On the unit ball of operator-valued H^2-functions

Oscar Blasco

Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain oblasco@uv.es[†]

Summary. Let X be a complex Banach space and let $H^2(\mathbb{D}, X)$ stand for the space of X-valued analytic functions in the unit disc such that $\sup_{0<r<1} \int_0^{2\pi} \|F(re^{it})\|^2 \frac{dt}{2\pi} < \infty$. It is shown that a function F belongs to the unit ball of $H^2(\mathbb{D}, X)$ if and only if there exist $f \in H^\infty(\mathbb{D}, X)$ and $\phi \in H^\infty(\mathbb{D})$ such that $\|f(z)\|^2 + |\phi(z)|^2 \leq 1$ and $F(z) = \frac{f(z)}{1-z\phi(z)}$ for $|z| < 1$.

1 Introduction.

Let $1 \leq p \leq \infty$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, let X be a complex Banach space and denote by $H^p(\mathbb{T}, X)$, $1 \leq p \leq \infty$, the set of functions $f \in L^p(\mathbb{T}, X)$ such that $\hat{f}(n) = 0$ for $n < 0$, and by $H^p(\mathbb{D}, X)$ the set of holomorphic functions $F : \mathbb{D} \to X$ such that, for $1 \leq p < \infty$,

$$\|F\|_{H^p(\mathbb{D},X)} = \sup_{0<r<1} \Big(\int_0^{2\pi} \|F(re^{it})\|^p \frac{dt}{2\pi} \Big)^{1/p} < \infty \tag{1}$$

and

$$\|F\|_{H^\infty(\mathbb{D},X)} = \sup_{|z|<1} \|F(z)\| < \infty.$$

Is is elementary to see that $H^p(\mathbb{T}, X) \subseteq H^p(\mathbb{D}, X)$ by means of the Poisson integral. That is to say if $f \in H^p(\mathbb{T}, X)$ then $F(re^{it}) = P_r * f(e^{it}) \in H^p(\mathbb{D}, X)$ and with the same norm. Actually $H^p(\mathbb{T}, X)$ can be identified with the closure of the polynomials in $H^p(\mathbb{D}, X)$ for $1 \leq p < \infty$. It is a well known fact that for $X = \mathbb{C}$ both the spaces $H^p(\mathbb{T}) = H^p(\mathbb{D})$ for $1 \leq p \leq \infty$ due to the fact that functions in $H^p(\mathbb{D})$ do have non-tangential boundary values almost everywhere which belong to $H^p(\mathbb{T})$. The reader should be aware that this is not longer true for infinite dimensional Banach spaces. Spaces satisfying that any function in $H^\infty(\mathbb{D}, X)$ has radial boundary limits almost everywhere are

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said to have the *analytic Radon Nikodym property* (ARNP for short). This property was first considered by A.V. Bukhvalov and A.A. Danilevich (see [3] or [2]) and it was shown to be equivalent to the fact that functions in $H^p(\mathbb{D},X)$ for some (and equivalently for all) $1\le p<\infty$ have radial boundary limits a.e. It was observed that the well known Radon-Nikodym property (RNP for short) (see [4] for the definition) was actually equivalent to the fact that any X-valued bounded harmonic functions in the unit disc has boundary limits a.e. Hence RNP implies ARNP. It was shown that $X=c_0$ fails to have ARNP and that $L^1(\mu)$ is a space with ARNP but without RNP. The fact that a Hilbert space $\mathcal{H}$ has the ARNP follows, among many other ways, from the description of functions $F\in H^2(\mathbb{D},\mathcal{H})$ as $F(z)=\sum_{n=0}^{\infty}h_nz^n$ where $h_n\in\mathcal{H}$ and

$$\|F\|_{H^2(\mathbb{T},\mathcal{H})}=\|F\|_{H^2(\mathbb{D},\mathcal{H})}=(\sum_{n=0}^{\infty}\|h_n\|^2)^{1/2}.$$

One can also consider the weak vector-valued Hardy spaces, denoted by $H^p_{weak}(\mathbb{T},X)$ and $H^p_{weak}(\mathbb{D},X)$, consisting in weakly measurable functions $f:\mathbb{T}\to X$ such that $\langle f(e^{it}),x^*\rangle\in H^p(\mathbb{T})$ for any $x^*\in X^*$ and X-valued analytic functions $F:\mathbb{D}\to X$ such that $\langle F(z),x^*\rangle\in H^p(\mathbb{D})$ for any $x^*\in X^*$, where we use the notation $\langle\cdot,\cdot\rangle$ for the duality pairing. Of course, from the scalar valued result, one has that $H^p_{weak}(\mathbb{T},X)=H^p_{weak}(\mathbb{D},X)$. The norm in the space is given by

$$\|F\|_{H^p_{weak}(\mathbb{D},X)}=\sup_{\|x^*\|=1}\|\langle F(\cdot),x^*\rangle\|_{H^p}. \tag{2}$$

