG = \sum_{k=1}^n \alpha_k G(E_k).$$

Since  $\|\int_{\mathbb{D}} \phi dG\| \leq \|G\|_1 \|\phi\|_{\infty}$ , using the density of simple functions we extend the definition of  $\int_{\mathbb{D}} \phi dG$  for any bounded function  $\phi$ .

**Definition 3.6.** Let  $G \in M(X)$ . We define the Bergman projection of the measure  $G$  as the analytic function in the disc given by

$$PG(z) = \int_{\mathbb{D}} K_z(\bar{w}) dG(w) \in X.$$

Since  $\sup_{w \in \mathbb{D}} \|K_z(w)\| \leq \frac{1}{(1-|z|^2)^2}$  then  $PG(z)$  is well defined. Actually since the series  $K_z = \sum_{n=0}^{\infty} (n+1)u_n z^n$  is absolutely convergent in  $L_{\infty}(m)$  for each  $|z| < 1$  then for  $z \in \mathbb{D}$  we have

$$PG(z) = \sum_{n=0}^{\infty} x_n z^n,$$

where  $x_n = (n+1) \int_{\mathbb{D}} \bar{w}^n dG(w)$ .

**Remark 3.7.** If  $f \in L_1(m, X)$  then  $Pf(z) = \int_{\mathbb{D}} f(w) K_z(\bar{w}) dm(w)$ .

In particular we have that  $Pf = f$  for  $f \in B_1(X)$ .

Indeed, if  $f(z) = \sum_{n=0}^{\infty} x_n z^n$  then

$$\begin{aligned} (n+1) \int_{\mathbb{D}} f(w) \bar{w}^n dm(w) &= (n+1) \int_0^1 2r^{n+1} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{in\theta} d\theta \right) dr \\ &= (n+1) \left( \int_0^1 2r^{2n+1} dr \right) x_n = x_n. \end{aligned}$$

This shows that the Taylor coefficients of  $f$  and  $Pf$  coincide.

**Theorem 3.8.** Let  $X$  be a complex Banach space and  $1 < p < \infty$ . Then the Bergman projection  $P$  is bounded from  $V_p(m, X)$  onto  $B_p(X)$ .

Proof. Since  $G \in V_p(m, X)$  there exists a nonnegative  $\phi$  in  $L_p(m)$  such that  $d|G| = \phi dm$  and  $\|\phi\|_p = \|G\|_p$  (see [14], page 243).

Now, for each  $z \in \mathbb{D}$  we have

$$\begin{aligned} \|PG(z)\| &= \left\| \int_{\mathbb{D}} K_z(\bar{w}) dG(w) \right\| \\ &\leq \int_{\mathbb{D}} |K_z(\bar{w})| d|G|(w) \\ &= \int_{\mathbb{D}} |K_z(\bar{w})| \phi(w) dm(w), \end{aligned}$$

Now to finish the proof, let us recall that if  $P^*(f)(z) = \int_{\mathbb{D}} |K(z, \bar{w})| f(w) dm(w)$  then  $P^* : L_p(m) \rightarrow L_p(m)$  defines a bounded operator for any  $1 < p < \infty$  (see for instance [22] for a proof).

Therefore  $\|PG\|_{B_p(X)} \leq \|P^*(\phi)\|_{L_p} \leq C\|\phi\|_{L_p} = C\|G\|_p$ .  $\square$

This allows, as in the scalar valued case, to get the duality result for vector-valued Bergman spaces.

**Theorem 3.9.** *Let  $X$  be a complex Banach space and  $1 < p < \infty$ . Then  $(B_p(X))^*$  is isometrically isomorphic to  $B_{p'}(X^*)$ .*

Proof. Let us define the linear operator  $J : B_{p'}(X^*) \rightarrow (B_p(X))^*$  given by

$$(Jg)(f) = \int_{\mathbb{D}} g(z) f(\bar{z}) dm(z).$$

It follows from Hölder's inequality that  $J$  is bounded. Let us see that it is injective. If  $g$  verifies that  $Jg = 0$ , then for each  $n \in \mathbb{N}$  and  $x \in X$  we have

$$(Jg)(f_n) = \left( \int_{\mathbb{D}} g(z) \bar{z}^n dm(z) \right) x = 0,$$

where  $f_n = u_n \otimes x$ . This shows that  $\int_{\mathbb{D}} g(z) \bar{z}^n dm(z) = \frac{1}{(n+1)!} g^{(n)}(0) = 0$  for all  $n \in \mathbb{N}$  and hence  $g = 0$ .

Let us now show that  $J$  is surjective.

Given  $\xi \in (B_p(X))^*$ , the Hahn-Banach theorem gives an extension  $\tilde{\xi} \in (L_p(m, X))^*$  with the same norm. Using duality (see Remark 3.5) there exists a vector valued measure  $G \in V_{p'}(m, X^*)$ , with  $p'$ -variation equal to  $\|\xi\|$ , for which  $\tilde{\xi}\varphi = \int_{\mathbb{D}} \varphi dG$  for every  $\varphi \in L_p(m, X)$ .

Let  $G_c$  be the measure defined by  $G_c(E) = G(\bar{E})$  for each measurable set  $E \subset \mathbb{D}$ . Clearly  $G_c$  has the same  $p'$ -variation as  $G$ , and

$$\int_{\mathbb{D}} \psi(z) dG_c(z) = \int_{\mathbb{D}} \psi(\bar{z}) dG(z).$$

for any simple function  $\psi$ . Define  $g = PG_c$ . From Theorem 3.8 we get  $g \in B_{p'}(X^*)$ . Let us see that  $Jg = \xi$ :

For any  $f \in \mathcal{P}(X)$  we can write

$$\begin{aligned}
(Jg)(f) &= \int_{\mathbb{D}} PG_c(z)f(\bar{z})dm(z) = \int_{\mathbb{D}} \left( \int_{\mathbb{D}} K_z(\bar{w})dG_c(w) \right) f(\bar{z})dm(z) \\
&= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} K_z(w)dG(w) \right) f(\bar{z})dm(z) = \int_{\mathbb{D}} \left( \int_{\mathbb{D}} K_z(w)f(\bar{z})dm(z) \right) dG(w) \\
&= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} K_w(\bar{z})f(z)dm(z) \right) dG(w) = \int_{\mathbb{D}} Pf(w)dG(w) \\
&= \int_{\mathbb{D}} f(w)dG(w) = \xi(f).
\end{aligned}$$

□

**Proposition 3.10.**  *$P$  is not bounded neither on  $M(X)$  nor on  $V_\infty(m, X)$ .*

*Proof.* Assume that  $P$  is bounded on  $V_\infty(m, X)$ . Using measures  $dG = (\phi \otimes x)dm$  for  $\phi \in L_\infty(m)$  and  $x \in X$  we also have that the corresponding Bergman projection is bounded on  $L_\infty(m)$ . In such case

$$\sup_{|z|<1} \left| \int_{\mathbb{D}} K_z(w)\phi(w)dm(w) \right| \leq C\|\phi\|_\infty$$

for all  $\phi \in L_\infty(m)$ . Hence  $\sup_{|z|<1} \|K_z\|_{L_1(m)} \leq C$ , but we have previously noticed that  $\|K_z\|_{L_1(m)} = \|K(z)\|_{B_1} \sim \log \frac{1}{1-|z|}$  as  $|z| \rightarrow 1$ .

