Introduction to vector valued Bergman spaces

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1 Introduction

This paper contains a little extended writting of the lectures that I gave at the Department of Mathematics of the Joensuu University during the summer school hold in May 2003. These lecture notes are based upon results in two papers in collaboration with J.L. Arregui (see [5] and [6]), although some proofs have been changed and extended to make them as selfcontained as possible.

In these notes we denote by $(\mathbb{D}, dA(z))$ the Lebesgue measure space over the disc $\mathbb{D} = \{|z| < 1\}$ where $dA(z)$ stands for the normalized area measure, that is $dA(z) = \frac{1}{\pi} dxdy = \frac{1}{\pi} drd\theta$ in rectangular and polar coordinates respectively. Throughout X will be a complex Banach space, $1 \leq p < \infty$, p' stands for the conjugate exponent of p, i.e. $1/p + 1/p' = 1$ and, as usual, C denotes a constant that may vary from line to line.

For $1 \leq p, q \leq \infty$, we consider the spaces $\ell(p, q, X)$ of sequences $(x_n)_n \subset X$ such that $(\|(x_n)_{n \in I_k}\|_{\ell_p})_k \in \ell_q$, where $I_k = \{n \in \mathbb{N};\ 2^{k-1} \leq n < 2^k\}$ for $k \in \mathbb{N}$ and $I_0 = \{0\}$, and we denote $\|(x_n)\|_{p,q} = \left(\sum_{k=0}^{\infty} (\sum_{n \in I_k} \|x_n\|^p)^{q/p}\right)^{1/q}$. We keep the notation $\ell_p(X)$ for the space $\ell(p, p, X)$.

We write $L_p(\mathbb{D}, X)$ for the Bochner space of measurable functions such that $(\int_{\mathbb{D}} \|f(z)\|^p dA(z))^{\frac{1}{p}} < \infty$ and we will denote by $B_p(\mathbb{D}, X)$ the Bergman space of functions $F \in \mathcal{H}(\mathbb{D}, X) \cap L_p(\mathbb{D}, X)$ where $\mathcal{H}(\mathbb{D}, X)$ stands for the space of X-valued holomorphic functions in the unit disc $f(z) = \sum_{n=0}^{\infty} x_n z^n$ for some sequence $(x_n) \in X$.

The notes are divided into four sections.

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In Section 2 we simply introduce the spaces $B_p(\mathbb{D}, X)$ and prove some generalities about them. Section 3 contains the duality $(B_p(\mathbb{D}, X))^* = B_{p'}(\mathbb{D}, X^*)$ and in Section 4 we give some results on Taylor coefficients of vector valued functions in $B_p(\mathbb{D}, X)$ and introduce the properties of Bergman type and cotype. Given $1 \leq p \leq 2 \leq q \leq \infty$, a Banach space X is said to have Bergman type p (respec. Bergman cotype q) if there exists a $C > 0$ such that

$$||(\frac{x_n}{n^{1/p}})_{1\leq n\leq N}||_{p',p} \leq C||f||_{B_p(\mathbb{D},X)}$$

(respec.

$$||f||_{B_q(X)} \leq C||(\frac{x_n}{n^{1/q}})_{1\leq n\leq N}||_{q',q})$$

for all $N \in \mathbb{N}$, $x_1, ..., x_N$ in X and $f(z) = \sum_{n=0}^{N} x_n z^n$..

It is shown that, for $1 \leq p \leq 2$, spaces of Fourier type p must also have Bergman type p and Bergman cotype p' where $1/p + 1/p' = 1$.

In Section 5 we consider the notion of operator-valued multiplier and prove some resuts about the subject. A sequence of bounded operators $(T_n)_n$ in $\mathcal{L}(X, Y)$ is said to be a multiplier between $B_p(\mathbb{D}, X)$ and $\ell_1(Y)$, to be denoted $(T_n) \in (B_p(\mathbb{D}, X), \ell_1(Y))$, if the sequence $(T_n(x_n))$ belongs to $\ell_1(Y)$ for any function $f(z) = \sum_{n=0}^{\infty} x_n z^n$ in $B_p(\mathbb{D}, X)$.

Of course, this is equivalent to the existence of a constant $C > 0$ such that

$$\sum_{n=0}^{N} ||T_n(x_n)|| \leq C||\sum_{n=0}^{N} x_n z^n||_{B_p(X)} \tag{1}$$

for any $N \in \mathbb{N}$ and $x_0, x_1, ..., x_N$ elements in X.

The infimum of the constants C verifying (1) is the multiplier norm, which coincides with the operator norm between $B_p(\mathbb{D}, X)$ and $\ell_1(Y)$ respectively.

We shall analyze for different values of p and under certain geometric properties the spaces $(B_p(\mathbb{D}, X), \ell_1(Y))$. The reader is referred to [6] and [14] for many other multiplier results vector-valued functions and a to [27, 28, 10, 9, 12] for a collection of scalar-valued results related to them.

2 Generalities

Definition 2.1 *Let* $1 \leq p < \infty$ *A function* $f \in \mathcal{H}(\mathbb{D}, X)$ *is said to belong to* $B^p(\mathbb{D}, X)$ *if*

$$\|F\|_{B_p(X)} = (\int_{\mathbb{D}} \|f(z)\| dA(z))^{\frac{1}{p}} < \infty.$$

Using the notation $M_p(f,r) = (\int_0^{2\pi} \|f(re^{it})\|^p \frac{dt}{2\pi})^{1/p}$ we can write

$$\|f\|^p_{B_p(X)} = \int_0^1 2M_p(f,r)^p r dr. \tag{2}$$

Proposition 2.2 *If $f \in \mathcal{H}(\mathbb{D}, X)$ then*

$$f(z) = \frac{1}{R^2} \int_{|w-z|<R} f(w) dA(w) \tag{3}$$

for all $|z| < 1$ and $0 < R < 1 - |z|$.

Proof: The formula follows easily from the Cauchy formula and integration in polar coordinates. ∎

Let us collect the first elementary properties of these spaces.

Proposition 2.3 *Let $1 \le p < \infty$ and let X be a complex Banach space.*
(i) $B_p(\mathbb{D}, X)$ is a closed subspace of $L_p(\mathbb{D}, X)$, that is $B_p(\mathbb{D}, X)$ is a Banach space.
(ii) If $f \in B^p(\mathbb{D}, X)$ then $\lim_{r\to 1} \|f - f_r\|_{B_p(\mathbb{D},X)} = 0$, where $f_r(z) = f(rz)$.
(iii) The X-valued analytic polynomials $\mathcal{P}(X)$ are dense in $B_p(\mathbb{D}, X)$.

