Hölder inequality for functions that are integrable with respect to bilinear maps.

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Abstract

Let (Ω, Σ, μ) be a finite measure space, $1 \leq p < \infty$, X be a Banach space X and $\mathsf{u} : X \times Y \to Z$ be a bounded bilinear map. We say that an X-valued function f is p-integrable with respect to u whenever $\sup_{\|y\|=1} \int_\Omega \|\mathsf{u}(f(w), y)\|^p d\mu < \infty$. We get an analogue to Hölder's inequality in this setting.

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1 Introduction

Throughout the paper $1 \leq p < \infty$, (Ω, Σ, μ) will be a finite complete measure space, X, Y and Z will stand for Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), and $\mathsf{u} : X \times Y \to Z$ will denote a bounded bilinear map. We denote by $L^0(X)$ and $L^0_{\text{weak}}(X)$ the spaces of strongly and weakly measurable functions with values in X and by $L^0_{\text{weak}^*}(X^*)$ the space of weak*-measurable functions with values in X^*. We write $L^p(X)$, $L^p_{\text{weak}}(X)$ and $L^p_{\text{weak}^*}(X^*)$ for the space of functions in $L^0(X)$, $L^0_{\text{weak}}(X)$ and $L^0_{\text{weak}^*}(X^*)$ such that $\|f\| \in L^p$, $\langle f, x^* \rangle \in L^p$ for $x^* \in X^*$ and $\langle x, f \rangle \in L^p(\mu)$ for $x \in X$ respectively. Finally we use the notation $\mathcal{S}(X)$ for the space of X-valued simple functions and by $P^p(X)$ for the space of Pettis p-integrable functions $P^p(X) = L^p_{\text{weak}}(X) \cap L^0(X)$.

Let us start mentioning the following basic examples of bilinear maps:

$$\mathcal{B}_X : X \times \mathbb{K} \to X, \qquad \mathcal{B}_X(x, \lambda) = \lambda x, \tag{1}$$

$$\mathcal{D}_X : X \times X^* \to \mathbb{K}, \qquad \mathcal{D}_X(x, x^*) = \langle x, x^* \rangle, \tag{2}$$

$$(\mathcal{D}_1)_X : X^* \times X \to \mathbb{K}, \qquad (\mathcal{D}_1)_X(x^*, x) = \langle x, x^* \rangle, \tag{3}$$

In this paper we shall consider some spaces of X-valued functions which are p-integrable with respect to a bounded bilinear map $\mathsf{u} : X \times Y \to Z$, that is to say X-valued functions f satisfying the condition $\mathsf{u}(f, y) \in L^p(Z)$ for all $y \in Y$. The cases $L^p(X)$, $L^p_{\text{weak}}(X)$ and $L^p_{\text{weak}^*}(X^*)$ correspond to the previous notion applied to the examples above.

Some other classes have been around for a long time for cases such us

$$\pi_Y : X \times Y \to X \hat{\otimes} Y, \qquad \pi_Y(x, y) = x \otimes y, \tag{4}$$

$$\tilde{\mathcal{O}}_Y : X \times \mathcal{L}(X,Y) \to Y, \qquad \tilde{\mathcal{O}}_Y(x,T) = T(x), \tag{5}$$

$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y,Z) \times Y \to Z, \qquad \mathcal{O}_{Y,Z}(T,y) = T(y). \tag{6}$$

A systematic study of theses spaces for general bilinear maps has been initiated in [6] and used to extend the results on boundedness from $L^p(Y)$ to $L^p(Z)$ of operator-valued kernels by M. Girardi and L. Weiss [10] corresponding to the bilinear map $\mathcal{O}_{Y,Z}$ to the case where $K : \Omega \times \Omega' \to X$ is measurable and the integral operators are defined by

$$T_K(f)(w) = \int_{\Omega'} \mathsf{u}(K(w,w'), f(w'))d\mu'(w').$$

The reader is also referred to [7] for an introduction of Fourier Analysis in the bilinear context. This allows to extend the results in [2, 4, 5] regarding convolution by means of bilinear maps and Fourier coefficients for functions in these wider classes.

The aim of this paper is the consideration of Hölder inequality in this general contex.

It is well known and easy to see the following analogues of Hölder's inequality in the vector-valued setting: Let $1 \leq p_1, p_2, p_3 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.

(1) If $f \in L^{p_1}_{\text{weak}}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}_{\text{weak}}(X)$.