The case  $p = 1$  follows now looking at the adjoint operator. □

**Theorem 3.11.** *The Bergman projection  $P$  defines a bounded operator from  $V_\infty(m, X)$  onto  $\mathcal{B}(X)$ .*

*Proof.* Let  $G$  belong to  $V_\infty(m, X)$ . Therefore there exists  $C > 0$  such that

$$|G|(A) \leq Cm(A)$$

for all measurable sets  $A$ . Now from the Radon-Nikodym theorem there exists  $\phi \in L_\infty(m)$  such that  $d|G| = \phi dm$  and  $\|\phi\|_{L_\infty} = \|G\|_\infty$ .

On the other hand  $PG(z) = \sum_{n=0}^{\infty} x_n z^n$ , where  $x_n = (n+1) \int_{\mathbb{D}} \bar{z}^n dG(z)$ .

Since  $(PG)'(z) = \int_{\mathbb{D}} \frac{2\bar{w}}{(1-\bar{w}z)^3} dG(w)$  we have

$$\|(PG)'(z)\| \leq \int_{\mathbb{D}} \frac{2}{|1-z\bar{w}|^3} \phi(w)dm(w) \leq C \frac{\|\phi\|_{L_\infty}}{1-|z|}.$$

Let us prove the surjectivity. Let  $f \in \mathcal{B}(X)$  with  $f(0) = f'(0) = 0$ . If  $f(z) = \sum_{n=2}^{\infty} x_n z^n$ , let  $g$  be given by

$$g(z) = \frac{(1-|z|^2)f'(z)}{\bar{z}}.$$

We have that  $g \in L_\infty(m, X)$  since  $f \in \mathcal{B}(X)$  and  $f'(0) = 0$ . Now write  $Pg(z) = \sum_{n=0}^{\infty} y_n z^n$  and take  $n \geq 1$

$$\begin{aligned} y_n &= (n+1) \int_{\mathbb{D}} g(z) \bar{z}^n dm(z) = (n+1) \int_{\mathbb{D}} (1-|z|^2) f'(z) \bar{z}^{n-1} dm(z) \\ &= (n+1) \int_{\mathbb{D}} f'(z) \bar{z}^{n-1} dm(z) - (n+1) \int_{\mathbb{D}} z f'(z) \bar{z}^n dm(z) = x_n. \end{aligned}$$

Also  $y_0 = 0$ . That is  $PG = f$  for  $dG = gdm$ .

The general case follows by writing  $f = f(0) + u_1 \otimes f'(0) + f_1$  where  $f_1$  is as above. So if  $Pg_1 = f_1$  then  $P(f(0) + u_1 \otimes f'(0) + g_1) = f$ .  $\square$

Let us recall that the Riesz projection  $R: L_p(\mathbb{T}) \rightarrow H_p(\mathbb{T})$  defined by  $R(f) = \sum_{n \geq 0} \hat{f}(n) u_n$  gives, as happens for the Bergman projection on  $L_p(m)$ , a bounded operator only for  $1 < p < \infty$ . Nevertheless  $H_1(\mathbb{T})$  is not isomorphic to any complemented subspace of  $L_1(\mathbb{T})$ , so we cannot define any bounded projection from  $L_1(\mathbb{T})$  to  $H_1(\mathbb{T})$ , while we can define several bounded projections from  $L_1(m)$  to  $B_1$  (see [22]). Let us extend this also to the vector valued setting.

**Definition 3.12.** For any  $G \in M(X)$ , we can also define

$$\tilde{P}G(z) = \int_{\mathbb{D}} \tilde{K}_z(\bar{w}) dG(w),$$

where the kernel  $\tilde{K}_z(w) = \frac{2(1-|w|^2)}{(1-wz)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)v_n(w)z^n$  and  $v_n(w) = (1-|w|^2)w^n$ .

Hence  $\tilde{P}(G)(z) = \sum_{n=0}^{\infty} x_n z^n$  where  $x_n = (n+1)(n+2) \int_{\mathbb{D}} (1-|w|^2) \bar{w}^n dG(w)$ .

**Theorem 3.13.**  $\tilde{P}$  defines a bounded projection from  $M(X)$  onto  $B_1(X)$ .

*Proof.* Let us first see that  $B_1(X)$  is left invariant under  $\tilde{P}$ .

Let  $dG(w) = f(w)dm(w)$  for some  $f(z) = \sum_{n=0}^{\infty} x_n z^n$  in  $B_1(X)$ . Let us show that the Taylor coefficients of  $f$  and  $\tilde{P}(f)$  coincide.

$$\begin{aligned} \int_{\mathbb{D}} (1-|w|^2) \bar{w}^n dG(w) &= \int_{\mathbb{D}} (1-|w|^2) \bar{w}^n f(w) dm(w) \\ &= \int_0^1 2r^{n+1} (1-r^2) \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta} d\theta \right) dr \\ &= \left( \int_0^1 2r^{2n+1} (1-r^2) dr \right) x_n \\ &= \frac{x_n}{(n+1)(n+2)}. \end{aligned}$$

Given now  $G \in M(X)$  we can write

$$\begin{aligned} \int_{\mathbb{D}} \|\tilde{P}G(z)\| dm(z) &= \int_{\mathbb{D}} \left\| \int_{\mathbb{D}} \tilde{K}_z(\bar{w}) dG(w) \right\| dm(z) \\ &\leq \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |\tilde{K}_z(\bar{w})| |dG(w)| \right) dm(z) \\ &= \int_{\mathbb{D}} \left( \int_{\mathbb{D}} |\tilde{K}_z(\bar{w})| dm(z) \right) |dG(w)|. \end{aligned}$$

Using Lemma 3.2 for  $\alpha = 0$  and  $\beta = 3$  we have

$$\int_{\mathbb{D}} |\tilde{K}_z(\bar{w})| dm(z) = \int_{\mathbb{D}} \frac{2(1-|w|^2)}{|1-wz|^3} dm(z) \leq C$$

and then  $\|\tilde{P}G\|_{B_1(X)} \leq C|G|(\mathbb{D})$ .  $\square$

**Theorem 3.14.**  $B_1(X)$  is isometrically isomorphic to  $B_1 \hat{\otimes} X$ .

*Proof.* Let  $\tilde{P}$  and  $\tilde{P}_X$  the respective projections from  $L_1(m)$  and  $L_1(m, X)$  onto  $B_1$  and  $B_1(X)$  given above. By the properties of the projective tensor product,  $\tilde{P} \otimes \text{id}_X$  is a projection from  $L_1(m) \hat{\otimes} X$  onto  $B_1 \hat{\otimes} X$ . Then the usual isometry  $J$  between  $L_1(m) \hat{\otimes} X$  and  $L_1(m, X)$  restricts to an operator  $\tilde{J}$  from  $B_1 \hat{\otimes} X$  such that  $\tilde{J}(\tilde{P} \otimes \text{id}_X) = \tilde{P}_X \tilde{J}$ , and  $\tilde{J}$  is an isometry between  $B_1 \hat{\otimes} X$  and  $B_1(X)$ .  $\square$

**Remark 3.15.** The Bloch space was first shown to be a dual space in [1]. In fact one has that  $(B_1)^* = \mathcal{B}$  and  $(\mathcal{B}_0)^* = B_1$  (see [22]) under the pairing

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} dm(z),$$

which is well defined for polynomials and then extends by density for functions in  $B_1$ .