Actually in the case of dual spaces X^* we can even consider another spaces $H^p_{weak^*}(\mathbb{T},X)$ and $H^p_{weak^*}(\mathbb{D},X)$ given by w^*-measurable functions $f:\mathbb{T}\to X^*$ such that $\langle f(e^{it}),x\rangle\in H^p(\mathbb{T})$ for any $x\in X$ and X^*-valued analytic functions $F:\mathbb{D}\to X^*$ such that $\langle F(z),x\rangle\in H^p(\mathbb{D})$ for any $x\in X^*$. Again we have $H^p_{weak^*}(\mathbb{T},X^*)=H^p_{weak^*}(\mathbb{D},X^*)$ and the norm now is given by

$$\|F\|_{H^p_{weak^*}(\mathbb{D},X^*)}=\sup_{\|x\|=1}\|\langle F(\cdot),x\rangle\|_{H^p}. \tag{3}$$

Clearly the $H^p_{weak}(\mathbb{D},X^*)$ is embedded into $H^p_{weak^*}(\mathbb{D},X^*)$ and both spaces coincide for reflexive spaces X but, in general, they are different.

For $p=2$ and $X=\mathcal{H}$ a Hilbert space, one has the following useful description: $F\in H^2_{weak}(\mathbb{D},\mathcal{H})$ if $F(z)=\sum_{n=0}^{\infty}h_nz^n$ where $h_n\in H$ and

$$\|F\|_{H^2_{weak}(\mathbb{D},\mathcal{H})}=\|F\|_{H^2_{weak^*}(\mathbb{D},\mathcal{H})}=\sup_{\|h\|=1}(\sum_{n=0}^{\infty}|\langle h_n,h\rangle|^2)^{1/2}.$$

The case $H^p_{weak*}(\mathbb{D},X^*)$ is a particular instance of a more general notion that is defined for operator-valued functions. Let $1\le p\le\infty$, let X_1,X_2 be Banach spaces and $X=\mathcal{L}(X_1,X_2)$ and denote by $H^p_{strong}(\mathbb{T},\mathcal{L}(X_1,X_2))$ the set of functions $f:\mathbb{T}\to\mathcal{L}(X_1,X_2)$ such that $e^{it}\to f(e^{it})(x_1)\in H^p(\mathbb{T},X_2)$ for all $x_1\in X_1$ and we write

$$\|f\|_{H^p_{strong}(\mathbb{T},\mathcal{L}(X_1,X_2))}=\sup_{\|x_1\|=1}\|f(\cdot)(x_1)\|_{L^p(\mathbb{T},X_2)}. \tag{4}$$

Similarly we denote by $H^p_{strong}(\mathbb{D},\mathcal{L}(X_1,X_2))$ the set of holomorphic functions $F:\mathbb{D}\to\mathcal{L}(X_1,X_2)$ such that $z\to F(z)(x_1)\in H^p(\mathbb{D},X_2)$ for all $x_1\in X_1$ and we write

$$\|F\|_{H^p_{strong}(\mathbb{D},\mathcal{L}(X_1,X_2))}=\sup_{\|x_1\|=1}\|F(\cdot)(x_1)\|_{H^p(\mathbb{D},X_2)}. \tag{5}$$

Of course $H^p_{strong}(\mathbb{T},\mathcal{L}(X_1,X_2))\subsetneq H^p_{strong}(\mathbb{D},\mathcal{L}(X_1,X_2))$ as it is shown by taking $X_1=\mathbb{C}$ and $X_2=X$ not having the analytic Radon-Nikodym property.