(2) If $f \in P^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in P^{p_3}(X)$.

(3) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}$ then $fg \in L^{p_3}(X)$.

(4) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g\rangle \in L^{p_3}$.

(5) If $f \in L^{p_1}(\mathcal{L}(X,Y))$ and $g \in L^{p_2}(X)$ then $f(\cdot)(g(\cdot)) \in L^{p_3}(Y)$.

We shall try to understand the situation when one considers integrability with respect to general bilinear maps.

Let us mention some notions that were relevant for developing the general theory (see [6]). Given $x \in X$ and $y \in Y$ we shall be denoting by $\mathsf{u}_x \in \mathcal{L}(Y,Z)$ and $\mathsf{u}^y \in \mathcal{L}(X,Z)$ the linear operators $\mathsf{u}_x(y) = \mathsf{u}(x,y)$ and $\mathsf{u}^y(x) = \mathsf{u}(x,y)$. A triple (Y, Z, u) is called *admissible* for X if the map $x \to \mathsf{u}_x$ is injective from $X \to \mathcal{L}(Y,Z)$ and X is said to be (Y, Z, u)-*normed* (or *normed by* u) if there exists $C > 0$ such that for all $\|x\| \leq C\|\mathsf{u}_x\|$, $x \in X$.

Given a bounded bilinear map $\mathsf{u} : X \times Y \to Z$, we can define the "adjoint" $\mathsf{u}^* : X \times Z^* \to Y^*$ by the formula

$$\langle y, \mathsf{u}^*(x, z^*)\rangle = \langle \mathsf{u}(x,y), z^*\rangle.$$

Note that for the just mentioned examples we have:

$$\mathcal{B}^*_X = \mathcal{D}_X, \ (\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*} \text{ and } (\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*).$$

Let us start with the following definitions:

Definition 1 *(see [6]) We say that $f : \Omega \to X$ belongs to $L^0_{\mathsf{u}}(X)$ if $\mathsf{u}(f, y) \in L^0(Z)$ for any $y \in Y$. We write $\mathfrak{L}^p_{\mathsf{u}}(X)$ for the space of functions f in $L^0_{\mathsf{u}}(X)$ such that*

$$\|f\|_{\mathfrak{L}^p_{\mathsf{u}}(X)} := \sup\{\|\mathsf{u}(f,y)\|_{L^p(Z)} : \|y\| = 1\} < \infty.$$

A function $f \in \mathfrak{L}^p_{\mathsf{u}}(X)$ is said to belong to $L^p_{\mathsf{u}}(X)$ if there exists a sequence of simple functions $(s_n)_n \in \mathcal{S}(X)$ such that

$$s_n \to f \text{ a.e.} \qquad \text{and} \qquad \|s_n - f\|_{\mathfrak{L}^p_{\mathsf{u}}(X)} \to 0.$$

For $f \in L^p_{\mathsf{u}}(X)$ we write $\|f\|_{L^p_{\mathsf{u}}(X)}$ instead of $\|f\|_{\mathfrak{L}^p_{\mathsf{u}}(X)}$. Clearly one has that

$$\|f\|_{L^p_{\mathsf{u}}(X)} = \lim_{n\to\infty} \|s_n\|_{L^p_{\mathsf{u}}(X)}.$$

In particular

$$L^0_{\mathcal{B}_X}(X) = L^0(X), L^0_{\mathcal{D}_X}(X) = L^0_{\text{weak}}(X) \text{ and } L^0_{\mathcal{D}_{1,X}}(X^*) = L^0_{\text{weak}*}(X).$$

$$\mathcal{L}^p_{\mathcal{B}_X}(X) = L^p(X), \mathcal{L}^p_{\mathcal{D}_X}(X) = L^p_{\text{weak}}(X) \text{ and } \mathcal{L}^p_{(\mathcal{D}_1)_X}(X^*) = L^p_{\text{weak}*}(X^*).$$

$$L^p_{\mathcal{B}_X}(X) = L^p(X) \text{ and } L^p_{\mathcal{D}_X}(X) = P^p(X) (\text{see [11], page 54 for the case } p = 1).$$

Observe that $L^p(X) \subseteq L^p_{\mathsf{u}}(X)$ for any u and that, in general, $L^p_{\mathsf{u}}(X) \subsetneq \mathcal{L}^p_{\mathsf{u}}(X)$ (see [8] page 53, for the case $\mathsf{u} = \mathcal{D}_X$). It was shown in [6] that $\mathcal{L}^p_{\mathsf{u}}(X) \subset L^p_{\text{weak}}(X)$ if and only if X is u-normed.