Proof:
(i) Let (f_n) be a sequence of holomorphic functions in $L^p(\mathbb{D}, X)$ converging to f in $L^p(\mathbb{D}, X)$. Let us show that $f \in \mathcal{H}(\mathbb{D}, X)$. Given $r < 1$ and $|z| < r$, from (3)

$$\begin{aligned}
\|f_n(z) - f_m(z)\| &= \frac{1}{(1-r)^2} \|\int_{|w-z|<1-r} (f_n(w) - f_m(w)) dA(w)\| \\
&\le \frac{1}{(1-r)^2} \int_{|w-z|<1-r} \|f_n(w) - f_m(w)\| dA(w) \\
&= \frac{1}{(1-r)^2} \|f_n - f_m\|_{L^p(X)}.
\end{aligned}$$

This clearly gives that (f_n) converges uniformly on compact sets to f and therefore $f \in \mathcal{H}(\mathbb{D}, X)$.
(ii) From (2) we have

$$\|f - f_r\|^p_{B_p(X)} \approx \int_0^1 M_p^p(f - f_r, s) ds.$$

Taking into account that $M_p^p(f - f_r, s) \leq \sup_{|z|=s} \|f(z) - f(rz)\|^p$ one gets

$$\lim_{r\to 1} M_p^p(f - f_r, s) = 0 \quad (0 < s < 1).$$

Since

$$M_p^p(f - f_r, s) \leq 2M_p^p(f, s)$$

we can apply the dominated convergence theorem to get $\|f - f_r\|_{B_p(X)} \to 0$ as $r \to 1$.

(iii) Now given $\varepsilon > 0$ we first choose $0 < r < 1$ such that $\|f - f_r\|_{B_p(X)} < \frac{\varepsilon}{2}$ and then we take a Taylor polynomial such that $sup_{z\in\mathbb{D}}\|f(rz) - P(z)\| < \frac{\varepsilon}{2}$ to get $\|f - P\|_{B_p(X)} < \varepsilon$.

■

Proposition 2.4 *Let $1 \leq p < \infty$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n$. Then $f \in B_p(\mathbb{D}, X)$ if and only if $f_1(z) = \frac{f(z)-f(0)}{z} \in B_p(\mathbb{D}, X)$.*

Morerover, there exists $A < 1$ such that

$$A(\|f(0)\| + \|f_1\|_{B_p(X)}) \leq \|f\|_{B_p(X)} \leq \|f(0)\| + \|f_1\|_{B_p(X)}.$$

Proof: Since $f(z) = f(0) + zf_1(z)$ we get $\|f\|_{B_p(X)} \leq \|f(0)\| + \|f_1\|_{B_p(X)}$. To get the other inequality, note that $x_n = (n+1)\int_{\mathbb{D}} f(w)\bar{w}^n dA(w)$ for $n \geq 0$. This allows to estimate $\|x_n\| \leq (n+1)\|f\|_{B_p(X)}$. Hence, since $f_1(z) = \sum_{n=0}^{\infty} x_{n+1} z^n$, we obtain

$$(\int_{\mathbb{D}} \|f_1(z)\|^p dA(z))^{1/p} \leq$$

$$(\int_{|z|<1/2} \|f_1(z)\|^p dA(z))^{1/p} + (\int_{1/2\leq|z|<1} \|f_1(z)\|^p dA(z))^{1/p} \leq$$

$$\leq \frac{1}{4^{1/p}}(\sum_{n=0}^{\infty} \frac{(n+1)}{2^n})\|f\|_{B_p(X)} + 2(\int_{1/2}^{1} (\|f(0)\| + M_p(f, r))^p r dr)^{1/p}.$$

This gives $\|f_1\|_{B_p(X)} \leq C\|f\|_{B_p(X)}$, and taking $A = 1/(C+1)$, the proof is finished. ■

Theorem 2.5 *Let $f \in \mathcal{H}(\mathbb{D}, X)$, $n \in \mathbb{N}$, $1 \leq p < \infty$. Then $f \in B_p(\mathbb{D}, X)$ if and only if the function $z \mapsto (1 - |z|^2)^n f^{(n)}(z) \in L_p(\mathbb{D}, X)$.*

Proof: Let us show that for any $g \in \mathcal{H}(\mathbb{D}, X)$ and $k \geq 0$, the function $(1-|z|^2)^k g(z)$ belongs to $L^p(\mathbb{D}, X)$ if and only if $(1-|z|^2)^{k+1} g'(z)$ also does. Then a recurrence argument gives the statement.

Note that $(1-|z|^2)^{k+1} g'(z) \in L^p(\mathbb{D}, X)$ if and only if

$$\int_{\mathbb{D}} (1-|z|^2)^{pk+p} \|zg'(z)\|^p dA(z) < \infty.$$

Let us denote $h(z) = zg'(z) = \sum_{n=0}^\infty n x_n z^n$, and observe that for each $r < 1$ one has that $h_{r^2} = g_r * \lambda_r$, where $\lambda(z) = \frac{z}{(1-z)^2}$.

Since $M_1(\lambda, r) = \frac{r}{1-r^2}$ and $M_p(h, r^2) \leq M_1(\lambda, r) M_p(g, r)$, one gets that

$$\begin{aligned}
\int_{\mathbb{D}} (1-|z|^2)^{pk+p} \|zg'(z)\|^p dA(z) &= \int_0^1 4r^3 (1-r^4)^{pk+p} M_p^p(h, r^2) dr \\
&\leq C \int_0^1 r(1-r^2)^{pk} M_p^p(g, r) dr \\
&= C \int_{\mathbb{D}} (1-|z|^2)^{pk} \|g(z)\|^p dA(z).
\end{aligned}$$

Conversely, let us take g such that $(1-r^2)^{k+1} M_p(g', r) \in L_p((0,1), dr)$. We may assume that $\int_0^1 (1-r)^{(k+1)p} M_p^p(g', r) dr = 1$ and also that $g(0) = 0$.

Since $M_p(g, r) \leq \int_0^r M_p(g', s) ds$ we have

$$\begin{aligned}
\int_{\mathbb{D}} (1-|z|^2)^{kp} \|g(z)\|^p dA(z) &= \int_0^1 2r(1-r^2)^{kp} M_p^p(g, r) dr \\
&\leq \int_0^1 2r(1-r^2)^{kp} \left(\int_0^r M_p(g', s) ds\right)^p dr \\
&\leq C \int_0^1 (1-r)^{kp} \left(\int_0^r M_p(g', s) ds\right)^p dr.
\end{aligned}$$

For $p = 1$ we get

$$\begin{aligned}
\int_{\mathbb{D}} (1-|z|^2)^k \|g(z)\| dA(z) &\leq C \int_0^1 (1-r)^k \left(\int_0^r M_1(g', s) ds\right) dr \\
&= C \int_0^1 (1-s)^{k+1} M_1(g', s) ds = C.
\end{aligned}$$

For $p > 1$, we write for each $t \in (0,1)$

$$I_t = \int_0^t (1-r)^{kp} \left(\int_0^r M_p(g', s) ds\right)^p dr.$$

Let $u(r) = -\frac{1}{pk+1}(1-r)^{pk+1}$ and $v(r) = (\int_0^r M_p(g', s)ds)^p$.
Since $u(t)v(t) < 0$ y $v(0) = 0$, we have

$$I_t = \int_0^t u'(r)v(r)dr \leq -\int_0^t u(r)v'(r)dr$$

That is

$$\begin{aligned} I_t &\leq \frac{p}{pk+1}\int_0^t (1-r)^{pk+1} M_p(g', r)(\int_0^r M_p(g', s)ds)^{p-1}dr \\ &= \frac{p}{pk+1}\int_0^t (1-r)^{k+1} M_p(g', r)(1-r)^{(p-1)k}(\int_0^r M_p(g', s)ds)^{p-1}dr. \end{aligned}$$

Then the assumption and Hölder's inequality shows that $I_t \leq C I_t^{1/p'}$. Hence $I_t \leq C$ for all t and the proof is finished. ∎