That it is well defined and bounded is seen as the first part in the following proposition:

**Proposition 3.16.**

- (i) If  $T \in \mathcal{L}(B_1, X)$  then  $f_T(z) = T(K_z) \in \mathcal{B}(X)$ .
- (ii) If  $f \in \mathcal{B}(X)$ , the linear operator defined by  $T_f(\phi) = \int_{\mathbb{D}} f(z) \phi(\bar{z}) dm(z)$  for each polynomial  $\phi$  extends to a bounded operator in  $\mathcal{L}(B_1, X)$ .
- (iii)  $\mathcal{B}(X)$  is isomorphic to  $\mathcal{L}(B_1, X)$ .

*Proof.* (i) Let  $g(z) = \frac{2w}{(1-zw)^3}$ . One easily sees that  $f'_T(z) = T(g_z)$  and  $\|g_z\|_1 \sim 1/(1-|z|)$ . This shows that  $f_T \in \mathcal{B}(X)$ .

(ii) From Remark 2.7  $x^*f \in \mathcal{B}$  and then  $x^*T_f \in (B_1)^*$  for all  $x^* \in X^*$ . Since  $\|T_f(\phi)\| = \sup_{\|x^*\|=1} |x^*T_f(\phi)|$ , we have that  $T_f$  is bounded. Moreover

$$\|T_f\| = \sup_{\|x^*\|=1} \|x^*T_f\|_{(B_1)^*} \approx \sup_{\|x^*\|=1} \|x^*f\|_{\mathcal{B}} \approx \|f\|_{\mathcal{B}(X)}.$$

(iii) follows easily from (i) and (ii).  $\square$

We will explore further this interplay between functions and operators in section 4.

**Corollary 3.17.** (see [7])  $\mathcal{B}(X^*)$  is isomorphic to  $(B_1(X))^*$ .

Proof. Since  $(X \hat{\otimes} Y)^* = \mathcal{L}(X, Y^*)$ , Proposition 3.16 and Theorem 3.14 give the result.  $\square$

#### 4. Vector-valued functions and operators.

Given two complex Banach spaces  $X$  and  $Y$  there are two natural ways of looking at a map  $F: \mathbb{D} \times X \rightarrow Y$ : it can be regarded as a map from  $X$  into  $Y^{\mathbb{D}}$  or, alternatively, from  $\mathbb{D}$  into  $Y^X$  (and vice-versa). More precisely, given  $F: \mathbb{D} \times X \rightarrow Y$ , we can define  $F_x: \mathbb{D} \rightarrow Y$  and  $F_z: X \rightarrow Y$  by

$$F_x(z) = F_z(x) = F(z, x)$$

for any  $x \in X$  and  $z \in \mathbb{D}$ .

**Proposition 4.1.** Let  $F: \mathbb{D} \times X \rightarrow Y$  be a continuous map such that  $F_z$  is linear for all  $z \in \mathbb{D}$ . Then  $F_z \in \mathcal{L}(X, Y)$  for all  $z$ , and the norm  $\|F_z\|$  is locally bounded.

Proof. First statement is immediate. To see the second one, let us assume there exists a compact set  $K \subset \mathbb{D}$  where  $\{\|F_z\|; z \in K\}$  is not bounded. By the Banach-Steinhaus theorem the set  $A = \{x \in X; \sup_{z \in K} \|F(z, x)\|_Y = \infty\}$  will be dense in  $X$ . Then we can take two sequences  $(x_j) \subset X$  and  $(z_j) \subset K$  such that  $x_j \rightarrow 0$  and  $\|F(z_j, x_j)\| \geq j$ . By the compactness of  $K$ , passing to a subsequence we see that we can assume that  $(z_j)$  converges to certain  $z_0 \in K$ . But  $(z_j, x_j)$  tends to  $(z_0, 0)$  and  $F(z_0, 0) = 0$ , so  $F$  cannot be continuous.  $\square$

**Theorem 4.2.** Let  $F: \mathbb{D} \times X \rightarrow Y$  be continuous, such that  $F_z$  is linear for all  $z \in \mathbb{D}$  and  $F_x$  is analytic for all  $x \in X$ . Then

- (i) The map  $z \mapsto F_z$  is an  $\mathcal{L}(X, Y)$ -valued analytic function.
- (ii) The operator  $x \mapsto F_x$  is linear, and continuous with respect to the topology of the uniform convergence on compact sets on the space  $\mathcal{H}(\mathbb{D}, Y)$ .

Proof. For each  $n \geq 0$  we define

$$T_n x = \frac{1}{2\pi i} \int_{|z|=r} \frac{F_z(x)}{z^{n+1}} dz.$$

Of course  $F_x(z) = \sum_{n=0}^{\infty} (T_n x) z^n$ .

It is clear that  $T_n$  is linear for each  $n \in \mathbb{N}$ . By the previous proposition, there exists  $C_r$  such that  $\|F_z\| \leq C_r$  for all  $z \in \overline{D(0, r)}$ , and then  $\|T_n x\| \leq C_r \|x\|/r^n$ .

Now for each  $z \in \mathbb{D}$  the series  $\sum_{n \geq 0} T_n z^n$  is absolutely convergent in  $\mathcal{L}(X, Y)$ . Hence  $z \mapsto \sum_{n \geq 0} T_n z^n$  defines an analytic function from  $\mathbb{D}$  into  $\mathcal{L}(X, Y)$ .

To see (ii), note that the linearity is immediate, so it suffices to see that if  $x_j \rightarrow 0$  then  $F_{x_j} \rightarrow 0$  uniformly on compact sets. If  $K \subset \overline{D(0, r)}$  is compact and we take  $s \in (r, 1)$  and  $C$  such that  $\|T_n\| \leq C/s^n$ , then for any  $z \in K$ ,

$$\|F_{x_j}(z)\|_Y \leq \sum_{n \geq 0} \|T_n\| \|x_j\| |z|^n \leq C \|x_j\| \sum_{n \geq 0} \left(\frac{r}{s}\right)^n = \frac{Cs}{s-r} \|x_j\|.$$

□

**Definition 4.3.** Let  $X, Y$  be two complex Banach spaces and let  $F(z) = \sum_{n=0}^{\infty} T_n z^n$  be a function in  $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$ . We denote by  $T_F : X \rightarrow \mathcal{H}(\mathbb{D}, Y)$  the linear operator given by

$$(T_F x)(z) = (F(z))(x) = \sum_{n=0}^{\infty} (T_n x) z^n.$$

**Theorem 4.4.**

$\mathcal{B}(\mathcal{L}(X, Y))$  is isomorphic to  $\mathcal{L}(X, \mathcal{B}(Y))$  (via the map  $F \mapsto T_F$ ).

Proof. Let  $F \in \mathcal{B}(\mathcal{L}(X, Y))$  we have that

$$\sup_{\|x\|=1} \|F_x(0)\|_Y = \|F(0)\|_{\mathcal{L}(X, Y)}$$

and also

$$\begin{aligned} \sup_{\|x\|=1} \sup_{z \in \mathbb{D}} (1 - |z|^2) \|F'(z)x\|_Y &= \sup_{z \in \mathbb{D}} \sup_{\|x\|=1} (1 - |z|^2) \|F'(z)x\|_Y \\ &= \sup_{z \in \mathbb{D}} (1 - |z|^2) \|F'(z)\|_{\mathcal{L}(X, Y)}. \end{aligned}$$

This shows that  $\|T_F\|_{\mathcal{L}(X, \mathcal{B}(Y))} \approx \|F\|_{\mathcal{B}(\mathcal{L}(X, Y))}$ .