Clearly $f \in L^0_{\mathsf{u}}(X)$ and $g \in L^0(Y)$ implies that $\mathsf{u}(f,g) \in L^0(Z)$. Hence a natural question that arises is the following:

Question 1. Does $\mathsf{u}(f,g)$ belong to $L^{p_3}(Z)$ for any $f \in \mathcal{L}^{p_1}_{\mathsf{u}}(X)$ and $g \in L^{p_2}(Y)$ for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.?

The answer is negative for infinite dimensional Banach spaces X.

Indeed, take $p_1 = p_2 = 2$ and $p_3 = 1$, let X be an infinite dimensional Banach space, $Y = X^*$ and $Z = \mathbb{K}$ and $\mathsf{u} = \mathcal{D}_X$. Take $(x_n) \in \ell^2_{\text{weak}}(X) \setminus \ell^2(X)$. This allows to find $(x_n^*) \in \ell^2(X^*)$ such that $\sum_n |\langle x_n, x_n^* \rangle| = \infty$. Consider now $\Omega = [0,1]$ with the Lebesgue measure, $I_k = (2^{-k}, 2^{-k+1}]$ and define the functions $f = \sum_{k=1}^\infty 2^{\frac{k}{2}} x_k \mathbf{1}_{I_k}$ and $g = \sum_{k=1}^\infty 2^{\frac{k}{2}} x_k^* \mathbf{1}_{I_k}$. It is clear that $f \in \mathcal{L}^2_{\mathcal{D}_X}(X)$ with $\|f\|^2_{\mathcal{L}^2_{\mathcal{D}_X}(X)} = \sup\{\sum_{n=1}^\infty |\langle x_n, x^* \rangle|^2 : \|x^*\| = 1\}$ and $g \in L^2(X^*)$ with $\|g\|^2_{L^2(X^*)} = \sum_{n=1}^\infty \|x_n^*\|^2$ but $\mathsf{u}(f,g) = \sum_{k=1}^\infty 2^k \langle x_k, x_k^* \rangle \mathbf{1}_{I_k} \notin L^1$.

One might think that the difficulty comes from allowing the functions to belong to $\mathcal{L}^{p_1}_{\mathsf{u}}(X)$ instead of $L^{p_1}_{\mathsf{u}}(X)$. Let us then modify the question:

Question 2. Does $\mathsf{u}(f,g)$ belong to $L^{p_3}(Z)$ for any $f \in L^{p_1}_{\mathsf{u}}(X)$ and $g \in L^{p_2}(Y)$ for $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$.?

The answer is again negative.

Assume the contrary. Hence there exists $M > 0$ such that

$$\|\mathsf{u}(s,t)\|_{L^1(Z)} \leq M \|s\|_{L^2_{\mathsf{u}}(X)} \|t\|_{L^2(Y)} \tag{7}$$

for any $s \in \mathcal{S}(X)$ and $t \in \mathcal{S}(Y)$.

Select $X = Y = \ell^2$, $Z = \ell^1$ and $\mathsf{u} : \ell^2 \times \ell^2 \to \ell^1$ given by $\mathsf{u}((\lambda_n)_n, (\beta_n)_n) = (\lambda_n \beta_n)_n$. Let us now consider $s_N = t_N = \sum_{k=1}^N 2^{\frac{k}{2}} e_k \mathbf{1}_{I_k}$ where e_k is the canonical basis and I_k are chosen as in the previous example. Hence $\mathsf{u}(s_N, y) = \sum_{k=1}^N 2^{\frac{k}{2}} \beta_k e_k \mathbf{1}_{I_k}$ for $y = (\beta_n)_n \in \ell^2$. Therefore $\|s_N\|_{L^2_{\mathsf{u}}(\ell^2)} \leq 1$, $\|s_N\|_{L^2(\ell^2)} = \sqrt{N}$ and $\|\mathsf{u}(s_N, s_N)\|_{L^1(\ell^1)} = N$. This contradicts (7).

Modifying the previous argument with $Z = \mathbb{K}$ and $\mathsf{u} = \mathcal{D}_X$ one can even show that there exist $f \in L^{p_1}_{\mathsf{u}}(X)$ and $g \in L^{p_2}(Y)$ such that $\mathsf{u}(f,g) \notin L^{p_3}_{\text{weak}}(Z)$.