Proposition 2.6 *Let* $f(z) = \sum_{k=0}^{\infty} x_k z^k \in B_1(\mathbb{D}, X)$ *and* $n \in \mathbb{N}$. *Then*

$$\int_{\mathbb{D}} f(z)\bar{z}^n dA(z) = \int_{\mathbb{D}} (1-|z|^2)f'(z)\bar{z}^{n-1}dA(z) = \frac{x_n}{n+1}.$$

Proof: Since $f \in L^1(\mathbb{D}, X)$

$$\begin{aligned} \int_D f(z)\bar{z}^n dA(z) &= \sum_{k=0}^{\infty}(\int_D z^k \bar{z}^n dA(z))x_k \\ &= x_n \int_{\mathbb{D}} |z|^{2n} dA(z) = x_n \int_0^1 2r^{2n+1}dr = \frac{x_n}{n+1}. \end{aligned}$$

Theorem 2.5 gives that $(1-|z|)^2 f'(z) \in L^1(\mathbb{D}, X)$ and arguing as above

$$\begin{aligned} \int_D (1-|z|^2)f'(z)\bar{z}^{n-1}dA(z) &= \sum_{k=1}^{\infty}(\int_D (1-|z|^2)z^{k-1}\bar{z}^{n-1}dA(z))kx_k \\ &= nx_n \int_{\mathbb{D}} (1-|z|^2)|z|^{2n-2}dA(z) \\ &= 2nx_n \int_0^1 (1-r^2)r^{2n-1}dr \\ &= nx_n \int_0^1 (1-r)r^{n-1}dr = \frac{x_n}{n+1}. \end{aligned}$$ ∎

Proposition 2.7 *If $f \in B_1(\mathbb{D}, X)$ then*

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\bar{w})^2} dA(w) = 2 \int_{\mathbb{D}} \frac{(1 - |w|^2) f(w)}{(1 - z\bar{w})^3} dA(w)$$

for all $|z| < 1$.

PROOF: Note that $\frac{1}{(1-z\bar{w})^2} = \sum_{n=0}^{\infty} (n+1) z^n \bar{w}^n$ where the series is absolutely convergent for each $z \in \mathbb{D}$, then the first formula follows from Proposition 2.6.

Taking derivatives we also have $\frac{2\bar{w}}{(1-z\bar{w})^3} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n \bar{w}^{n+1}$ and the series also converges absolutely. Hence

$$\begin{aligned} 2 \int_{\mathbb{D}} \frac{(1 - |w|^2) f(w)}{(1 - z\bar{w})^3} dA(w) &= \sum_{n=0}^{\infty} (n+1)(n+2) z^n \int_{\mathbb{D}} (1 - |w|^2) f(w) \bar{w}^n dA(w) \\ &= \sum_{n=0}^{\infty} (n+1)(n+2) x_n z^n \int_{\mathbb{D}} (1 - |w|^2) |w|^{2n} dA(w) \\ &= \sum_{n=0}^{\infty} (n+1)(n+2) x_n z^n \int_0^1 (1 - r) r^n dr \\ &= \sum_{n=0}^{\infty} x_n z^n = f(z). \end{aligned}$$

■

3 Duality

In this section we shall get the duality result between the spaces $B_p(\mathbb{D}, X)$ and $B_{p'}(\mathbb{D}, X^*)$ without conditions on the Banach space X. Let us start with some preliminary results.

Proposition 3.1 *Let $1 \leq p < \infty$, $z \in \mathbb{D}$ and $x^* \in X^*$.*

(i) The point evaluation δ_z is a bounded linear operator from $B_p(\mathbb{D}, X)$ into X with norm $\|\delta_z\| \leq (1 - |z|)^{\frac{-2}{p}}$.

(ii) $\delta_z \otimes x^*$ belongs to $(B_p(\mathbb{D}, X))^*$ and $\|\delta_z \otimes x^*\| \leq (1 - |z|)^{\frac{-2}{p}} \|x^*\|$.

PROOF: (i) follows by applying Hölder's inequality in the formula (3) and taking limits as R goes to $1 - |z|$.

(ii) It is immediate from (i). ■

We need the following estimates.

Lemma 3.2 *Let $\alpha > 1$. Then $J_\alpha(r) = \int_{-\pi}^{\pi} \frac{dt}{|1-re^{it}|^\alpha} \sim \frac{1}{(1-r)^{\alpha-1}}$ as $r \to 1$.*

PROOF: Note that for $|t| < \pi$ we have

$$|1 - re^{it}|^\alpha = ((1-r)^2 + 4r\sin^2 t/2)^{\alpha/2} \approx ((1-r)^2 + 4r|t|^2)^{\alpha/2}.$$

Use that

$$\int_{-\pi}^{\pi} \frac{dt}{((1-r)^2 + 4r|t|^2)^{\alpha/2}} = \frac{1}{r^{1/2}(1-r)^{\alpha-1}} \int_0^{\frac{2r^{1/2}\pi}{1-r}} \frac{dt}{(1+t^2)^{\alpha/2}}.$$

This gives $J_\alpha(r) \approx \frac{1}{(1-r)^{\alpha-1}}$ for $\alpha > 1$. ■

Lemma 3.3 *Let $\alpha > -1$ and $\beta > \alpha + 1$.*
Then $I_{\alpha,\beta}(r) = \displaystyle\int_0^1 \frac{(1-s)^\alpha}{(1-rs)^\beta} ds \sim \frac{1}{(1-r)^{\beta-\alpha-1}}$ as $r \to 1$.

PROOF: Let us rewrite the integral as follows

$$\begin{aligned} I_{\alpha,\beta}(r) &= \int_0^1 \frac{(1-s)^\alpha}{((1-s)+(1-r)s)^\beta} ds \\ &= \int_0^1 \frac{u^\alpha}{(u+(1-r)(1-u))^\beta} du \\ &= \frac{1}{(1-r)^\beta} \int_0^1 \frac{u^\alpha}{(\frac{u}{1-r} + (1-u))^\beta} du \\ &= \frac{1}{(1-r)^{\beta-\alpha-1}} \int_0^{\frac{1}{1-r}} \frac{v^\alpha}{(1+rv)^\beta} dv \end{aligned}$$

This gives $I_{\alpha,\beta}(r) \approx \frac{1}{(1-r)^{\beta-\alpha-1}}$ for $\beta - \alpha > 1$.

■

Combining Lemmas 3.2 and 3.3 we obtain

Corollary 3.4 *Let $\beta - \alpha > 2$ and $A_{\alpha,\beta}(r) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-rw|^\beta} dA(w)$ for $\beta > 0$ and $\alpha > -1$. Then*

$$A_{\alpha,\beta}(r) \sim \frac{1}{(1-r)^{\beta-\alpha-2}} \quad (r \to 1).$$

Theorem 3.5 *Let $1 < p < \infty$ and define*

$$T(\phi) = \int_{\mathbb{D}} \frac{\phi(w)}{|1-\bar{w}z|^2} dA(w).$$

Then T is a bounded operator on $L^p(\mathbb{D})$.