Now given  $T \in \mathcal{L}(X, \mathcal{B}(Y))$  we can define  $F : \mathbb{D} \times X \rightarrow Y$  by  $F(z, x) = Tx(z)$ . Now by Theorem 4.2 we have that  $z \mapsto F_z$  belongs to  $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$ . Since  $T_F = T$  the previous identities complete the proof. □

**Proposition 4.5.** Let  $1 \leq p < \infty$ .  $B_p(\mathcal{L}(X, Y)) \subset \mathcal{L}(X, B_p(Y))$  (via the mapping  $F \mapsto T_F$ ). In general  $B_p(\mathcal{L}(X, Y)) \neq \mathcal{L}(X, B_p(Y))$ .

Proof. Given  $F \in B_p(\mathcal{L}(X, Y))$  we clearly have

$$\begin{aligned} \|T_F x\|_{B_p(Y)} &= \left( \int_{\mathbb{D}} \|F(z)x\|_Y^p dm(z) \right)^{1/p} \\ &\leq \|x\| \left( \int_{\mathbb{D}} \|F(z)\|_{\mathcal{L}(X, Y)}^p dm(z) \right)^{1/p} = \|F\|_{B_p(X)} \|x\|. \end{aligned}$$

To see that this inclusion is not surjective, let us take  $X = \ell_1$  and  $Y = \mathbb{C}$ . The inclusion becomes  $B_p(\ell_\infty) = B_p(\mathcal{L}(\ell_1, \mathbb{C})) \hookrightarrow \mathcal{L}(\ell_1, B_p) = \ell_\infty(B_p)$ .

Let us consider the function  $f(z) = \frac{1}{(1-z)^{1/p}}$ . Clearly

$$\|f\|_{B_p}^p \sim \int_0^1 \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|1-re^{i\theta}|} d\theta \right) dr \sim \int_0^1 \log \frac{1}{1-r} dr < \infty.$$

Now let  $(\zeta_n)$  be a dense sequence in the unit circle, and define  $f_n(z) = f(\zeta_n z)$  for each  $n$  and  $F(z) = (f_n(z))_{n \in \mathbb{N}}$ .

By the density of  $(\zeta_n)$  one gets

$$\|F(z)\|_{\ell_\infty} = \sup_n |f_n(z)| = \sup_n |f(\zeta_n z)| = M_\infty(f, |z|) = \frac{1}{(1-|z|)^{1/p}}$$

for every  $z \in \mathbb{D}$ . Then, despite  $(f_n)$  obviously belongs to  $\ell_\infty(B_p)$ , we get that the vector valued function  $F$  does not belong to  $B_p(\ell_\infty)$  since

$$\int_0^1 M_p^p(F, r) dr = \int_0^1 \frac{1}{1-r} dr = \infty.$$

□

Let us now introduce an interesting ideal of operators that play an important role in understanding the interpretation of vector-valued Bergman functions as operators.

**Definition 4.6.** Let  $X$  and  $Y$  be Banach spaces and  $1 \leq p < \infty$ . A linear operator  $T \in \mathcal{L}(X, Y)$  is said to be  $p$ -summing (denoted  $T \in \Pi_p(X, Y)$ ) if there is a constant  $C > 0$  such that for every  $k \in \mathbb{N}$  and  $x_1, x_2, \dots, x_k \in X$  we have

$$\left( \sum_{i=1}^k \|T(x_i)\|^p \right)^{1/p} \leq C \sup_{\|x^*\|_{X^*} \leq 1} \left( \sum_{i=1}^k |\langle x_i, x^* \rangle|^p \right)^{1/p}.$$

Its norm is given by the infimum of the constants  $C$  satisfying the previous inequality and is denoted by  $\pi_p(T)$ .

The reader is referred to [13], [21], [19] or [18] for results and references on these classes of operators. We simply include the following remark to be used in the sequel.

**Remark 4.7.** (see for instance [21]) Let  $(\Omega, \Sigma, \mu)$  be a measure space, let  $f : \Omega \rightarrow X$  be a measurable function such that  $x^* f \in L_p(\mu)$  for all  $x^* \in X^*$  and  $T \in \Pi_p(X, Y)$ . Then  $Tf : \Omega \rightarrow Y$  given by  $Tf(\omega) = T(f(\omega))$  belongs to  $L_p(\mu, Y)$ .

**Proposition 4.8.** Let  $1 \leq p < \infty$ . Then  $B_p(\Pi_p(X, Y)) \subset \Pi_p(X, B_p(Y))$  (via the mapping  $F \mapsto T_F$ ).

There exist infinite dimensional Banach spaces  $X$  and  $Y$  such that

$$B_p(\Pi_p(X, Y)) \neq \Pi_p(X, B_p(Y)).$$

Proof. Let  $x_1, x_2, \dots, x_n$  be elements in  $X$ . Then

$$\begin{aligned} \sum_{k=1}^n \|T_F x_k\|_{B_p(Y)}^p &= \sum_{k=1}^n \int_{\mathbb{D}} \|F(z)x\|_Y^p dm(z) \\ &= \int_{\mathbb{D}} \sum_{k=1}^n \|F(z)(x_k)\|^p dm(z) \\ &\leq \int_{\mathbb{D}} \pi_p^p(F(z)) \sup_{\|x^*\|=1} \sum_{k=1}^n |\langle x^*, x_k \rangle|^p dm(z) \\ &= \|F\|_{B_p(\Pi_p(X,Y))}^p \sup_{\|x^*\|=1} \sum_{k=1}^n |\langle x^*, x_k \rangle|^p. \end{aligned}$$

To see that the embedding is not surjective even for infinite dimensional Banach space we can take  $p = 2$ ,  $X = \ell_1$  and  $Y = \ell_2$ . It is well known (see [18]) that  $\Pi_2(\ell_1, H) = \mathcal{L}(\ell_1, H)$  for any Hilbert space (actually  $\Pi_1 \subseteq \Pi_2$ , and Grothendieck theorem (see [13] or [18]) even says that  $\Pi_1(\ell_1, H) = \mathcal{L}(\ell_1, H)$ ). Hence, in our situation  $\Pi_2(X, Y) = \mathcal{L}(X, Y)$  and  $\Pi_2(X, B_2(Y)) = \mathcal{L}(X, B_2(Y))$ . Therefore we simply need to show that  $B_2(\ell_\infty(\ell_2))$  is strictly contained in  $\ell_\infty(B_2(\ell_2))$ .

Let us define now  $f_n(z) = \frac{1}{\log(n+1)} \sum_{k=1}^{\infty} (1 - 1/n)^k e_k z^k$  where  $e_k$  is the canonical basis of  $\ell_2$ .

Using (2.1) one has that

$$\|f_n\|_{B_2(\ell_2)} = \frac{1}{\log(n+1)} \left( \sum_{k=1}^{\infty} \frac{(1 - 1/n)^{2k}}{k+1} \right)^{1/2}.$$

Hence  $\sup_n \|f_n\|_{B_2(\ell_2)} < \infty$ .