The objective of this paper is to present an analogue to Hölder inequality in the setting of vector-valued functions that are integrable with respect to bilinear maps. We shall then study the following general problem: Let $1 \leq p_1, p_2, p_3 \leq \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$ and let $\mathsf{u} : X \times Y \to Z$ be a bounded bilinear map. If $\mathsf{u}_1 : X \times X_1 \to X_2$ and $\mathsf{u}_2 : Y \times Y_1 \to Y_2$ are bounded bilinear maps, find $\mathsf{u}_3 : Z \times Z_1 \to Z_2$ such that for any $f \in \mathcal{L}^{p_1}_{\mathsf{u}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)$ one has $\mathsf{u}(f,g) \in \mathcal{L}^{p_3}_{\mathsf{u}_3}(Z)$.

2 A bilinear version of Hölder's Inequality.

The notion that will fit to our purposes is the following.

Definition 2 *We say that* $(\mathsf{u}, \mathsf{u}_1, \mathsf{u}_2)$ *is a compatible triple if* $\mathsf{u} : X \times Y \to Z$, $\mathsf{u}_1 : X \times X_1 \to X_2$ *and* $\mathsf{u}_2 : Y \times Y_1 \to Y_2$ *are bounded bilinear maps and there exist a Banach space* F *and two bounded bilinear maps* $\mathcal{P} : X_2 \times Y_2 \to F$ *and* $\widetilde{\mathcal{P}} : Z \times (X_1 \hat{\otimes} Y_1) \to F$ *such that*

$$\widetilde{\mathcal{P}}(\mathsf{u}(x,y), x_1 \otimes y_1) = \mathcal{P}(\mathsf{u}_1(x, x_1), \mathsf{u}_2(y, y_1))$$

for all $x \in X, y \in Y$, $x_1 \in X_1$ *and* $y_1 \in Y_1$.

A general procedure of construction of such compatible triples of bilinear maps can be obtained as follows:

Proposition 1 Let U be a Banach space, $\mathsf{u}_1 : X \times X_1 \to U$ and $\mathsf{u}_2 : Y \times Y_1 \to U^*$ be bounded bilinear maps . Define the bilinear map $\mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2} : X \times Y \to \mathcal{L}(X_1, Y_1^*)$ by the formula

$$\langle \mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2}(x,y)(x_1), y_1\rangle = \langle \mathsf{u}_1(x,x_1), \mathsf{u}_2(y,y_1)\rangle$$

for $x \in X$, $y \in Y$, $x_1 \in X_1$ and $y_1 \in Y_1$.

Then $(\mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2}, \mathsf{u}_1, \mathsf{u}_2)$ is a compatible triple.

PROOF. Using that $\mathcal{L}(X_1, Y_1^*) = (X_1 \hat{\otimes} Y_1)^*$ we also can write

$$\langle \mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2}(x,y), x_1 \otimes y_1\rangle = \langle \mathsf{u}_1(x,x_1), \mathsf{u}_2(y,y_1)\rangle.$$

This shows that $(\mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2}, \mathsf{u}_1, \mathsf{u}_2)$ is compatible by selecting $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_X : U \times U^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = (\mathcal{D}_1)_{X_1 \hat{\otimes} Y_1} : \mathcal{L}(X_1, Y_1^*) \times (X_1 \hat{\otimes} Y_1) \to \mathbb{K}$.

□

Let us now give some more concrete examples of compatible triples:

Example 1 $(\mathsf{u}, \mathcal{B}_X, \mathcal{B}_Y)$ is a compatible triple for any $\mathsf{u} : X \times Y \to Z$.

In particular, $(\mathcal{D}_X, \mathcal{B}_X, \mathcal{B}_{X^*})$ or $(\mathcal{O}_{X,Y}, \mathcal{B}_X, \mathcal{B}_Y)$ are compatible triples.

Indeed, if $\mathsf{u} : X \times Y \to Z$, $\mathsf{u}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathsf{u}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then select $F = Z$, $\mathcal{P} = \mathsf{u} : X \times Y \to Z$ and $\widetilde{\mathcal{P}} = \mathcal{B}_Z : Z \times \mathbb{K} \to Z$. Observe that $\widetilde{\mathcal{P}}(\mathsf{u}(x,y), \lambda\beta) = \mathcal{P}(\mathcal{B}(x,\lambda), \mathcal{B}(y,\beta))$.