It is also elementary to see that, for $1\le p\le\infty$,

$$H^p(\mathbb{D},\mathcal{L}(X_1,X_2))\subseteq H^p_{strong}(\mathbb{D},\mathcal{L}(X_1,X_2))\subseteq H^p_{weak}(\mathbb{D},\mathcal{L}(X_1,X_2)) \tag{6}$$

with continuous inclusions. Actually in the case $p=\infty$, taking into account that

$$\|T\|=\sup_{\|x_1\|=1}\|T(x_1)\|=\sup_{\|x_1\|=1,\|x_2^*\|=1}|\langle T(x_1),x_2^*\rangle|$$

for $T\in\mathcal{L}(X_1,X_2)$, one can conclude that

$$H^\infty(\mathbb{D},\mathcal{L}(X_1,X_2))=H^\infty_{strong}(\mathbb{D},\mathcal{L}(X_1,X_2))=H^\infty_{weak}(\mathbb{D},\mathcal{L}(X_1,X_2)) \tag{7}$$

Let us see the expressions of the norms in these different spaces in the particular case $X=\mathcal{L}(\mathcal{H},\mathcal{H}')$ where $\mathcal{H}$ and $\mathcal{H}'$ are Hilbert spaces.

Let $F:\mathbb{D}\to\mathcal{L}(\mathcal{H},\mathcal{H}')$ be an analytic function given by $F(z)=\sum_{k=0}^\infty T_kz^k$ where $T_k\in\mathcal{L}(\mathcal{H},\mathcal{H}')$.

$F\in H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ if and only if

$$\|F\|_{H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))}=\sup_{0<r<1}\Big(\int_0^{2\pi}\|\sum_{k=0}^\infty T_kr^ke^{ikt}\|^2_{\mathcal{L}(\mathcal{H},\mathcal{H}')}\frac{dt}{2\pi}\Big)^{1/2}<\infty. \tag{8}$$

$F\in H^2_{strong}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ if and only if

$$\|F\|_{H^2_{strong}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))}=\sup_{\|h\|=1}\Big(\sum_{k=0}^\infty\|T_k(h)\|^2\Big)^{1/2}<\infty. \tag{9}$$

$F\in H^2_{weak^*}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ if and only if

$$\|F\|_{H^2_{weak^*}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))}=\sup_{\|h\|=1,\|h'\|=1}\Big(\sum_{k=0}^\infty|\langle T_k(h),h'\rangle|^2\Big)^{1/2}<\infty \tag{10}$$

which follows from the fact $(\mathcal{H}\hat{\otimes}\mathcal{H}')^*=\mathcal{L}(\mathcal{H},\mathcal{H}')$.

Let us mention that for infinite dimensional Hilbert spaces the Hardy spaces considered above are different.

Proposition 1. *Let $\mathcal{H}$ and $\mathcal{H}'$ be separable infinite dimensional Hilbert spaces. Then*

$$H^2(\mathbb{T}, \mathcal{L}(\mathcal{H}, \mathcal{H}')) \subsetneq H^2(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}')).$$

Proof. Let (e_k) and (e'_k) be orthonormal basis of $\mathcal{H}$ and $\mathcal{H}'$, and write $e_k \otimes e'_k \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ for the rank 1 operator given by $h \to \langle h, e_k\rangle e'_k$. Define $T_k = e_k \otimes e'_k$ and $F(z) = \sum_{k=0}^{\infty}(e_k \otimes e'_k)z^k$. Therefore

$$F(z)(h) = \sum_k \langle h, e_k\rangle z^k e'_k.$$

It follows that $\|F(z)\|_{\mathcal{L}(\mathcal{H},\mathcal{H}')} = \sup_{n\geq 0} |z^n| = 1$. This shows that F is bounded and therefore $F \in H^2(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}')$.

To see that $F \notin H^2(\mathbb{T}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$ note that if there exists $\lim_{r\to 1} F(re^{it}) = F(e^{it})$ in $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ for some $t \in [0, 2\pi)$ then $F(e^{it})(h) = \sum_k \langle h, e_k\rangle e^{ik\theta} e'_k$. On the other hand $F(z)$ is compact for any $z \in \mathbb{D}$ (because $(z^n) \in c_0$ for $|z| < 1$) but $F(e^{it})$ is not compact. □

Proposition 2. *Let $\mathcal{H}$ and $\mathcal{H}'$ be separable infinite dimensional Hilbert spaces. Then*

$$H^2(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}')) \subsetneq H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}')) \subsetneq H^2_{weak}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}')).$$

Proof. Fix $h' \in \mathcal{H}'$ with $\|h'\|_{\mathcal{H}'} = 1$ and (e_k) an orthonormal basis of $\mathcal{H}$. Define $T_k = e_k \otimes h'$ and $F(z) = \sum_{k=0}^{\infty}(e_k \otimes h')z^k = (\sum_{k=0}^{\infty} e_k z^k) \otimes h'$. Therefore

$$F(z)(h) = (\sum_{k=0}^{\infty} \langle e_k, h\rangle z^k)h'$$

for any $h \in \mathcal{H}$. This implies $\|F(z)\|_{\mathcal{L}(\mathcal{H},\mathcal{H}')} = \frac{1}{(1-|z|^2)^{1/2}}$ and $\|F(z)(h)\|_{\mathcal{H}'} = |\sum_{k=0}^{\infty}\langle e_k, h\rangle z^k)|$ which shows that $F \in H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$ but $F \notin H^2(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$.