On the other hand, for all  $n \in \mathbb{N}$

$$\begin{aligned} \|f_n(z)\|_{\ell_2} &= \frac{1}{\log(n+1)} \left( \sum_{k=1}^{\infty} (1 - 1/n)^{2k} |z|^{2k} \right)^{1/2} \\ &\geq C \frac{1}{\log(n+1)} \left( \sum_{k=n}^{\infty} |z|^{2k} \right)^{1/2} \\ &\geq C \frac{1}{\log(n+1)} \frac{|z|^n}{(1 - |z|^2)^{1/2}}. \end{aligned}$$

This shows that  $F \notin B_2(\ell_\infty(\ell_2)) = B_2(\Pi_2(X, Y))$  while  $(f_n) \in \ell_\infty(B_2(\ell_2))$  and therefore  $T_F \in \Pi_2(X, B_2(Y))$ .  $\square$

**Definition 4.9.** Let  $X, Y$  be two complex Banach spaces and let  $F(z) = \sum_{n=0}^{\infty} T_n z^n$  be a function in  $\mathcal{H}(\mathbb{D}, \mathcal{L}(X, Y))$ . We denote by  $S_F : \mathcal{P}(X) \rightarrow Y$  the linear operator given by

$$S_F(g) = \int_{\mathbb{D}} F(z)(g(\bar{z})) dm(z) = \sum_{n \geq 0} \frac{T_n(x_n)}{n+1}$$

for  $g = \sum_{n \geq 0} u_n \otimes x_n$ .

**Theorem 4.10.** (see [7])  $\mathcal{B}(\mathcal{L}(X, Y)) = \mathcal{L}(B_1(X), Y)$  (via the map  $F \mapsto S_F$ ) with equivalent norms.

Proof. Theorem 3.14, Proposition 3.16 and  $\mathcal{L}(X \hat{\otimes} Y, Z) = \mathcal{L}(X, \mathcal{L}(Y, Z))$  imply that

$$\mathcal{L}(B_1 \hat{\otimes} X, Y) = \mathcal{L}(B_1, \mathcal{L}(X, Y)) = \mathcal{B}(\mathcal{L}(X, Y)).$$

It is rather clear that the mapping which gives the isomorphism is actually  $F \mapsto S_F$ .  
□

**Proposition 4.11.** Let  $1 < p < \infty$  and let  $X$  and  $Y$  be complex Banach spaces. Then  $B_p(\mathcal{L}(X, Y))$  (resp.  $\mathcal{B}_0(\mathcal{L}(X, Y))$ ) is isomorphically embedded in  $\mathcal{K}(B_{p'}(X), Y)$  (resp.  $\mathcal{K}(B_1(X), Y)$ ), via the map  $F \mapsto S_F$ .

Proof. For the case  $F \in \mathcal{B}_0(\mathcal{L}(X, Y))$ , Theorem 4.10 gives  $\|F\|_{\mathcal{B}(\mathcal{L}(X, Y))} \approx \|S_F\|$ . In the case  $F \in B_p(\mathcal{L}(X, Y))$ . Clearly

$$\|S_F(g)\| \leq \int_{\mathbb{D}} \|F(z)\| \|g(\bar{z})\| dm(z) \leq \|F\|_{B_p(\mathcal{L}(X, Y))} \|g\|_{B_{p'}(X)}.$$

So  $\|S_F\| \leq C \|F\|_{B_p(\mathcal{L}(X, Y))}$ . The compactness of  $S_F$  in both cases follows from the fact that  $\mathcal{P}(X)$  is dense in the corresponding spaces and for polynomials  $F$  then  $S_F$  is finite rank operator.  
□

**Remark 4.12.** Let  $X, Y$  be two complex Banach spaces. If  $T : \mathcal{P}(X) \rightarrow Y$  is a linear operator such that the linear operators  $T_n : X \rightarrow Y$  given by  $T_n(x) = T(u_n \otimes x)$  are bounded and  $\limsup_{n \rightarrow \infty} \|T_n\|^{1/n} \leq 1$  then we can define the  $\mathcal{L}(X, Y)$ -valued analytic function

$$F_T(z) = \sum_{n \geq 0} (n+1) T_n z^n.$$

It is worth mentioning that this is the inverse map of  $F \mapsto S_F$ , so that  $F_{S_F} = F$  and  $S_{F_T} = T$ .

**Definition 4.13.** Let  $1 < p < \infty$ , and let  $X$  be a complex Banach space and  $T \in \mathcal{L}(B_{p'}, X)$ . We define  $f_T \in \mathcal{H}(\mathbb{D}, X)$  given by

$$f_T(z) = T(K_z).$$

**Remark 4.14.**  $T \in \mathcal{L}(B_1, X)$  if and only if  $f_T \in \mathcal{B}(X)$  (see Proposition 3.16).

If  $p > 2$  and  $T \in \mathcal{L}(B_{p'}, X)$  then  $f_T \in B_q(X)$  for  $1 \leq q < \frac{p}{2}$  (use Proposition 3.3).

We would like to find some properties of  $T$  to get that  $f_T \in B_p(X)$ .  
For that purpose we need to use the following class of operators.

**Definition 4.15.** (see [6] and [5]) Let  $E$  be a Banach lattice and  $Y$  a Banach space. A linear operator  $T \in \mathcal{L}(E, Y)$  is said to be positive  $p$ -summing (denoted  $T \in \Lambda_p(E, Y)$ ) if there is a constant  $C > 0$  such that for every  $k \in \mathbb{N}$  and positive elements  $e_1, e_2, \dots, e_k \in E$  we have

$$\left( \sum_{i=1}^k \|T(e_i)\|^p \right)^{1/p} \leq C \sup_{\|e^*\|_{E^*} \leq 1} \left( \sum_{i=1}^k |\langle e_i, e^* \rangle|^p \right)^{1/p}.$$

Its norm is given by the infimum of the constants  $C$  satisfying the previous inequality and denoted by  $\lambda_p(T)$ .

**Remark 4.16.** In the case  $p = 1$  these operators are also known as cone absolutely summing (c.a.s) operators (see [20]).

In this case,  $T \in \Lambda_1(E, Y)$  if and only if there is a constant  $C > 0$  such that for every  $k \in \mathbb{N}$  and positive elements  $e_1, e_2, \dots, e_k \in E$  we have

$$(4.1) \quad \sum_{i=1}^k \|T(e_i)\| \leq C \left\| \sum_{i=1}^k e_i \right\|.$$

It is easy to see that  $\Lambda_{p_1}(E, Y) \subset \Lambda_{p_2}(E, Y)$  if  $p_1 < p_2$ , and it was shown in [5] that, for  $E = L_p(\mu)$ , we have  $\Lambda_r(E, Y) = \Lambda_1(E, Y)$  for all  $1 \leq r \leq p'$ .

**Theorem 4.17.** Let  $1 < p < \infty$  and  $X$  a Banach space.

- (i) If  $T \in \mathcal{L}(B_{p'}, X)$  and  $f_T \in B_p(X)$  then  $T$  is compact.
- (ii) If  $T \in \Pi_p(B_{p'}, X)$  then  $f_T \in B_p(X)$ .
- (iii) If  $T \in \Lambda_p(L_{p'}(m), X)$  and  $T_1$  denotes its restriction to  $B_{p'}$  then  $f_{T_1} \in B_p(X)$ .

*Proof.* To see (i) we show that  $T = S_{f_T}$  and then (ii) in Proposition 4.11 gives the compactness.