□

Example 2 $(\mathsf{u}, \mathsf{u}^*, \mathcal{B}_Y)$ is a compatible triple.

Indeed, if $\mathsf{u} : X \times Y \to Z$, $\mathsf{u}_1 = \mathsf{u}^* : X \times Z^* \to Y^*$ given by

$$\langle y, \mathsf{u}_1(x,z^*)\rangle = \langle \mathsf{u}(x,y), z^*\rangle$$

and $\mathsf{u}_2 = \mathcal{B}_Y : Y \times \mathbb{K} \to Y$ then we can select $F = \mathbb{K}$, $\mathcal{P} = (\mathcal{D}_1)_Y : Y^* \times Y \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$.

□

Example 3 $(\pi_Y, \mathcal{B}_X, \widetilde{\mathcal{O}}_{X^*})$ is a compatible triple.

Indeed, if $\mathsf{u} = \pi_Y : X \times Y \to X \hat{\otimes} Y$, $\mathsf{u}_1 = \mathcal{B}_X : X \times \mathbb{K} \to X$ and $\mathsf{u}_2 = \widetilde{\mathcal{O}}_{X^*} : Y \times \mathcal{L}(Y, X^*) \to X^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_X : X \times X^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = \mathcal{D}_{X \hat{\otimes} Y} : X \hat{\otimes} Y \times \mathcal{L}(Y, X^*) \to \mathbb{K}$. The compatibility now follows from

$$\widetilde{\mathcal{P}}(\mathsf{u}(x,y), \lambda T) = \langle x \otimes y, \lambda T\rangle = \langle \lambda x, Ty\rangle = \mathcal{P}(\mathsf{u}_1(x,\lambda), \mathsf{u}_2(y,T)).$$

□

Example 4 Let $\mathcal{C} : \mathcal{L}(X,Z) \times \mathcal{L}(Y,Z^*) \to \mathcal{L}(Y,X^*)$ be given by $(T,S) \to T^*S$. Then $(\mathcal{C}, \mathcal{O}_{X,Z}, \mathcal{O}_{Y,Z^*})$ is a compatible triple.

Indeed, if $\mathsf{u}_1 = \mathcal{O}_{X,Z} : \mathcal{L}(X,Z) \times X \to Z$ and $\mathsf{u}_2 = \mathcal{O}_{Y,Z^*} : \mathcal{L}(Y,Z^*) \times Y \to Z^*$ then we can take $F = \mathbb{K}$, $\mathcal{P} = \mathcal{D}_Z : Z \times Z^* \to \mathbb{K}$ and $\widetilde{\mathcal{P}} = (\mathcal{D}_1)_{X \hat{\otimes} Y} : \mathcal{L}(Y, X^*) \times X \hat{\otimes} Y \to \mathbb{K}$ given by $\widetilde{\mathcal{P}}(T, x \otimes y) = \langle x, Ty\rangle$.

Observe that the compatibility follows from the formula

$$\widetilde{\mathcal{P}}(\mathcal{C}(T,S), x \otimes y) = \langle x, T^*Sy\rangle = \langle Tx, Sy\rangle = \mathcal{P}(\mathsf{u}_1(T,x), \mathsf{u}_2(S,y)).$$

□

Theorem 1 (Hölder's inequality I) *Let $1 \le p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$. Assume that $(\mathsf{u}, \mathsf{u}_1, \mathsf{u}_2)$ is a compatible triple for some F, $\mathcal{P}$ and $\widetilde{\mathcal{P}}$.*

(1) If $f \in \mathcal{L}^{p_1}_{\mathsf{u}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)$ then $\mathsf{u}(f,g) \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

(2) If $f \in L^{p_1}_{\mathsf{u}_1}(X)$ and $g \in L^{p_2}_{\mathsf{u}_2}(Y)$ then $\mathsf{u}(f,g) \in L^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

Moreover $\|\mathsf{u}(f,g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \le \|\mathcal{P}\| \|f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)}$.

PROOF. *(1)* Let us first show that if $f \in L^0_{\mathsf{u}_1}(X)$ and $g \in L^0_{\mathsf{u}_2}(Y)$ then $h = \mathsf{u}(f,g) \in L^0_{\widetilde{\mathcal{P}}}(Z)$.