Fix $h_0 \in \mathcal{H}$ with $\|h_0\|_{\mathcal{H}} = 1$ and (e'_k) an orthonormal basis of $\mathcal{H}'$. Define $T_k = h_0 \otimes e'_k$ and $F(z) = \sum_{k=0}^{\infty}(h_0 \otimes e'_k)z^k = h_0 \otimes (\sum_{k=0}^{\infty} z^k e'_k)$. Therefore

$$F(z)(h) = \langle h_0, h\rangle(\sum_{k=0}^{\infty} z^k e'_k)$$

for any $h \in \mathcal{H}$. This gives $\|F(z)(h)\|_{\mathcal{H}'} = |\langle h_0, h\rangle| \frac{1}{(1-|z|^2)^{1/2}}$ and $|\langle F(z)(h), h'\rangle| = |\langle h_0, h\rangle|| \sum_{k=0}^{\infty}\langle e'_k, h'\rangle z^k)|$. Hence $F \in H^2_{weak}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$ but $F \notin H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$.
□

Our objective is to describe the elements in the unit ball of these spaces for the case $p = 2$.

Our starting point is the following factorization result due D. Sarason for scalar-valued functions.

Theorem 1. *(See [9] page 490) Let h be an analytic function in the unit disc $\mathbb{D}$. The following are equivalent:*

(i) $h \in H^2(\mathbb{D})$ and $\|h\|_{H^2(\mathbb{D})} \leq 1$.

(ii) There exist $\phi, \psi \in H^\infty(\mathbb{D})$ such that $|\phi(z)|^2 + |\psi(z)|^2 \leq 1$ and

$$h(z) = \frac{\phi(z)}{1 - z\psi(z)}.$$

This gives a very interesting and useful factorization of functions in the unit ball of H^2, that is any $h \in H^2$ and $\|h\|_{H^2} \leq 1$ can be written as $h(z) = \frac{f(z)}{1-zg(z)}$ where $z \to (f(z), g(z))$ belongs to the unit ball of $H^\infty(\mathbb{D}, \mathbb{C}^2)$.

Sarason's result was extended to matrix-valued functions by D. Alpay, V. Bolotnikov and Y. Peretz in [1]. Let us set the notation to establish the result. Given $p, q \in \mathbb{N}$ the authors denoted by $H_2^{p\times q}$ the Hilbert space of $\mathbb{C}^{p\times q}$-valued functions with H^2 entries endowed with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} tr(g(e^{it})^* f(e^{it}))\frac{dt}{2\pi},$$

and denote $H_2^{p\times q}(I_q)$ for the set of functions $f \in H_2^{p\times q}$ such that

$$\sum_{k=0}^{\infty} f_k^* f_k \leq I_q$$

where I_q is the identity operator on $\mathbb{C}^q$ and $f(z) = \sum_{k=0}^\infty f_k z^k$ with $f_k \in \mathbb{C}^{p\times q}$.

Theorem 2. *(See [1, Theorem 2.2]) Let f be a $\mathbb{C}^{p\times q}$-valued function analytic in $\mathbb{D}$. Then f belongs to $H_2^{p\times q}(I_q)$ if and only if it can be written as*

$$f(z) = s_1(z)(I_q - zs_2(z))^{-1}$$

for some Schur function $S(z) = (s_1(z), s_2(z))$, that is s_1 and s_2 are $\mathbb{C}^{p\times q}$-valued and $\mathbb{C}^{q\times q}$-valued analytic functions in $\mathbb{D}$ with

$$\|s_1(z)\| + \|s_2(z)\| \leq 1, \quad |z| < 1,$$

and conversely.