PROOF: Therefore

$$\begin{aligned} |T(\phi)(z)| &\leq \int_{\mathbb{D}} \frac{(1-|w|)^{\frac{1}{pp'}}|\phi(w)|}{|1-z\bar{w}|^2(1-|w|)^{\frac{1}{pp'}}} dA(w) \\ &\leq (\int_{\mathbb{D}} \frac{|\phi(w)|^p(1-|w|)^{\frac{1}{p'}}}{|1-z\bar{w}|^2} dA(w))^{\frac{1}{p}} (\int_{\mathbb{D}} \frac{(1-|w|)^{\frac{-1}{p}}}{|1-z\bar{w}|^2} dA(w))^{\frac{1}{p'}} \end{aligned}$$

Now, using Corollary 3.4 for $\alpha = \frac{-1}{p}$ and $\beta = 2$, we have

$$(\int_{\mathbb{D}} \frac{(1-|w|)^{\frac{-1}{p}}}{|1-z\bar{w}|^2} dA(w))^{\frac{1}{p'}} \leq C \frac{1}{(1-|z|)^{\frac{1}{pp'}}}$$

Now

$$\begin{aligned} \int_{\mathbb{D}} \|T(\phi)(z)\|^p dA(z) &\leq \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|)^{\frac{1}{p'}}|\phi(w)|^p}{|1-z\bar{w}|^2} dA(w)(1-|z|)^{\frac{-1}{p'}} dA(z) \\ &\leq \int_{\mathbb{D}} (1-|w|)^{\frac{1}{p'}}|\phi(w)|^p (\int_D \frac{(1-|z|)^{\frac{-1}{p'}}}{|1-z\bar{w}|^2} dA(z)) dA(w) \end{aligned}$$

Applying again Corollary 3.4 for $\alpha = \frac{-1}{p'}$ and $\beta = 2$, we conclude that $\|T(\phi)\|_p \leq C\|\phi\|_p$. ∎

Theorem 3.6 *Let $1 < p < \infty$. Then $(B_p(\mathbb{D}, X))^* = B^{p'}(\mathbb{D}, X^*)$ with equivalent norms.*

PROOF: Let us define $J : B^{p'}(\mathbb{D}, X^*) \to (B_p(\mathbb{D}, X))^*$ given by

$$J(f)(g) = \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z). \tag{4}$$

It is linear and clearly bounded with $\|J\| \leq 1$.

Let us see that J is injective. Indeed, assume that $f(z) = \sum_{n=0}^{\infty} x_n^* z^n \in B^{p'}(\mathbb{D}, X^*)$ and $J(f) = 0$. Hence if $u_n(z) = (n+1)z^n$ then, for any $n \in \mathbb{N}$ and $x \in X$, we have

$$J(f)(x \otimes u_n) = < (n+1) \int_{\mathbb{D}} f(z)\bar{z}^n dA(z), x > = < x_n^*, x > = 0.$$

This implies that $f = 0$.

Let us now show that J is surjective. Given $\Phi \in (B_p(\mathbb{D}, X))^*$ we define

$$f(z) = \sum_{n=0}^{\infty} x_n^* z^n \tag{5}$$

where $x_n^* \in X^*$ are given by

$$< x_n^*, x > = \Phi(x \otimes u_n), \tag{6}$$

for $u_n(z) = (n+1)z^n$.

Due to the fact that $\|x_n^*\| \leq \|\Phi\| \|u_n\|_{B_p} \approx \|\Phi\| \frac{1}{(n+1)^{1/p}}$, one gets $f \in \mathcal{H}(\mathbb{D}, X)$.

To see that $f \in B^{p'}(\mathbb{D}, X^*)$, we use that

$$\|f\|_{L^{p'}(X)} = \sup\{ | \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z) | : \|g\|_{L^p(X)} = 1 \}.$$

Hence

$$\langle f(z), g(\bar{z}) \rangle = \sum_{n=0}^{\infty} \Phi(u_n \otimes g(\bar{z}) z^n = \Phi(\sum_{n=0}^{\infty} u_n z^n \otimes g(\bar{z})) = \Phi(K_z \otimes g(\bar{z})),$$

where $K_z(w) = \frac{1}{(1-wz)^2}$ stands for the Bergman kernel.

In particular we have that $\int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z) = \Phi(\int_{\mathbb{D}} K_z g(\bar{z}) dA(z))$.

This gives that

$$| \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z) | \leq \|\Phi\| \| \int_{\mathbb{D}} |K_z| \|g(\bar{z})\| dA(z) \|_p = \|\Phi\| \|T(\|g\|)\|_p.$$

An application of Theorem 3.5 gives $\|f\|_{L^p(X)} \leq C\|\Phi\|$.

Since $J(f)(u_n \otimes x) = \langle x_n^*, x \rangle = \Phi(u_n \otimes x)$ for all $n \geq 0$ and $x \in X$ we get that $J(f) = \Phi$ and the proof is finished. ∎

Remark 3.1 *The reader is referred to the papers [5, 8, 9] for some duality results in the case $p = 1$.*

4 Taylor coefficients of functions in $B_p(\mathbb{D}, X)$.

Proposition 4.1 *Let* $1 \leq p < \infty$ *and* $f(z) = \sum_{n=0}^{\infty} x_n z^n \in B_p(\mathbb{D}, X)$. *Then*
(i) $\|x_n\| = o(n^{1/p})$.
(ii) $\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{n^2} < \infty$.

Proof: For each n and $r \in (0,1)$, we have that

$$x_n r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

This implies that for any $n \in \mathbb{N}$ and $0 < r < 1$ we have

$$\|x_n\| r^n \leq M_1(f, r). \tag{7}$$

Since $M_p(f, \cdot)$ is increasing in $(0,1)$, from (7) one gets that

$$(1-r)||x_n||^p r^{np} \leq (1-r) M_p^p(f, r) \leq \int_r^1 M_p^p(f, s) ds$$

for each $r \in (0,1)$.

Hence, for any n, by taking $r = 1 - 1/n$, we see that

$$\frac{1}{n} \|x_n\|^p \sim \frac{1}{n} \|x_n\|^p \Big(1 - \frac{1}{n}\Big)^{np} \leq \int_{1-1/n}^1 M_p^p(f, s) ds.$$

This shows that $\dfrac{\|x_n\|}{n^{1/p}} \to 0$.

Now observe that the norm in $B_p(\mathbb{D}, X)$ can be estimated from below as follows

$$\begin{aligned}
\sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} M_p^p(f, r) dr \quad &\geq \quad \sum_{n=1}^{\infty} \int_{1-1/n}^{1-1/(n+1)} \|x_n\|^p r^{np} dr \\
&\geq C \quad \sum_{n=1}^{\infty} \|x_n\|^p \frac{1}{n(n+1)} (1 - \frac{1}{n})^{np} \\
&\sim \quad \sum_{n=1}^{\infty} \frac{\|x_n\|^p}{n^2}.
\end{aligned}$$

∎

Actually Proposition 4.1 can be easily improved by considering a new family of vector-valued sequence spaces.