Indeed, since  $f_T(z) = T(K_z) = \sum_{n=0}^{\infty} (n+1)T u_n z^n$ , we have for any  $m \in \mathbb{N}$

$$S_{f_T}(u_m) = \int_{\mathbb{D}} \sum_{n=0}^{\infty} (n+1)T u_n \bar{z}^n z^m dm(z) = T(u_m).$$

To prove (ii) let us first observe that if  $\phi \in B_p$

$$\langle K(z), \phi \rangle = \int_{\mathbb{D}} K_z(w) \phi(\bar{w}) dm(w) = \phi(z).$$

Hence it follows that the function  $K: \mathbb{D} \rightarrow B_{p'}$  verifies, for all  $\phi \in (B_{p'})^*$ , that  $z \mapsto \langle K(z), \phi \rangle$  belongs to  $L_p(m)$ . Now Remark 4.7 gives that  $f_T(z) = T(K_z) \in L_p(m, X)$ .

To see (iii) let us observe first that the measure  $G(E) = T(\chi_E)$  belongs to  $V_p(m, X)$ .

Indeed, for any partition  $\pi$  we have

$$\begin{aligned} \sum_{A \in \pi} \frac{\|G(A)\|^p}{m(A)^{p-1}} &= \sum_{A \in \pi} \left\| T\left(\frac{\chi_A}{m(A)^{1/p'}}\right) \right\|^p \\ &\leq \lambda_p^p(T) \sup \left\{ \sum_{A \in \pi} \left| \int_A g(z) dm(z) \frac{1}{m(A)^{1/p'}} \right|^p : \|g\|_p = 1 \right\} \\ &= \lambda_p^p(T) \sup \left\{ \left\| \sum_{A \in \pi} \frac{\int_A g(z) dm(z)}{m(A)} \chi_A \right\|_p^p : \|g\|_p = 1 \right\} \\ &\leq \lambda_p^p(T) \end{aligned}$$

Given  $z \in \mathbb{D}$  we get, taking  $G_c(E) = G(\bar{E})$ ,

$$f_{T_1}(z) = T(K_z) = \int_{\mathbb{D}} K_z(w) dG(w) = \int_{\mathbb{D}} K_z(\bar{w}) dG_c(w) = PG_c(z),$$

and then  $F_T = PG_c \in B_p(X)$  according to Theorem 3.8.  $\square$

**Theorem 4.18.** *Let  $1 < p < \infty$ , and let  $X$  be a complex Banach space and  $F \in B_1(X)$ . Then  $F \in B_p(X)$  if and only if the linear operator  $\Phi_F(\phi) = \int_{\mathbb{D}} F(z)\phi(\bar{z})dm(z)$  defined on the subspace of simple functions extends to an operator in  $\Lambda_p(L_{p'}(m), X)$ .*

*Moreover  $\|F\|_{B_p(X)} \sim \lambda_p(\Phi_F)$ .*

*Proof.* Let us assume that  $F \in B_p(X)$ , which ensures that  $\Phi_F \in \mathcal{L}(L_{p'}(m), X)$ . Now take positive functions  $\phi_1, \phi_2, \dots, \phi_n \in L_{p'}(m)$ . We have that

$$\begin{aligned} \sum_{k=1}^n \|\Phi_F \phi_k\|^p &= \sum_{k=1}^n \left| \int_{\mathbb{D}} F(\bar{z}) \phi_k(z) dm(z) \right|^p \\ &\leq \|F\|_{B_p(X)}^p \sum_{k=1}^n \left( \int_{\mathbb{D}} \frac{\|F(z)\|}{\|F\|_{B_p(X)}} \phi_k(z) dm(z) \right)^p \\ &\leq \|F\|_{B_p(X)}^p \sup_{\|\psi\|_{p'}=1} \sum_{k=1}^n |\langle \psi, \phi_k \rangle|^p. \end{aligned}$$

This shows that  $\lambda_p(\Phi_F) \leq C\|F\|_{B_p(X)}$ .

To see the converse let us observe that  $f_S = F$ , where  $S$  denotes the restriction of  $\Phi_F$  to  $B_{p'}$ . Indeed, for all  $z \in \mathbb{D}$

$$S(K_z) = \Phi_F(K_z) = \int_{\mathbb{D}} F(w) K_z(\bar{w}) dm(w) = F(z).$$

Now (iii) in Theorem 4.17 gives that  $F \in B_p(X)$  and  $\|F\|_{B_p(X)} \leq C\lambda_p(\Phi_F)$ .  $\square$

From Theorem 3.16 we have that  $\mathcal{B}(X) = \mathcal{L}(B_1, X)$ . The next result covers the cases  $1 < p < \infty$ .

**Corollary 4.19.** *Let  $1 < p < \infty$ . Then*

$$B_p(X) = \{T : B_{p'} \rightarrow X : TP \in \Lambda_p(L_{p'}(m), X)\}.$$

Moreover  $\lambda_p(TP) \approx \|f_T\|_{B_p(X)}$ .

*Proof.* If  $TP$  is positive  $p$ -summing then (iii) in Theorem 4.17 gives that  $F \in B_p(X)$  for  $F(z) = T(K_z)$ .

To see the converse, assume that  $F \in B_p(X)$ . Let  $\Phi_F$  as in Theorem 4.18 and let  $T$  be its restriction to  $B_{p'}$ . Now take the vector measure defined by  $G(E) = T(P(\chi_E))$  and denote  $G_c(E) = G(\bar{E})$  for all measurable set  $E$ . We have that

$$\begin{aligned} G_c(E) &= T\left(\int_{\bar{E}} K(\cdot, \bar{w}) dm(w)\right) = T\left(\int_E K(\cdot, w) dm(w)\right) \\ &= \int_E T(K(\cdot, w)) dm(w) = \int_E \tilde{\Phi}_F(K_w) dm(w) = \int_E F(w) dm(w). \end{aligned}$$

Therefore  $dG_c = F dm$ .

This obviously implies that  $TP(\phi) = \int_{\mathbb{D}} F(z)\phi(\bar{z}) dm(z)$  for all  $\phi \in L_{p'}(m)$ , and, in particular

$$\|TP(\phi)\| \leq \int_{\mathbb{D}} \|F(z)\|\phi(\bar{z}) dm(z)$$

for all positive  $\phi \in L_{p'}(m)$ .

A simple computation using (4.1) now shows that  $TP$  is cone absolutely summing and hence also positive  $p$ -summing.  $\square$

## 5. $B_p(X)$ is complemented in $\ell_p(X)$ .

A classical result in the theory of Bergman spaces is the isomorphism between  $B_p$  and  $\ell_p$  for each  $p \geq 1$  (see [21]). It is enough to see that  $B_p$  is isomorphic to a complemented subspace of  $\ell_p$ , since then it is automatically isomorphic to  $\ell_p$ . In the vector case, Theorem 3.14 gives the isomorphism for  $p = 1$ :

**Theorem 5.1.** *For any complex Banach space  $X$ ,  $B_1(X)$  is isomorphic to  $\ell_1(X)$ .*

*Proof.*  $B_1(X)$  is isomorphic to  $B_1 \hat{\otimes} X$ , and then to  $\ell_1 \hat{\otimes} X = \ell_1(X)$ .  $\square$

As for  $p > 1$ , we will show next that  $B_p(X)$  is isomorphic to a complemented subspace of  $\ell_p(X)$ . The proof follows similar ideas to the ones used to get a so-called atomic decomposition of  $B_p$  (see [22], theorem 4.4.6).

For each  $z \in \mathbb{D}$ , let  $\varphi_z$  the involutive Möbius transformation fixing the unit disc and verifying  $\varphi_z(0) = z$  and  $\varphi_z(z) = 0$ , that is

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

The Bergman metric between  $z$  and  $w$  is defined by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

Note that  $|\varphi_z(w)|$  is the hyperbolic tangent of  $\beta(z, w)$ .