We can rephrase the result using our vector-valued Hardy spaces. Let us observe that

$$\begin{aligned}\|f\|^2_{H_2^{p\times q}} &= \int_0^{2\pi} tr(f(e^{it})^* f(e^{it}))\frac{dt}{2\pi} = tr(\int_0^{2\pi} f(e^{it})^* f(e^{it})\frac{dt}{2\pi})\\ &= tr(\sum_{k=0}^{\infty} f_k^* f_k) = \sum_{k=0}^{\infty}\sum_{i=1}^{q}\sum_{j=1}^{p}(f_k^*)_{i,j}(f_k)_{j,i}\\ &= \sum_{k=0}^{\infty}\sum_{i=1}^{q}\sum_{j=1}^{p}|(f_k)_{i,j}|^2 = \sum_{k=0}^{\infty}\sum_{i=1}^{q}\sum_{j=1}^{p}|\langle f_k(e_i), e'_j\rangle|^2,\end{aligned}$$

where e_i and e'_j stand for the canonical basis of $\mathbb{C}^q$ and $\mathbb{C}^p$ respectively. Hence

$$\|f\|^2_{H_2^{p\times q}} = \sup_{\|h\|_{\mathbb{C}^q}=1, \|h'\|_{\mathbb{C}^p}=1} \sum_{k=0}^{\infty} |\langle f_k(h), h'\rangle|^2.$$

This shows that $H_2^{p\times q}$ is nothing else than $H^2_{weak}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$.

Observe now that $H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$ is now defined by the condition

$$\sup_{\|h\|_{\mathbb{C}^q}=1} \sum_{k=0}^{\infty} \|f_k(h)\|^2 < \infty,$$

which is equivalent to

$$\sup_{\|h\|_{\mathbb{C}^q}=1} \sum_{k=0}^{\infty} |\langle f_k^* f_k(h), h\rangle| < \infty.$$

In particular $H_2^{p\times q}(I_q)$ coincides with the unit ball of $H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$. This point of view also shows that $f \in H^{p\times q}(I_q)$ if and only if the operator of multiplication by f is a contraction from $\mathbb{C}^q$ to $H^2(\mathbb{D}, \mathbb{C}^p)$ (see [1, Theorem 2.1]).

The Schur class $\mathbf{S}(\mathcal{H}, \mathcal{H}')$ is the set of $\mathcal{L}(\mathcal{H}, \mathcal{H}')$-valued analytic functions S in the unit disc such that $S(z)$ are contractions. In other words $\mathbf{S}(\mathcal{H}, \mathcal{H}')$ coincides with the unit ball of $H^\infty(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$ or $H^\infty_{strong}(\mathbb{D}, \mathcal{L}(\mathcal{H}, \mathcal{H}'))$.

It is well known that a $\mathbb{C}^{p\times q}$-valued analytic function in $\mathbb{D}$ is a Schur function is equivalent to the operator of multiplication by the function is a contraction from $H^2(\mathbb{D}, \mathbb{C}^q)$ to $H^2(\mathbb{D}, \mathbb{C}^p)$. This, in our terminology, corresponds to the fact that elements in the unit ball of $H^\infty_{strong}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$ define contractions from $H^2(\mathbb{D}, \mathbb{C}^q)$ to $H^2(\mathbb{D}, \mathbb{C}^p)$ via multiplication and clearly exhibits that the difference between the Schur class and $H_2^{p\times q}(I_q)$ is nothing else but the difference between $H^\infty_{strong}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$ and $H^2_{strong}(\mathbb{D}, \mathcal{L}(\mathbb{C}^q, \mathbb{C}^p))$.

A further generalization of Theorem 2 to the infinite dimensional case is due to A.E. Frazho, S. ter Horst, M.A. Kaashoek (see [5, 6]).

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ the authors denote $H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$ the set of $\mathcal{L}(\mathcal{H}, \mathcal{H}')$-valued analytic functions F such that $F(z)h \in H^2(\mathbb{D}, \mathcal{H}')$ for any $h \in \mathcal{H}$ and $\|F(z)h\|_{H^2(\mathcal{H}')} \le \|h\|$. Hence $H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$ is the unit ball of $H^2_{strong}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$ and, in particular, $H^2_{ball}(\mathbb{C}^q, \mathbb{C}^p) = H_2^{p\times q}(I_q)$.

Theorem 3. *(See [5, Corollary 0.3]) Let F be an $\mathcal{L}(\mathcal{H}, \mathcal{H}')$- valued analytic function in the unit disc $\mathbb{D}$. Then $F \in H^2_{ball}(\mathcal{L}(\mathcal{H}, \mathcal{H}'))$ if and only if there exist $S = (s_1, s_2) \in \mathbf{S}(\mathcal{H}, \mathcal{H}' \bigotimes \mathcal{H})$ such that*

$$F(z) = s_1(z)(I_{\mathcal{H}} - z s_2(z))^{-1}.$$

The reader is referred to the papers [1, 9, 5, 6] for the use of these theorems in interpolation theory, computing exposed points and connections with the relaxed commutant lifting problem.