Indeed, if $x_1 \in X_1$ and $y_1 \in Y_1$ then $\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) = \mathcal{P}(\mathsf{u}_1(f, x_1), \mathsf{u}_2(g, y_1))$. Now since $\mathsf{u}_1(f, x_1) \in L^0(X_2)$, $\mathsf{u}_2(g, y_1) \in L^0(Y_2)$ and $\mathcal{P}$ is continuous then $\widetilde{\mathcal{P}}(h, x_1 \otimes y_1) \in L^0(F)$. For general $\varphi \in X_1 \hat{\otimes} Y_1$, assume $\varphi = \sum_n x_1^n \otimes y_1^n$ with $\sum_n \|x_1^n\| \|y_1^n\| < \infty$. Then, using the continuity of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, one has

$$\widetilde{\mathcal{P}}(h, \varphi) = \lim_{N \to \infty} \sum_{k=1}^{N} \widetilde{\mathcal{P}}(\mathsf{u}_1(f, x_1^k), \mathsf{u}_2(g, y_1^k)) \in L^0(F).$$

Assume $f \in \mathcal{L}^{p_1}_{\mathsf{u}_1}(X)$ and $g \in \mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)$. Let us show that $h \in \mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)$.

If $x_1 \in X_1$ and $y_1 \in Y_1$ then

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, x_1 \otimes y_1)\|^{p_3} d\mu)^{\frac{1}{p_3}} &= (\int_\Omega \|\mathcal{P}(\mathsf{u}_1(f, x_1), \mathsf{u}_2(g, y_1))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\| (\int_\Omega (\|\mathsf{u}_1(f, x_1)\| \|\mathsf{u}_2(g, y_1)\|)^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\| (\int_\Omega \|\mathsf{u}_1(f, x_1)\|^{p_1} d\mu)^{\frac{1}{p_1}} (\int_\Omega \|\mathsf{u}_2(g, y_1)\|^{p_2} d\mu)^{\frac{1}{p_2}} \\
&\le \|\mathcal{P}\| \|f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)} \|x_1\| \|y_1\|.
\end{aligned}$$

In general, for each $\varphi = \sum_n x_1^n \otimes y_1^n \in X_1 \hat{\otimes} Y_1$, one has $\widetilde{\mathcal{P}}(h, \sum_n x_1^n \otimes y_1^n) = \sum_n \widetilde{\mathcal{P}}(h, x_1^n \otimes y_1^n)$. Therefore

$$\begin{aligned}
(\int_\Omega \|\widetilde{\mathcal{P}}(h, \sum_n x_1^n \otimes y_1^n)\|^{p_3} d\mu)^{\frac{1}{p_3}} &\le \sum_n (\int_\Omega \|\mathcal{P}(\mathsf{u}_1(f, x_1^n), \mathsf{u}_2(g, y_1^n))\|^{p_3} d\mu)^{\frac{1}{p_3}} \\
&\le \|\mathcal{P}\| (\sum_n \|x_1^n\| \|y_1^n\|) \|f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)}.
\end{aligned}$$

This gives $\|\mathsf{u}(f,g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \le \|\mathcal{P}\| \|f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)}$.

(2) Assume that f and g are simple functions. If $f = \sum_k x_k \mathbf{1}_{E_k} \in \mathcal{S}(X)$ and $g = \sum_p y_p \mathbf{1}_{F_p} \in \mathcal{S}(Y)$ then

$$h = \mathsf{u}(f,g) = \sum_{k,p} \mathsf{u}(x_k, y_p) \mathbf{1}_{E_k \cap F_p} \in \mathcal{S}(Z).$$

Now, if we take $f \in L^{p_1}_{\mathsf{u}_1}(X)$ and $g \in L^{p_2}_{\mathsf{u}_2}(Y)$ then there exist $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathsf{u}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathsf{u}_2}(Y)} \to 0$. Clearly $\mathsf{u}(f_n, g_n)$ are simple functions and converge to $\mathsf{u}(f,g)$ a.e.

Due to the previous result

$$\begin{aligned}
\|\mathsf{u}(f_n, g_n) - \mathsf{u}(f,g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} &\le \|\mathsf{u}(f_n - f, g_n)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} + \|\mathsf{u}(f, g_n - g)\|_{\mathcal{L}^{p_3}_{\widetilde{\mathcal{P}}}(Z)} \\
&\le \|\mathcal{P}\| \|f_n - f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g_n\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)} \\
&+ \|\mathcal{P}\| \|f\|_{\mathcal{L}^{p_1}_{\mathsf{u}_1}(X)} \|g_n - g\|_{\mathcal{L}^{p_2}_{\mathsf{u}_2}(Y)}
\end{aligned}$$

Taking limits the proof is completed.