This distance  $\beta$  is not bounded on  $\mathbb{D}$ , and for any  $z \in \mathbb{D}$  and  $r > 0$  the  $\beta$ -ball

$$E(z, r) = \{w \in \mathbb{D}; \beta(w, z) < r\}$$

is the euclidean disc with center  $\frac{1-s^2}{1-s^2|z|^2}z$  and radius  $\frac{1-|z|^2}{1-s^2|z|^2}s$ , where  $s = \tanh r$ .

One relevant connection between Bergman metric and Bloch spaces is the following result:

**Theorem 5.2.** (see [22], 5.1.6.)  $\beta(z, w) \sim \sup\{|f(z) - f(w)|; \|f\|_{\mathcal{B}} \leq 1\}$  (with constants independent from  $z$  and  $w$  in  $\mathbb{D}$ ).

In particular this allows us to get the following remark.

**Corollary 5.3.** If  $F \in \mathcal{B}(X)$  then  $F : \mathbb{D} \rightarrow X$  is a Lipschitz map with respect to the Bergman metric.

*Proof.* A look at Proposition 3.16 gives that  $F(z) = T(K_z)$  for some  $T \in \mathcal{L}(B_1, X)$ . Hence

$$\begin{aligned} \|F(z) - F(w)\| &\leq C\|T\|\|K_z - K_w\|_{B_1} \\ &\leq C\|T\| \sup\{|\xi(K_z - K_w)|; \xi \in B_1^*\} \\ &\sim \sup\{|f(z) - f(w)|; \|f\|_{\mathcal{B}} = 1\} \\ &\sim \beta(z, w). \end{aligned}$$

□

The key point in order to relate  $B_p(X)$  to  $\ell_p(X)$  is the use of sequences in  $\mathbb{D}$  with good separation properties with respect to Bergman metric. The next lemma resumes some well known results (see for instance [22]):

**Lemma 5.4.** There exists a number  $N \in \mathbb{N}$  such that, for any  $r \leq 1$ , we can take a sequence  $(\lambda_n)$  in  $\mathbb{D}$  and a decomposition of  $\mathbb{D}$  into a disjoint union of measurable sets  $E_n$  such that

- (i)  $E(\lambda_n, r/4) \subseteq E_n \subseteq E(\lambda_n, r)$  for every  $n$ ,
  - (ii) every point in  $\mathbb{D}$  belongs to no more than  $N$  discs from  $\{E(\lambda_n, 2r)\}$ ,
  - (iii)  $|E_n| \sim |E(\lambda_n, r)| \sim |E(\lambda_n, 2r)| \sim (1 - |\lambda_n|^2)^2 \sim |E(w, r)|$  for any  $w \in E(\lambda_n, 2r)$
- and
- (iv)  $1 - |\lambda_n|^2 \leq C(1 - |z|^2)$  for each  $z \in E(\lambda_n, 2r)$ .

The well known fact that, for all  $0 < p < \infty$ ,  $|f|^p$  is a subharmonic function with respect to  $\beta$ -balls for any analytic function  $f$ , also holds true in the vector valued setting.

**Lemma 5.5.** *Let  $X$  be any complex Banach space, let  $f \in \mathcal{H}(\mathbb{D}, X)$  and  $p > 0$ . There exists a constant  $C > 0$  such that we have*

$$\|f(z)\|^p \leq \frac{C}{|E(z, r)|} \int_{E(z, r)} \|f(w)\|^p dm(w)$$

for any  $r \leq 1$  and  $z \in \mathbb{D}$ .

Proof. From the scalar valued case we get  $C > 0$  such that

$$|x^* f(z)|^p \leq \frac{C}{|E(z, r)|} \int_{E(z, r)} |x^* f(w)|^p dm(w) \leq \frac{C}{|E(z, r)|} \int_{E(z, r)} \|f(w)\|^p dm(w)$$

for all  $r \leq 1$  and  $z \in \mathbb{D}$ .

Now take the supremum over the unit ball of  $X^*$  to finish the proof.  $\square$

**Corollary 5.6.** *Let  $r < 1$  and  $p > 1$ , and let  $X$  be a Banach space. Let  $Q_r = Q_{r,p,X}: B_p(X) \rightarrow L_p(m, X)$  be defined by*

$$Q_r(f) = \sum_{n=1}^{\infty} f(\lambda_n) \chi_{E_n}.$$

Then  $Q_r$  is a bounded operator.

Proof. By Lemmas 5.5 and 5.4

$$\begin{aligned} \sum_{n=1}^{\infty} |E_n| \|f(\lambda_n)\|^p &\leq C \sum_{n=1}^{\infty} \int_{E(\lambda_n, r)} \|f(z)\|^p dm(z) \\ &= \int_{\mathbb{D}} \|f(z)\|^p \sum_{n=1}^{\infty} \chi_{E(\lambda_n, r)}(z) dm(z) \\ &\leq CN \int_{\mathbb{D}} \|f(z)\|^p dm(z). \end{aligned}$$

This shows the boundedness of  $Q_{r,p,X}$ .  $\square$

**Lemma 5.7.** *Let  $r \leq 1$ . The linear operator*

$$f \mapsto \sum_{n=1}^{\infty} (f - f(\lambda_n)) \chi_{E_n}$$

is bounded from  $B_p(X)$  to  $L_p(m, X)$ , and its norm is less or equal than  $C \tanh r$ .

Proof. Let  $z \in E_n$ , and observe that

$$\|f(z) - f(\lambda_n)\| = \left\| \int_{[\lambda_n, z]} f'(w) dw \right\| \leq \left( \sup_{w \in [\lambda_n, z]} \|f'(w)\| \right) |z - \lambda_n|.$$

Since  $E(w, r) \subset E(\lambda_n, 2r)$  for any  $z \in E_n$  and  $w \in [\lambda_n, z]$ , by Lemma 5.5 and the properties of  $(\lambda_n)$  we have that

$$\|f'(w)\|^p \leq \frac{C}{|E(w, r)|} \int_{E(w, r)} \|f'\|^p dm \leq \frac{C}{|E_n|} \int_{E(\lambda_n, 2r)} \|f'\|^p dm.$$

Hence if  $w \in [\lambda_n, z]$

$$\|f(z) - f(\lambda_n)\|^p \leq \frac{C}{|E_n|} \left( \int_{E(\lambda_n, 2r)} \|f'\|^p dm \right) |z - \lambda_n|^p.$$

Let  $s = \tanh r$ . As  $E(\lambda_n, r)$  is a disc with center  $z_0 = \frac{1-s^2}{1-s^2|\lambda_n|^2} \lambda_n$  and radius  $R = \frac{1-|\lambda_n|^2}{1-s^2|\lambda_n|^2} s$ , for any  $z$  in it

$$|z - \lambda_n| \leq R + |\lambda_n - z_0| = \frac{1-|\lambda_n|^2}{1-s^2|\lambda_n|^2} s(1+s|\lambda_n|) \leq Cs(1-|\lambda_n|^2),$$

and then

$$\|f(z) - f(\lambda_n)\|^p \leq \frac{C}{|E_n|} s^p \left( \int_{E(\lambda_n, 2r)} \|f'\|^p dm \right) (1-|\lambda_n|^2)^p.$$