The aim of this paper is to get the analogue to Theorem 3 for functions in $H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ instead of $H^2_{strong}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$. It will be shown that if $F \in H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ and $\|F\|_{H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))} \leq 1$ then the operator-valued function $s_2(z)$ can be chosen to be $\phi(z)I_{\mathcal{H}}$ for some scalar-valued function ϕ.

Namely we will show the following:

Theorem 4. *Let $\mathcal{H}$ and $\mathcal{H}'$ be complex Hilbert spaces . The following are equivalent*

(i) $F \in H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ and $\|F\|_{H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))} \leq 1$.

(ii) There exist $f \in H^\infty(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ and $\phi \in H^\infty(\mathbb{D})$ such that

$$\|f(z)\|^2 + |\phi(z)|^2 \leq 1$$

and

$$F(z) = \frac{f(z)}{1 - z\phi(z)}.$$

Let us point out that there is nothing special in the space $X = \mathcal{L}(\mathcal{H},\mathcal{H}')$ for this result to be true and it actually holds in any Banach space X.

Before embarking in the proof let me point out the following general observation which shows which is the main point in the converse implication of the theorem.

Proposition 3. *Let $1 \leq p < \infty$, X and Y be Banach spaces, A be a Banach algebra and $\mathcal{B} : X \times A \to Y$ a bounded bilinear map. If $f \in H^\infty(\mathbb{D}, X)$ and $g \in H^\infty(\mathbb{D}, A)$ such that*

$$\|f(z)\|_X^p + \|g(z)\|_A^p \leq 1. \tag{11}$$

Then

$$F(z) = \mathcal{B}(f(z), (1 - zg(z))^{-1}) \in H^p(\mathbb{D}, Y)$$

and

$$\|F\|_{H^p(\mathbb{D},Y)} \leq \frac{\|\mathcal{B}\|\|f\|_{H^\infty(\mathbb{D},X)}}{1 - \|g\|_{H^p(\mathbb{D},A)}}.$$

Proof. From (11) one gets $\|f\|^p_{H^p(\mathbb{D},X)} + \|g\|^p_{H^p(\mathbb{D},A)} \leq 1$. Now we can assume that $\|g\|_{H^p(\mathbb{D},A)} < 1$, otherwise $F = 0$.

We first observe that $\|g^n\|_{H^p(\mathbb{D},A)} \leq \|g\|^n_{H^p(\mathbb{D},A)}$.

Indeed,

$$\int_0^{2\pi} \|g(re^{it})^n\|_A^p \frac{dt}{2\pi} \leq \int_0^{2\pi} \|g(re^{it})\|_A^{pn} \frac{dt}{2\pi} \leq$$

$$\leq \|g\|^{p(n-1)}_{H^\infty(\mathbb{D},A)} \int_0^{2\pi} \|g(re^{it})\|_A^p \frac{dt}{2\pi} \leq \|g\|^{pn}_{H^p(\mathbb{D},A)}.$$

Hence $z \to \sum_{n=0}^\infty z^n g(z)^n$ defines an absolutely convergent series in $H^p(\mathbb{D}, A)$ and we denote by $(1 - zg(z))^{-1}$ its sum which satisfies

$$\|(1-zg(z))^{-1}\|_{H^p(\mathbb{D},A)} \leq \frac{1}{1-\|g\|_{H^p(\mathbb{D},A)}}.$$

Define $F(z) = \mathcal{B}(f(z),(1-zg(z))^{-1})$. Is is elementary to see that F is a Y-valued analytic function. Moreover

$$\int_0^{2\pi} \|F(re^{it})\|_Y^p \frac{dt}{2\pi} \leq \int_0^{2\pi} \|\mathcal{B}\|^p \|f(re^{it})\|_X^p \|(1-re^{it}g(re^{it}))^{-1}\|_A^p \frac{dt}{2\pi}$$

$$\leq \|\mathcal{B}\|^p \|f\|^p_{H^\infty(\mathbb{D},X)} \|(1-zg(z))^{-1}\|^p_{H^p(\mathbb{D},A)} \leq \frac{\|\mathcal{B}\|^p\|f\|^p_{H^\infty(\mathbb{D},X)}}{(1-\|g\|_{H^p(\mathbb{D},A)})^p}.$$

□

When applying Proposition 3 for $p=2$, $X=Y=\mathcal{L}(\mathcal{H},\mathcal{H}')$, $A=\mathcal{L}(\mathcal{H},\mathcal{H})$ and $\mathcal{B}: X\times A\to Y$ given by $B(T,S)=TS$ one realizes that the point in the reverse implication of Theorem 3 is actually to get that F belongs to the unit ball of $H^2_{strong}(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ because the function $F\in H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H}'))$ but with norm $\frac{1}{1-\|g\|_{H^2(\mathbb{D},\mathcal{L}(\mathcal{H},\mathcal{H})}} \geq 1$.