□

Let us point out a little improvement that can be achieved for the compatible triples in Proposition 1. Let us recall the following fact that will be used in the proof.

Lemma 1 Let X be a Banach space, $1 \leq p < \infty$ and $(x_n^*)_n \subseteq X^*$. Then

$$\sup\{(\sum_n |\langle x_n^*, x^{**}\rangle|^p)^{\frac{1}{p}} : \|x^{**}\| = 1\} = \sup\{(\sum_n |\langle x, x_n^*\rangle|^p)^{\frac{1}{p}} : \|x\| = 1\}$$

Corollary 1 (Hölder's inequality II) Let X, X_1, Y, Y_1 and U be Banach spaces and $1 \leq p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3}$. Let $\mathsf{u}_1 : X \times X_1 \to U$, $\mathsf{u}_2 : Y \times Y_1 \to U^*$ be bounded bilinear maps and let $\mathsf{u}_{\mathsf{u}_1,\mathsf{u}_2} = \tilde{\mathsf{u}} : X \times Y \to \mathcal{L}(X_1, Y_1^*)$ be defined by the formula

$$\langle \tilde{\mathsf{u}}(x,y)(x_1), y_1\rangle = \langle \mathsf{u}_1(x,x_1), \mathsf{u}_2(y,y_1)\rangle.$$

If $f \in L^{p_1}_{\mathsf{u}_1}(X)$ and $g \in L^{p_2}_{\mathsf{u}_2}(Y)$ then $\tilde{\mathsf{u}}(f,g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*))$.
Moreover $\|\tilde{\mathsf{u}}(f,g)\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1,Y_1^*))} \leq \|f\|_{L^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{L^{p_2}_{\mathsf{u}_2}(Y)}$.

Proof. Assume first that f and g are simple functions. If $f = \sum_k x_k \mathbf{1}_{E_k} \in \mathcal{S}(X)$ and $g = \sum_p y_p \mathbf{1}_{F_p} \in \mathcal{S}(Y)$ then $h = \tilde{\mathsf{u}}(f,g) = \sum_{k,p} \tilde{\mathsf{u}}(x_k, y_p)\mathbf{1}_{E_k \cap F_p} \in \mathcal{S}(\mathcal{L}(X_1, Y_1^*))$. Note that $\mathcal{L}(X_1, Y_1^*) = (X_1 \hat{\otimes} Y_1)^*$. Hence from Lemma 1

$$\begin{aligned}
\|h\|_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^*)} &= \sup\{(\sum_{k,p} |\langle \tilde{\mathsf{u}}(x_k,y_p), \psi\rangle|^{p_3} \mu(E_k \cap F_p))^{\frac{1}{p_3}} : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&= \sup\{(\sum_{k,p} |\langle \varphi, \tilde{\mathsf{u}}(x_k,y_p)\rangle|^{p_3} \mu(E_k \cap F_p))^{\frac{1}{p_3}} : \|\varphi\|_{X_1\hat{\otimes}Y_1} = 1\} \\
&= \|h\|_{L^{p_3}_{\text{weak}*}((X_1\hat{\otimes}Y_1)^*)}.
\end{aligned}$$

We conclude, using Theorem 1, that

$$\|h\|_{L^{p_3}_{\text{weak}}(\mathcal{L}(X_1,Y_1^*))} \leq \|f\|_{L^{p_1}_{\mathsf{u}_1}(X)} \|g\|_{L^{p_2}_{\mathsf{u}_2}(Y)}.$$