Definition 4.2 *Let* $1 \leq p,q \leq \infty$*, and* X *be a Banach space.* $\ell(p,q,X)$ *denotes the space of sequences* $(x_n)_n \subset X$ *such that* $(\|(x_n)_{n\in I_k}\|_{\ell_p})_k \in \ell_q$*, where* $I_k = \{n \in \mathbb{N};\ 2^{k-1} \leq n < 2^k\}$ *for* $k \in \mathbb{N}$ *and* $I_0 = \{0\}$*.*

For $1 \leq p,q \leq \infty$*, the spaces* $\ell(p,q,X)$ *become Banach spaces under the norm* $\|\cdot\|_{p,q}$ *given by, for* $1 < p,q < \infty$

$$\|(x_n)\|_{p,q} = \Big(\sum_{k=0}^{\infty}(\sum_{n\in I_k}\|x_n\|^p)^{q/p}\Big)^{1/q},$$

$$\|(x_n)\|_{p,\infty} = \sup_{k\geq 0}(\sum_{n\in I_k}\|x_n\|^p)^{1/p},$$

$$\|(x_n)\|_{\infty,q} = \Big(\sum_{k=0}^{\infty}\sup_{n\in I_k}\|x_n\|^q\Big)^{1/q}.$$

Of course $\ell(p,p,X) = \ell^p(X)$ *for* $1 \leq p \leq \infty$ *and, as usual, when* $X = \mathbb{C}$ *we simply write* $\ell(p,q)$*.*

Now we list some useful reformulations for the norms in the spaces $\ell(p,q)$.

Lemma 4.3 *(see [12],or [31]) Let* (α_n) *be a sequence of nonnegative numbers and* $0 < q, \beta < \infty$*. Then*

$$\Big(\alpha_n\Big) \in \ell(1,\infty) \Leftrightarrow \sum_{n=1}^{\infty} n^\beta \alpha_n r^n = O((\frac{1}{1-r})^\beta)\ . \tag{8}$$

$$\Big(\alpha_n\Big) \in \ell(1,q) \Leftrightarrow \int_0^1 (1-r)^{\beta q-1}(\sum_{n=1}^{\infty} n^\beta \alpha_n r^n)^q dr < \infty. \tag{9}$$

In particular,

$$\int_0^1 (\sum_{n=1}^{\infty}\alpha_n r^n)^q dr \sim \sum_{k=1}^{\infty}(\sum_{n\in I_k}\frac{\alpha_n}{n^{1/q}})^q\,. \tag{10}$$

Proof:
To see (8) assume first that $\Big(\alpha_n\Big) \in \ell(1,\infty)$. This implies that

$$\begin{aligned}\sum_{n=0}^{\infty} n^\beta \alpha_n r^n &\leq C\sum_{k=0}^{\infty} 2^{k\beta}\sum_{n\in I_k}\alpha_n r^{2^k}\\ &\leq (\sup_k \sum_{n\in I_k}\alpha_n)(\sum_{k=0}^{\infty}2^{k\beta}r^{2^k})\end{aligned}$$

$$
\begin{aligned}
&\leq C(\sup_k \sum_{n\in I_k} \alpha_n)(\sum_{n=0}^{\infty} n^{\beta-1} r^n) \\
&\leq (\sup_k \sum_{n\in I_k} \alpha_n)\frac{C}{(1-r)^{\beta}}.
\end{aligned}
$$

Conversely, for fixed $N \in \mathbb{N}$, take $r = 1 - 1/N$ to obtain

$$
\begin{aligned}
CN^{\beta} &\geq \sum_{n=0}^{N} n^{\beta}\alpha_n(1-\frac{1}{N})^n \\
&\geq C(\sum_{n=0}^{N} n^{\beta}\alpha_n)(1-\frac{1}{N})^N \\
&\geq C(\sum_{n=0}^{N} n^{\beta}\alpha_n).
\end{aligned}
$$

Therefore $\sum_{n\in I_k} n^{\beta}\alpha_n \leq C2^{k\beta}$, which gives $\left(\alpha_n\right) \in \ell(1,\infty)$.

To prove (9) we have the case $q = 1$ trivialy, since

$$
\int_0^1 (1-r)^{\beta-1}(\sum_{n=1}^{\infty} n^{\beta}\alpha_n r^n)dr = \sum_{n=1}^{\infty}(\int_0^1 (1-r)^{\beta-1} r^n dr) n^{\beta}\alpha_n \approx \sum_{n=1}^{\infty} \alpha_n.
$$

Now define the operator T by $T((\alpha_n)_n) = (1-r)^{\beta}\sum_{n=1}^{\infty} n^{\beta}\alpha_n r^n$. We have just shown that $T : \ell(1,1) \to L^1((0,1), \frac{dr}{1-r})$ is bounded. Now (8) gives the boundedness of $T : \ell(1,\infty) \to L^{\infty}((0,1), \frac{dr}{1-r})$. Now we can use interpolation to obtain $T : \ell(1,q) \to L^q((0,1), \frac{dr}{1-r})$, which means

$$
\int_0^1 (1-r)^{\beta q-1}(\sum_{n=1}^{\infty} n^{\beta}\alpha_n r^n)^q dr \leq C\sum_{k=0}^{\infty}(\sum_{n\in I_k} \alpha_n)^q.
$$

The converse inequality is simpler,

$$
\begin{aligned}
\sum_{k=0}^{\infty}(\sum_{n\in I_k} \alpha_n)^q &\leq C\sum_{k=0}^{\infty}(\int_{1-2^{-k}}^{1-2^{-(k+1)}} 2^{k\beta q}(1-r)^{\beta q-1} r^{2^k} dr)(\sum_{n\in I_k} \alpha_n)^q \\
&\leq C\int_0^1 (1-r)^{\beta q-1}(\sum_{k=0}^{\infty}(\sum_{n\in I_k} n^{\beta}\alpha_n r^n)^q)dr.
\end{aligned}
$$

∎

Theorem 4.4 *Let $1 \leq p < \infty$. There exist $C_1, C_2 > 0$ such that*

$$C_1 \Big\|(\frac{x_n}{n^{1/p}})\Big\|_{\infty,p} \leq ||f||_{B_p(\mathbb{D},X)} \leq C_2 \Big\|(\frac{x_n}{n^{1/p}})\Big\|_{1,p}$$

for any $f \in B_p(\mathbb{D}, X)$ with Taylor coefficients (x_n).

Proof: Since $||f||_{B_p(\mathbb{D},X)} \leq (\int_0^1 M_\infty^p(f,r)dr)^{1/p}$ and $M_\infty(f,r) \leq \sum_{n=0}^\infty ||x_n|| r^n$ then (10) implies

$$||f||_{B_p(\mathbb{D},X)} \leq (\int_0^1 (\sum_{n=0}^\infty ||x_n|| r^n)^p dr)^{1/p} \leq C \Big\|(\frac{x_n}{n^{1/p}})\Big\|_{1,p}.$$

For the other inequatily, we observe that

$$\begin{aligned}
\int_0^1 M_p^p(f,r)dr &\geq \sum_{k=0}^\infty \int_{1-2^{-k}}^{1-2^{-(k+1)}} M_p^p(f,r)dr \\
&\geq \sum_{k=0}^\infty \int_{1-2^{-k}}^{1-2^{-(k+1)}} (\sup_{n\in I_k} r^{np} \|x_n\|^p)dr \\
&\geq \sum_{k=0}^\infty (1-2^{-k})^{p2^{k+1}} 2^{-(k+1)} \sup_{n\in I_k} \|x_n\|^p.
\end{aligned}$$