2 Proof of the main theorem

We start working with analytic functions that can be approached by polynomials in $H^2(\mathbb{D},X)$, that is to say functions with boundary values belonging to the space $H^2(\mathbb{T},X)$. This actually allows us to give an alternative proof of Theorem 1 for $X=\mathbb{C}$. We use the same notation for the function defined at the boundary and its Poisson extension to the disc.

Theorem 5. *Let X be a complex Banach space. The following are equivalent:*
(i) $F\in H^2(\mathbb{T},X)$ and $\|F\|_{H^2(\mathbb{T},X)}\leq 1$
(ii) There exist $f\in H^\infty(\mathbb{T},X)$ and $\phi\in H^\infty(\mathbb{T})$ such that for any $|z|<1$,

$$\|f(z)\|^2+|\phi(z)|^2\leq 1$$

and

$$F(z)=\frac{f(z)}{1-z\phi(z)}.$$

Proof. (i) $\Longrightarrow$ (ii) Assume $F\in H^2(\mathbb{T},X)$ and $\|F\|_{H^2(\mathbb{T},X)}=1$. Then $u(z)=\|F(z)\|^2$ is a subharmonic function in $\mathbb{D}$ and $\int_0^{2\pi} u(e^{it})\frac{dt}{2\pi}=1$. Consider the harmonic function $U(z)$ given by the Poisson extension of $u(e^{it})=\lim u(re^{it})$ and write $U=\Re(k)$ where k is holomorphic which takes values in $\{\Re w\geq 0\}$ and $k(0)=1$. One has that $\int_0^{2\pi} U(e^{it})\frac{dt}{2\pi}=\int_0^{2\pi} u(e^{it})\frac{dt}{2\pi}=1$ and $\|F(z)\|^2\leq Re(k(z))$ for $|z|<1$.

Define $G(z) = \frac{1-k(z)}{1+k(z)}$, that is $k(z) = \frac{1+G(z)}{1-G(z)}$ and $Re(k(z)) = \frac{1-|G(z)|^2}{|1-G(z)|^2}$. Hence

$$\|F(z)\|^2 \leq \frac{1-|G(z)|^2}{|1-G(z)|^2}.$$

Clearly $G \in \mathcal{H}(\mathbb{D})$, $|G(z)| < 1$ and $G(0) = 0$. So we have $G(z) = z\phi(z)$ for some $\phi \in \mathcal{H}(\mathbb{D})$. Define now $f(z) = F(z)(1 - z\phi(z))$.

Obviously $f \in \mathcal{H}(\mathbb{D}, X)$. Let us see that $\|f(z)\|^2 + |\phi(z)|^2 \leq 1$ for $z \in \mathbb{D}$.

Indeed, on the boundary

$$\|f(e^{it})\|^2 + |\phi(e^{it})|^2 = \|F(e^{it})(1 - e^{it}\phi(e^{it}))\|^2 + |\phi(e^{it})|^2 \leq$$

$$\leq (1 - |\phi(e^{it})|^2) + |\phi(e^{it})|^2 = 1.$$

Consider now the Banach space $X_1 = X \otimes_2 \mathbb{C}$ and look at the function $h(e^{it}) = (f(e^{it}), \phi(e^{it}))$. We have shown that h belongs to the unit ball of $H^\infty(\mathbb{T}, X_1)$. Therefore $h(z) = (f(z), \phi(z)) \in H^\infty(\mathbb{D}, X_1)$ and $\|f(z)\|^2 + |\phi(z)|^2 \leq 1$ for any $z \in \mathbb{D}$.