Now, if we take $f \in L^{p_1}_{\mathsf{u}_1}(X)$ and $g \in L^{p_2}_{\mathsf{u}_2}(Y)$ then there exist $(f_n)_n \subseteq \mathcal{S}(X)$ and $(g_n)_n \subseteq \mathcal{S}(Y)$ such that $f_n \to f$ a.e., $g_n \to g$ a.e., $\|f_n - f\|_{L^{p_1}_{\mathsf{u}_1}(X)} \to 0$ and $\|g_n - g\|_{L^{p_2}_{\mathsf{u}_2}(Y)} \to 0$. Clearly $\tilde{\mathsf{u}}(f_n, g_n) \to \tilde{\mathsf{u}}(f,g)$ a.e. and therefore $\tilde{\mathsf{u}}(f,g)$ is strongly measurable and

$$|\langle \tilde{\mathsf{u}}(f_n,g_n), \psi\rangle|^{p_3} \to |\langle \tilde{\mathsf{u}}(f,g), \psi\rangle|^{p_3} \text{ a.e.}$$

for all $\psi \in (X_1 \hat{\otimes} Y_1)^{**}$.
To see that $\tilde{\mathsf{u}}(f,g) \in P^{p_3}(\mathcal{L}(X_1, Y_1^*))$ it suffices to show that $\tilde{\mathsf{u}}(f,g) \in L^{p_3}_{\text{weak}}(\mathcal{L}(X_1, Y_1^*))$.
Then using $(X_1 \hat{\otimes} Y_1)^* = \mathcal{L}(X_1, Y_1^*)$, Fatou's Lemma and the inequality for simple functions we have that

$$\begin{aligned}
\|\tilde{\mathsf{u}}(f,g)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^*)} &= \sup\{\int_\Omega |\langle \tilde{\mathsf{u}}(f,g), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&= \sup\{\int_\Omega \lim_n |\langle \tilde{\mathsf{u}}(f_n,g_n), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \sup\{\liminf_n \int_\Omega |\langle \tilde{\mathsf{u}}(f_n,g_n), \psi\rangle|^{p_3} d\mu : \|\psi\|_{(X_1\hat{\otimes}Y_1)^{**}} = 1\} \\
&\leq \liminf_n \|\tilde{\mathsf{u}}(f_n,g_n)\|^{p_3}_{L^{p_3}_{\text{weak}}((X_1\hat{\otimes}Y_1)^{**})} \\
&\leq \liminf_n \|f_n\|^{p_3}_{L^{p_1}_{\mathsf{u}_1}(X)} \|g_n\|^{p_3}_{L^{p_2}_{\mathsf{u}_2}(Y)} \\
&= \|f\|^{p_3}_{L^{p_1}_{\mathsf{u}_1}(X)} \|g\|^{p_3}_{L^{p_2}_{\mathsf{u}_2}(Y)}.
\end{aligned}$$

□

Applying Theorem 1 to the examples given above one obtains the following applications.

Corollary 2 Let $1 \le p_1, p_2, p_3 < \infty$ such that $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$.
Let $\mathsf{u} : X \times Y \to Z$ be a bounded bilinear map.

(1) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}(X^*)$ then $\langle f, g \rangle \in L^{p_3}$.

(2) If $f \in L^{p_1}(X)$ and $g \in L^{p_2}_{\widetilde{\mathsf{u}}_*}(Y)$ then $\mathsf{u}(f,g) \in L^{p_3}_{\mathrm{weak}}(Z)$, where $\widetilde{\mathsf{u}}_* : Y \times Z^* \to X^*$ is given by $\langle x, \widetilde{\mathsf{u}}_*(y, z^*) \rangle = \langle \mathsf{u}(x,y), z^* \rangle$.

(3) If $f \in L^{p_1}_{\mathsf{u}}(X)$ and $g \in L^{p_2}(Z^*)$ then $\mathsf{u}^*(f,g) \in L^{p_3}_{\mathrm{weak}^*}(Y^*)$, where

$\mathsf{u}^* : X \times Z^* \to Y^*$ is given by $\langle y, \mathsf{u}^*(x, z^*) \rangle = \langle \mathsf{u}(x,y), z^* \rangle$.

(4) If $f \in L^{p_1}_{\widetilde{\mathcal{O}}_{Y^*}}(X)$ and $g \in L^{p_2}(Y)$ then $f \otimes g \in L^{p_3}_{\mathrm{weak}}(X \hat{\otimes} Y)$.

(5) If $f \in L^{p_1}_{\mathcal{O}_{X,Z}}(\mathcal{L}(X,Z))$ and $g \in L^{p_2}_{\mathcal{O}_{Y,Z^*}}(\mathcal{L}(Y,Z^*))$ and if we put $f^*(t) = f(t)^* \in \mathcal{L}(Z^*, X^*)$ then $f^*g \in L^{p_3}_{\mathrm{weak}^*}(\mathcal{L}(Y, X^*))$.

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