Therefore

$$||f||^p_{B_p(\mathbb{D},X)} \geq \sum_{k=0}^\infty 2^{-k} \sup_{n\in I_k} \|x_n\|^p \sim \sum_{k=0}^\infty \Big(\sup_{n\in I_k} \frac{\|x_n\|^p}{n}\Big) \sim ||(\frac{x_n}{n^{1/p}})||^p_{\infty,p}.$$

■

Corollary 4.5 *Let $1 \leq p < \infty$ and $f(z) = \sum_{n=0}^\infty x_n z^{2^n}$. Then*

$$||f||_{B_p(\mathbb{D},X)} \approx (\sum_{n=0}^\infty ||x_n||^p 2^{-n})^{1/p}.$$

Theorem 4.6 *Let X be a Hilbert space and $f(z) = \sum_{n=0}^\infty x_n z^n \in B^2(\mathbb{D}, X)$. Then*

$$||f||_{B_2(X)} = (\sum_{n=0}^\infty \frac{\|x_n\|^2}{n+1})^{1/2}. \tag{11}$$

PROOF: Since for Hilbert-valued functions we have Plancherel's theorem at our disposal we get

$$||f||^2_{B_2(X)} = 2\int_0^1 \sum_{n=0}^{\infty} \|x_n\|^2 r^{2n} r dr = \sum_{n=0}^{\infty} \frac{\|x_n\|^2}{n+1}.$$

Proposition 4.7 *Let $1 \leq p \leq 2 \leq q < \infty$ and let X be a Hilbert space. Then there exist positive constants C_p, C'_p and C''_q such that*

$$||(\frac{x_n}{n^{1/p}})||_{p',p} \leq C_p ||f||_{B_p(\mathbb{D},X)}, \tag{12}$$

$$(\sum_{n=1}^{\infty} \frac{\|x_n\|^p}{n^{3-p}})^{1/p} \leq C'_p ||f||_{B_p(\mathbb{D},X)} \tag{13}$$

and

$$||f||_{B_q(X)} \leq C''_q ||(\frac{x_n}{n^{1/q}})||_{q',q} \tag{14}$$

for all finite sequences $(x_n) \in X$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

PROOF: (12) follows by interpolation between Theorem 4.4 for $p = 1$ and (11).

Let us see that (13) actually follows from (12).

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{||x_n||^p}{n^{3-p}} &\leq C \sum_{k=0}^{\infty} (\sum_{n\in I_k} ||x_n||^p) 2^{-k(3-p)} \\
&\leq C \sum_{k=0}^{\infty} (\sum_{n\in I_k} ||x_n||^{p'})^{p/p'} 2^{k(1-p/p')} 2^{-k(3-p)} \\
&= C \sum_{k=0}^{\infty} (\sum_{n\in I_k} ||x_n||^{p'})^{p/p'} 2^{-k} \\
&\approx ||(\frac{x_n}{n^{1/p}})||^p_{p',p}.
\end{aligned}$$

(14) is the dual estimate of (12). Indeed, if $f(z) = \sum_{n=0}^{\infty} x_n z^n$ then

$$\begin{aligned}
||f||_{B_q(X)} &= \sup\{|\int_D \langle f(w), g(\bar{w})\rangle dA(w)| : g \in \mathcal{P}(X^*), \|g\|_{B_{q'}(X^*)} = 1\} \\
&= \sup\{|\sum_{n=0}^{\infty} \frac{\langle x_n, x_n^*\rangle}{n+1}| : g(z) = \sum x_n^* z^n \in \mathcal{P}(X^*), \|g\|_{B_{q'}(X^*)} = 1\} \\
&\leq \sup\{\sum_{n=0}^{\infty} \frac{\|x_n\|\|x_n^*\|}{n+1} : g(z) = \sum x_n^* z^n \in \mathcal{P}(X^*), \|g\|_{B_{q'}(X^*)} = 1\}
\end{aligned}$$

$$\leq \sup\{\sum_{k\geq 0}\sum_{n\in I_k}\frac{\|x_n\|}{(n+1)^{1/q}}\frac{\|x_n^*\|}{(n+1)^{1/q'}} : \|g\|_{B_{q'}(X^*)}=1\}$$

$$\leq \sup\{\sum_{k\geq 0}(\sum_{n\in I_k}\frac{\|x_n\|^{q'}}{(n+1)^{q'/q}})^{1/q'}(\sum_{n\in I_k}\frac{\|x_n^*\|^{q}}{(n+1)^{q/q'}})^{1/q} : \|g\|_{B_{q'}(X^*)}=1\}$$

$$\leq C_q\Big(\sum_{k\geq 0}(\sum_{n\in I_k}\frac{\|x_n\|^{q'}}{(n+1)^{q'/q}})^{q/q'}\Big)^{1/q}.$$

■

The previous estimates do not hold for general Banach spaces.

Proposition 4.8 *Let $1 < p \leq 2 \leq q < \infty$.*

(i) For any $r > p'$ there exists $f \in B_p(\mathbb{D}, \ell^r)$ such that $\sum_{n=1}^\infty \frac{\|x_n\|^p}{n^{3-p}} = \infty$ where $f(z) = \sum_{n=0}^\infty x_n z^n$.

(ii) For any $s < q'$ there exists $(x_n) \in \ell(q', q, \ell^s)$ such that if $f(z) = \sum_{n=0}^\infty x_n z^n$ then $f \notin B_q(\mathbb{D}, \ell^s)$.

Proof: Let $\beta > 1, \alpha > 0$ and consider $X = \ell^\beta$ and $F_{\alpha,\beta}(z) = \sum_{n=1}^\infty x_n z^n$ with $x_n = n^\alpha e_n$ where (e_n) stands for the canonical basis of ℓ^β.

Hence $\|x_n\| = n^\alpha$ and $\|F_{\alpha,\beta}(z)\| = (\sum_{n=1}^\infty n^{\alpha\beta}|z|^{\beta n})^{1/\beta} \approx C(1-|z|)^{-\alpha-\frac{1}{\beta}}$. Then

$$(\frac{x_n}{n^{1/p}}) \in \ell(p', p, \ell^\beta) \text{ if and only if } \alpha < 1/p - 1/p' \tag{15}$$

and

$$F_{\alpha,\beta} \in B_p(\mathbb{D}, \ell^\beta) \text{ if and only if } \alpha < 1/p - 1/\beta. \tag{16}$$

Assume $1 < p \leq 2$ and take $\alpha = 1/p - 1/p'$ and $f = F_{\alpha,\beta}$. Now (16) implies that $f \in B_p(\mathbb{D}, \ell^r)$ but $\sum_{n=1}^\infty \frac{\|x_n\|^p}{n^{3-p}} = \sum_{n=1}^\infty \frac{1}{n} = \infty$.

Assume $2 \leq q < \infty$, $s < q'$ and take $\alpha = 1/q - 1/s$ and $f = F_{\alpha,\beta}$. Now (15) and (16) give that $(\frac{x_n}{n^{1/q}}) \in \ell(q', q, \ell^s)$ but $f \notin B_q(\mathbb{D}, \ell^s)$. ■

The previous proposition leads to the following definitions:

Definition 4.9 *Let X be a complex Banach space and $1 \leq a \leq 2$. X is said to have Bergman type a if there exists a constant $C_a > 0$ such that*

$$||(\frac{x_n}{n^{1/a}})||_{a',a} \leq C_a ||f||_{B_a(X)} \tag{17}$$

for all $f(z) = \sum_{n=0}^\infty x_n z^n \in B_a(\mathbb{D}, X)$,

Remark 4.1 *Any Banach space has Bergman type* 1 *(see Theorem 4.4).*

Definition 4.10 *Let X be a complex Banach space and $2 \leq b \leq \infty$. X is said to have Bergman cotype b if there exists a constant $C_b > 0$ such that*

$$||f||_{B_b(X)} \leq C_b||(\frac{x_n}{n^{1/b}})||_{b',b} \tag{18}$$

for all $(x_n) \in \ell(b', b, X)$ where $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

Let us show that they are dual notions.