Therefore

$$\int_{E_n} \|f(z) - f(\lambda_n)\|^p dm(z) \leq Cs^p (1-|\lambda_n|^2)^p \int_{E(\lambda_n, 2r)} \|f'\|^p dm.$$

We use now that  $(1-|\lambda_n|^2)^p \leq C(1-|z|^2)^p$  for each  $z \in E(\lambda_n, 2r)$ , and then

$$\int_{E_n} \|f(z) - f(\lambda_n)\|^p dm(z) \leq Cs^p \int_{E(\lambda_n, 2r)} (1-|z|^2)^p \|f'(z)\|^p dm(z).$$

Hence

$$\sum_{n=1}^{\infty} \int_{E_n} \|f(z) - f(\lambda_n)\|^p dm(z) \leq CNs^p \int_{\mathbb{D}} (1-|z|^2)^p \|f'(z)\|^p dm(z),$$

which is bounded by  $Cs^p \|f\|_{B_p(X)}^p$  in view of Theorem 2.5.  $\square$

**Corollary 5.8.** *There exist  $r_0 > 0$  such that  $PQ_{r,p,X} : B_p(X) \rightarrow B_p(X)$  is an isomorphism for all  $r < r_0$ ,  $1 < p < \infty$  and all Banach spaces  $X$ .*

*Proof.* We shall show this by noting that, if  $I$  denotes the identity in  $B_p(X)$ , then  $\|I - PQ_r\|$  tends to zero as  $r \rightarrow 0$ . Recall that then, if  $r$  is such that  $\|I - PQ_r\| < 1$ , the inverse of  $PQ_r$  is just  $\sum_{n=0}^{\infty} (I - PQ_r)^n$ .

Now from Lemma 5.7 one has that  $I - PQ_r \in \mathcal{L}(B_p(X), B_p(X))$  and  $\|I - PQ_r\| \leq C\|P\| \tanh r$ .  $\square$

**Theorem 5.9.** *For every  $p > 1$  and every complex Banach space  $X$ , the Bergman space  $B_p(X)$  is isomorphic to a complemented space of  $\ell_p(X)$ .*

Proof. We take  $r$  small enough to have that  $PQ_r$  is an isomorphism on  $B_p(X)$ . Then the identity in  $B_p(X)$  factorizes as  $I = (PQ_r)^{-1}PQ_r$ . Now write  $\tilde{Q}_r : B_p(X) \rightarrow \ell_p(X)$  for the operator given by

$$\tilde{Q}_r(f) = (|E_n|^{1/p}f(\lambda_n))$$

and  $J : \ell_p(X) \rightarrow L_p(m, X)$  for the one given by

$$J((x_n)) = \sum_{n=1}^{\infty} |E_n|^{-1/p}x_n\chi_{E_n}.$$

Since  $J$  is an embedding and  $\tilde{Q}_r$  is bounded due to Corollary 5.6 we can factorize the identity as  $I = (PQ_r)^{-1}PJ\tilde{Q}_r$  and therefore  $B_p(X)$  is isomorphic to the image of  $\tilde{Q}_r$  in  $\ell_p(X)$ .  $\square$

**Theorem 5.10.** *Let  $r < 1$ ,  $p > 1$  and  $X$  be a Banach space. Let  $P_r = P_{r,p,X}$  the linear operator  $P_r : V_p(m, X) \rightarrow B_p(X)$  defined by*

$$P_r(G) = \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G(\bar{E}_n).$$

*Then the linear operator  $P_r$  is bounded.*

*Moreover  $\|P_{r,p,X}\| = \|Q_{r,p',X^*}\|$ .*

Proof. For any polynomial  $g \in \mathcal{P}(X^*)$  we have that

$$\left\langle \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G(\bar{E}_n), g \right\rangle = \sum_{n=1}^{\infty} G(\bar{E}_n)g(\lambda_n) = \left\langle \sum_{n=1}^{\infty} g(\lambda_n)\chi_{E_n}, G \right\rangle = \langle Q_{r,p',X^*}(g), G \rangle.$$

Since  $V_p(m, X)$  is isometrically embedded in  $(L_{p'}(m, X^*))^*$  it follows that

$$\left| \left\langle \sum_{n=1}^{\infty} K_{\lambda_n} \otimes G(E_n), g \right\rangle \right| \leq \|G\|_{V_p(m, X)} \left\| \sum_{n=1}^{\infty} g(\lambda_n)\chi_{E_n} \right\|_{L_{p'}(m, X^*)}.$$

Now Corollary 5.6 gives that  $\|P_{r,p,X}\| \leq \|Q_{r,p',X^*}\|$ .

A similar argument shows that  $Q_{r,p',X^*}^* = P_{r,p,X}$ , giving the other inequality.  $\square$

Let us now compare  $\bar{P}_r$  and the Bergman projection on  $V_p(m, X)$ .

**Theorem 5.11.** *Let  $p > 1$  and  $X$  be a Banach space. Then*

*(i)  $\lim_{r \rightarrow 0} P_r = P$ .*

*(ii) The restriction of  $P_r$  to  $B_p(X)$  given by*

$$P_r(g) = \sum_{n=1}^{\infty} K_{\lambda_n} \otimes \int_{E_n} g(z)dm(z)$$

*is an isomorphism for  $r$  close enough to zero.*

Proof. An easy computation shows that  $\langle P(G), g \rangle = \langle g, G \rangle$  for any  $G \in V_p(m, X)$  and  $g \in \mathcal{P}(X^*)$ .

Therefore for all  $G \in V_p(m, X)$  and  $g \in \mathcal{P}(X^*)$  we have

$$\langle (P - P_r)(G), g \rangle = \langle g - \sum_{n=1}^{\infty} g(\lambda_n) \chi_{E_n}, G \rangle.$$

Now applying Lemma 5.7 we get  $\|P - P_r\| \leq C \tanh r$  and (i) follows.

(ii) is proved the same way as Corollary 5.8.  $\square$

Note that  $\beta(z, w) = \beta(\bar{z}, \bar{w})$ , and then  $(\bar{\lambda}_n)$  and  $(\bar{E}_n)$  satisfy the same estimates and properties as  $(\lambda_n)$  and  $(E_n)$ .

**Theorem 5.12.** *Let  $X$  be a Banach space and  $p > 1$ . For each  $f \in B_p(X)$  we denote*

$$S_r(f) = \sum_{n=1}^{\infty} |E_n| K_{\lambda_n} \otimes f(\bar{\lambda}_n).$$

Then  $f = \lim_{r \rightarrow 0} S_r(f)$  in  $B_p$ .

Proof. We shall see that  $S_r = S_{r,p,X} : B_p(X) \rightarrow B_p(X)$  are bounded operators and  $\lim_{r \rightarrow 0} S_r = I$ .

Let us denote by  $\bar{Q}_r$  the operator associated to  $(\bar{\lambda}_n)$  and  $(\bar{E}_n)$ , that is  $\bar{Q}_r(f) = \sum_{n=1}^{\infty} f(\bar{\lambda}_n) \chi_{\bar{E}_n}$ . We actually have  $S_r = P_r \bar{Q}_r$  and

$$\|I - P_r \bar{Q}_r\| \leq \|I - P \bar{Q}_r\| + \|P \bar{Q}_r - P_r \bar{Q}_r\| \leq \|I - P \bar{Q}_r\| + \|P - P_r\| \|Q_r\|.$$

The result follows from Corollary 5.8 and Theorem 5.11.  $\square$

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