(ii) $\Longrightarrow$ (i) Assume that there exist $f \in H^\infty(\mathbb{T}, X)$ and $\phi \in H^\infty$ such that $\|f(z)\|^2 + |\phi(z)|^2 \leq 1$ and

$$F(z) = \frac{f(z)}{1 - z\phi(z)}.$$

Let us show that $F \in H^2(\mathbb{T}, X)$ with norm bounded by 1. Indeed,

$$\begin{aligned}
\int_0^{2\pi} \|F(e^{it})\|^2 \frac{dt}{2\pi} &= \int_0^{2\pi} \frac{\|f(e^{it})\|^2}{|1 - e^{it}\phi(e^{it})|^2} \frac{dt}{2\pi} \\
&\leq \int \frac{1 - |e^{it}\phi(e^{it})|^2}{|1 - e^{it}\phi(e^{it})|^2} dt \\
&= \lim_{r\to 1} \int_0^{2\pi} \frac{1 - |re^{it}\phi(re^{it})|^2}{|1 - re^{it}\phi(re^{it})|^2} \frac{dt}{2\pi} \\
&= \lim_{r\to 1} Re(\int_0^{2\pi} \frac{1 + re^{it}\phi(re^{it})}{1 - re^{it}\phi(re^{it})}) \frac{dt}{2\pi} \\
&= Re(\lim_{r\to 1} \int_0^{2\pi} \frac{1 + re^{it}\phi(re^{it})}{1 - re^{it}\phi(re^{it})} \frac{dt}{2\pi}) = 1
\end{aligned}$$

□

In order to extend the result for functions in $H^2(\mathbb{D}, X)$ we are going to use the following extension of a classical factorization for vector-valued functions. The reader is also referred to the work by G. Pisier [8] (and references thereby) for other factorization results of vector-valued analytic functions which depend on the geometry of the space X.

Lemma 1. *(see [2] or [7, Theorem 2.10]) Let $1 \le p < \infty$ and $F \in H^p(\mathbb{D}, X)$. There exist $F_1 \in H^\infty(\mathbb{D}, X)$ with $\|F_1\|_{H^\infty(\mathbb{D},X)} = 1$ and $\phi \in H^p(\mathbb{D})$ with $\|\phi\|_{H^p(\mathbb{D})} = \|F\|_{H^p(\mathbb{D},X)}$ such that $F(z) = \phi(z)F_1(z)$.*

Theorem 6. *Let X be a complex Banach space $F \in \mathcal{H}(\mathbb{D}, X)$. The following are equivalent:*

(i) $F \in H^2(\mathbb{D}, X)$ and $\|F\|_{H^2(\mathbb{D},X)} \le 1$.

(ii) There exist $f \in H^\infty(\mathbb{D}, X)$ and $\phi \in H^\infty(\mathbb{D})$ such that

$$\|f(z)\|^2 + |\phi(z)|^2 \le 1, \quad |z| < 1$$

and

$$F(z) = \frac{f(z)}{1 - z\phi(z)}.$$

Proof. (i) $\Longrightarrow$ (ii) Use Lemma 1 to find $F_1 \in H^\infty(\mathbb{D}, X)$ with $\|F_1\|_{H^\infty(\mathbb{D},X)} = 1$ and $\phi \in H^2(\mathbb{D})$ with $\|\phi\|_{H^2(\mathbb{D})} = \|F\|_{H^2(\mathbb{D},X)}$ such that $F(z) = \phi(z)F_1(z)$.

Apply Sarason's result (or Theorem 5 for $X = \mathbb{C}$) to obtain $\phi_1, \phi_2 \in H^\infty(\mathbb{D})$ with $|\phi_1(z)| + |\phi_2(z)| \le 1$ and $\phi(z) = \frac{\phi_1(z)}{1 - z\phi_2(z)}$. Define $f(z) = F_1(z)\phi_1(z)$ and $g(z) = \phi_2(z)$ to get this implication.

(ii) $\Longrightarrow$ (i) Assume now that there exist $f \in H^\infty(\mathbb{D}, X)$ and $g \in H^\infty(\mathbb{D})$ such that $\|f(z)\|^2 + |g(z)|^2 \le 1$ and

$$F(z) = \frac{f(z)}{1 - zg(z)}.$$

Arguing as above we have

$$\begin{aligned}
\int \|F(re^{it})\|^2 dt &= \int \frac{\|f(re^{it})\|^2}{|1 - re^{it}g(re^{it})|^2} dt \\
&\le \int \frac{1 - |g(re^{it})|^2}{|1 - re^{it}g(re^{it})|^2} dt \\
&\le \int \frac{1 - |re^{it}g(re^{it})|^2}{|1 - re^{it}g(re^{it})|^2} dt \\
&= Re\Big(\int \frac{1 + re^{it}g(re^{it})}{1 - re^{it}g(re^{it})}\Big) dt = 1
\end{aligned}$$

□

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Departamento de Análisis Matemático
Universidad de Valencia
46100 Burjassot
Valencia
Spain
oblasco@uv.es