Proposition 4.11 *Let $1 < a \leq 2$ and X be a Banach space. Then X has Bergman type a if and only if X^* has Bergman cotype a'.*

Proof: Observe first that

$$\sum_{n=0}^{\infty} \frac{\langle x_n^*, x_n \rangle}{n+1} = \int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z), \tag{19}$$

for any $g(z) = \sum_{n=0}^{\infty} x_n^* z^n$, $f(z) = \sum_{n=0}^{\infty} x_n z^n$ and any of those being a polynomial.

Let us assume X has Bergman type a, take $(x_n^*) \in \ell(a, a', X^*)$ and define $f(z) = \sum_{n=0}^{\infty} x_n^* z^n$.

Then, for any X-valued polynomial $g(z) = \sum_{n=0}^{\infty} x_n z^n$, using (19) one gets

$$\begin{aligned} |\int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z)| &\leq ||(\frac{x_n}{(n+1)^{1/a'}})||_{a',a}||(\frac{x_n^*}{(n+1)^{1/a}})||_{a,a'} \\ &\leq C||g||_{B_a(X)}||(\frac{x_n^*}{n^{1/a}})||_{a,a'}. \end{aligned}$$

By the duality $(B_a(X))^* = B_{a'}(X^*)$ one has

$$||f||_{B_{a'}(X^*)} = sup\{|\int_{\mathbb{D}} \langle f(z), g(\bar{z}) \rangle dA(z)| : g \in \mathcal{P}(X), ||g||_{B_a(X)} = 1\}.$$

Therefore $||f||_{B_{a'}(X^*)} \leq C||(\frac{x_n^*}{n^{1/a}})||_{a,a'}$.

Let us assume X^* has Bergman cotype a' and take $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

For any $(x_n^*)_n$ such that $||(x_n^*)||_{a,a'} = 1$, using (19), for any $N \in \mathbb{N}$ we have

$$\begin{aligned} |\sum_{n=0}^{N} \langle n^{-1/a} x_n, x_n^* \rangle| &\leq ||f||_{B_a(X)} || \sum_{n=0}^{\infty} (n+1) n^{-1/a} x_n z^n ||_{B_{a'}(X^*)} \\ &\leq C ||f||_{B_a(X)} || (\frac{n+1}{n} x_n^*) ||_{a,a'} \\ &\leq C ||f||_{B_a(X)}. \end{aligned}$$

Now use

$$\left\|(\frac{x_n}{n^{1/a}})\right\|_{a',a} = \sup\{|\sum_{n=0}^{N} \langle n^{-1/a} x_n, x_n^* \rangle| : \left\|(x_n^*)\right\|_{a,a'} = 1, N \in \mathbb{N}\},$$

to get that X has Bergman type a. ∎

Let $1 \leq p \leq 2$. A Banach space X is said to have Fourier type p if there exists a constant C such that

$$(\sum_{n=-\infty}^{\infty} ||\hat{f}(n)||^{p'})^{1/p'} \leq C ||f||_{L^p(X)} \tag{20}$$

for all function $f \in L^p(\mathbb{T}, X)$.

It was first introduced by J. Peetre (see [32]). We refer the reader to the survey [23] for a complete study and references about this property.

We just point out here the equivalent formulation:

There exists a constant $C > 0$ such that for all $(x_n) \in \ell_p(X)$ the function $f(t) = \sum_{n=-\infty}^{\infty} x_n e^{int}$ belongs to $L^{p'}(\mathbb{T}, X)$ and

$$||f||_{L^{p'}(X)} \leq C ||(x_n)||_p. \tag{21}$$

It is not difficult to see that X has Fourier type p if and only if X^* does have it. The main examples are $L^r(\mu)$ for any $p \leq r \leq p'$ or interpolation spaces between any Banach space X_0 and any Hilbert space X_1, $[X_0, X_1]_\theta$ where $1/p = 1 - \theta/2$.

Theorem 4.12 *Let $1 < p \leq 2$. If X has Fourier type p then X has Bergman type p and Bergman cotype p'.*

PROOF: Using (10) we have

$$\sum_{k=1}^{\infty}(\sum_{n\in I_k}\frac{\|x_n\|^{p'}}{n^{p'/p}})^{p/p'}\approx\int_0^1(\sum_{n=1}^{\infty}\|x_n\|^{p'}r^{np'})^{p/p'}dr.$$

Since X has Fourier type p

$$\Big\|(\frac{x_n}{n^{1/p}})\Big\|_{p',p}^p\leq C\int_0^1\Big(\sum_{n=1}^{\infty}\|x_n\|^{p'}r^{p'n}\Big)^{p/p'}dr\leq C\int_0^1 M_p^p(f,r)dr\,.$$

To get that X has Bergman cotype p', one can either use the dual formulation of Bergman type of X^* or repeat the previous argument using now $M_{p'}(f,r)\leq C\Big(\sum_{n=1}^{\infty}\|x_n\|^p r^{np}\Big)^{1/p}$. ∎

Corollary 4.13 *Let X_0 be a complex Banach space, X_1 be a Hilbert space and let $0<\theta<1$. Then $[X_0,X_1]_\theta$ has Bergman type $p=\frac{2}{2-\theta}$ adn Bergbman cotype $p'=\frac{2}{\theta}$.*

In particular, for any σ-finite measure μ we have $L^p(\mu)$ has Bergman type $\min\{p,p'\}$ *and Bergman cotype* $\max\{p,p'\}$.

5 Vector-valued multipliers on vector-valued Bergman spaces

Theorem 5.1 *Let X and Y two complex Banach spaces. A sequence $(T_n)\in(B_1(X),\ell_1(Y))$ if and only if the sequence (nT_n) defines a bounded operator from X to $\ell(1,\infty,Y)$.*

PROOF: Assume that $(T_n)\in(B_1(X),\ell_1(Y))$. In particular $(T_n(x))\in(B_1,\ell_1(Y))$ for all $x\in X$. Now for each $z\in\mathbb{D}$ we define $K_z(w)=\frac{1}{(1-wz)^3}$. Since $||K_z||_{B_1}\approx\frac{1}{1-|z|}$, the assumption implies that

$$\sum_{n=0}^{\infty}n^2||T_n(x)|||z|^n\leq\frac{C}{1-|z|}$$

for each $z\in\mathbb{D}$ and $||x||\leq 1$. Invoking now (8) one gets that $(n||T_n(x)||)\in\ell(1,\infty,Y)$, or equivalently $x\to(nT_n(x))$ is a bounded operator from X into $\ell(1,\infty,Y)$.

Let us assume now, that

$$\sup_{||x||\leq 1} \sup_{k\geq 0} \sum_{n\in I_k} n||T_n(x)|| = A < \infty.$$

Then, if $f(z) = \sum_{n=1}^{n} x_n z^n$, using Theorem 4.4 one gets

$$\begin{aligned}
\sum_{n=0}^{\infty} ||T_n(x_n)|| &= \sum_{k=0}^{\infty} \sum_{n\in I_k} ||T_n(\frac{x_n}{||x_n||})|| ||x_n|| \\
&\leq \sum_{k=0}^{\infty} (\sum_{n\in I_k} n||T_n(\frac{x_n}{||x_n||})||) sup_{n\in I_k} \frac{||x_n||}{n} \\
&\leq \sup_{k\geq 0} \sum_{n\in I_k} n||T_n(\frac{x_n}{||x_n||})|| \sum_{k=0}^{\infty} (sup_{n\in I_k} \frac{||x_n||}{n}) \\
&\leq A(\sum_{k=0}^{\infty} sup_{n\in I_k} \frac{||x_n||}{n}) \leq A||f||_{B_1(X)} \quad \blacksquare
\end{aligned}$$

Theorem 5.2 *Let $1 \leq p \leq 2$ and let X be a complex Banach space. The following statements are equivalent:*
(i) X has Bergman type p.
(ii) For any other Banach space Y,

$$\{(T_n) : (n^{1/p} T_n) \in \ell(p, p', \mathcal{L}(X, Y))\} \subset (B_p(X), \ell_1(Y)).$$

PROOF: Let us assume X has Bergman type p. Now (ii) follows from the embedding $B_p(X) \subset \{(x_n) : (n^{-1/p} x_n) \in \ell(p', p, X)\}$.

Assume (ii) for $Y = \mathbb{C}$. Then any (x_n^*) such that $||(n^{1/p} x_n^*)||_{p,p'} < \infty$ gives a multiplier in $(B_p(X), \ell_1)$. Therefore there exists $C > 0$ such that

$$\sum_{n=0}^{\infty} |\langle x_n^*, x_n\rangle| \leq C||f||_{B_p(X)} ||(n^{1/p} x_n^*)||_{p,p'}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n$.

From duality now one gets $||(n^{-1/p} x_n)||_{p',p} \leq C||f||_{B_p(X)}$. ∎

Theorem 5.3 *Let $2 \leq q < \infty$ and let X be a complex Banach space of Bergman cotype q. Then for any other Banach space Y,*

$$(B_q(X), \ell_1(Y)) \subset \{(T_n) : (n^{1/q} T_n) \in \ell(q, q', \mathcal{L}(X, Y))\}.$$

PROOF: The assumption means that $\{(x_n) : (\frac{x_n}{n^{1/q}}) \in \ell(q', q, X)\} \subset B_q(X)$. Therefore $(T_n) \in (B_q(X), \ell_1(Y))$ gives $(n^{1/q}T_n) \in (\ell(q', q, X), \ell_1(Y))$. Using now that $\|T_n\| \approx \|T_n(x'_n)\|$ for some $\|x'_n\| = 1$ we get

$$\|(n^{1/q}\|T_n\|)\|_{q,q'} \approx \|(n^{1/q}\|T_n(x'_n)\|)\|_{q,q'} \approx \sum_n \|T_n(n^{1/p}\lambda_n x'_n)\|$$

for some $(\lambda_n) \in l(q', q)$ of norm 1. Taking $x_n = n^{1/q}\lambda_n x'_n$ for $n \in \mathbb{N}$ we have $n^{-1/q}x_n \in l(q', q, X)$ with norm bounded by a constant. This finishes the proof. ∎

Recall the well known notion of Rademacher type (see [31]).

For $1 \leq p \leq 2$ a Banach space X is said to have Rademacher type p if there exists a constant C such that

$$\int_0^1 ||\sum_{j=1}^n x_j r_j(t)||dt \leq C(\sum_{j=1}^n \|x_j\|^p)^{1/p}$$

for any finite family $x_1, x_2, \dots x_n$ of vectors in X where r_j stand for the Rademacher functions on $[0, 1]$.

It is known and easy to see that Fourier type p implies Rademacher type p.

Theorem 5.4 *Let $1 \leq p < \infty$, $1 \leq a \leq 2$, and let X be a complex Banach space of Rademacher type a and Y be any Banach space. Then*

$$(B_p(X), \ell_1(Y)) \subset \{(T_n) : (n^{1/p}T_n) \in \ell(a', p', \mathcal{L}(X, Y))\}.$$

PROOF: Let $\widetilde{T}: B_p(X) \to \ell_1(Y)$ the bounded linear operator defined by (T_n) as a multiplier, i.e. $\widetilde{T}f = (T_n x_n)$ for every X-valued polynomial $f(z) = \sum_{n=0}^\infty x_n z^n$.

For any $t \in [0, 1]$, let f_t the polynomial given by $f_t(z) = \sum_{n=0}^\infty r_n(t)x_n z^n$, where (r_n) is the sequence of Rademacher functions.

It's clear that $\|\widetilde{T}f_t\| = \|\widetilde{T}f\|$ for every t, and then

$$\begin{aligned} \|(T_n x_n)\|_1^p &= \|\widetilde{T}f\|_1^p = \int_0^1 \|\widetilde{T}f_t\|^p dt \leq \|\widetilde{T}\|^p \int_0^1 \|f_t\|^p_{B_p(X)} dt \\ &= \frac{\|\widetilde{T}\|^p}{2\pi} \int_0^1 \int_{-\pi}^{\pi} \int_0^1 \|\sum_{n=0}^\infty r_n(t) x_n r^n e^{in\theta}\|^p dt d\theta dr\,. \end{aligned}$$

Since X is of type a, we have for every θ that

$$\int_0^1 \| \sum_{n=0}^{\infty} r_n(t) x_n r^n e^{in\theta} \|^p dt \leq C(\sum_{n=0}^{\infty} \|x_n\|^a r^{na})^{p/a} ,$$

and integrating this in $[-\pi, \pi]$ we get

$$\|(T_n x_n)\|_1^p \leq C \|\widetilde{T}\|^p \int_0^1 (\sum_{n=0}^{\infty} \|x_n\|^a r^{na})^{p/a} dr.$$

Now (9) yields

$$\|(T_n x_n)\|_q^p \leq C \int_0^1 (\sum_{n=0}^{\infty} \|x_n\|^a r^n)^{p/a} dr \leq C \sum_k \Big(\sum_{n \in I_k} \frac{\|x_n\|^a}{n^{a/p}} \Big)^{p/a} ,$$

which gives $\|(T_n x_n)\| \leq C \Big\| (\frac{x_n}{n^{1/p}}) \Big\|_{\ell(a,p,X)}$.

We have thus shown that $(n^{1/p} T_n) \in (\ell(a,p,X), \ell^1(Y))$, and a simple argument as above this gives $(n^{1/p} T_n) \in \ell(a', p', \mathcal{L}(X,Y))$. ∎

Theorems 5.2 and 5.4 give a characterization of the multipliers from $B_2(\mathbb{D}, X)$ to $\ell_1(Y)$ whenever X is a Hilbert space.

Corollary 5.5 *Let X be a Hilbert space. Then $(T_n) \in (B_2(X), \ell_1(Y))$ if and only if the sequence $(\sqrt{n}\, \|T_n\|) \in \ell_2$.*

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