Vector-valued functions integrable with respect to bilinear maps.

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Abstract

Let (Ω, Σ, μ) be a $\sigma-$finite measure space, $1 \leq p < \infty$, X be a Banach space X and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We say that an X-valued function f is $p-$integrable with respect to $\mathcal{B}$ whenever $\sup\{\int_\Omega \|\mathcal{B}(f(w), y)\|^p d\mu : \|y\| = 1\}$ is finite. We identify the spaces of functions integrable with respect to the bilinear maps arising from Hölder's and Young's inequalities. We apply the theory to give conditions on X-valued kernels for the boundedness of integral operators $T_\mathcal{B}(f)(w) = \int_{\Omega'} \mathcal{B}(k(w, w'), f(w'))d\mu'(w')$ from $\mathrm{L}^p(Y)$ into $\mathrm{L}^p(Z)$, extending the results known in the operator-valued case, corresponding to $\mathcal{B} : \mathrm{L}(X, Y) \times X \to Y$ given by $\mathcal{B}(T, x) = Tx$.

1 Introduction.

In this paper we shall consider spaces of X-valued functions which are integrable with respect to bilinear maps, that is to say functions f satisfying the condition $\mathcal{B}(f, y) \in \mathrm{L}^1(Z)$ for all $y \in Y$ for some bounded bilinear map $\mathcal{B} : X \times Y \to Z$. The motivation for our study comes from two different sources: On the one hand, the recent paper by M. Girardi and L. Weiss [9], where conditions on operator-valued kernels $K : \Omega \times \Omega' \to \mathrm{L}(X, Y)$ for the integral operator

$$T_K(f)(w) = \int_{\Omega'} K(w, w')(f(w'))d\mu'(w')$$

to be bounded from $\mathrm{L}^p(X)$ to $\mathrm{L}^p(Y)$ were given, and, on the other hand, the papers [2, 4, 5] where the notion of convolution by means of bilinear maps was introduced and applied in different contexts.

Operator-valued multipliers and operator-valued singular integrals has been considered by different authors. An introduction to the general theory and its applications can be found in [1, 8]. We shall deal here with more general bilinear maps in our study and present a basic introduction to the spaces which can be defined with this notion of integrability. These will allow, among other things, to get that the conditions appearing on the kernels for the boundedness of integral operators can be understood as certain integrability conditions with respect to the corresponding bilinear maps. This approach also shows that between the class of Pettis integrable functions and the Bochner integrable ones, there are many others,

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corresponding to integrable with respect to other bilinear maps. These classes are the natural ones where the results on convolution by means of bilinear maps obtained in [2, 4, 5] still hold true.

The paper is organized as follows: First we introduce the spaces, consider the basic properties on the triples $(Y, Z, \mathcal{B})$ formed by two Banach spaces Y and Z and a bounded bilinear map $\mathcal{B} : X \times Y \to Z$ which play some important role in the development of the theory and present the examples of natural triples that naturally appear for any Banach space X. Next we identify the spaces of $p-$integrable functions with respect to concrete examples of bilinear map arising from on Hölder's and Young's inequalities. The last section concludes with the analogues of the results in [9] in our more general situation.

Throughout the paper $1 \le p < \infty$, (Ω, Σ, μ) stands for a $\sigma-$finite complete measure space and X denotes a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Recall that an X-valued function $f : \Omega \to X$ is said to be strongly measurable if there exists a sequence of simple functions, $(s_n)_n \subseteq \mathcal{S}(X)$, which converges to f a.e. and to be weakly measurable if $\langle f, x^*\rangle$ is measurable for any $x^* \in X^*$. In the case of dual spaces X^* a function is called weak*-measurable if $\langle x, f\rangle$ is measurable for any $x \in X$. We denote by $\mathrm{L}^0(X)$, $\mathrm{L}^0_{\text{weak}}(X)$ and $\mathrm{L}^0_{\text{weak}^*}(X^*)$ the spaces of strongly, weakly measurable and weak*-measurable functions. We write $\mathrm{L}^p(X)$, $\mathrm{L}^p_{\text{weak}}(X)$ and $\mathrm{L}^p_{\text{weak}^*}(X^*)$ for the space of functions in $\mathrm{L}^0(X)$, $\mathrm{L}^0_{\text{weak}}(X)$ and $\mathrm{L}^0_{\text{weak}^*}(X^*)$ such that $\|f\| \in \mathrm{L}^p(\mu)$, $\langle f, x^*\rangle \in \mathrm{L}^p(\mu)$ for $x^* \in X^*$ and $\langle x, f\rangle \in \mathrm{L}^p(\mu)$ for $x \in X$ respectively. Finally we use the notation $P^p(X)$ for the space of Pettis $p-$integrable functions $P^p(X) = \mathrm{L}^p_{\text{weak}}(X) \cap \mathrm{L}^0(X)$.

2 Integrability with respect to bilinear maps.

Definition 1 *Let Y and Z be Banach spaces and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. We say that $f : \Omega \to X$ is $(Y, Z, \mathcal{B})$-measurable if $\mathcal{B}(f, y) \in \mathrm{L}^0(Z)$ for any $y \in Y$. We shall denote the class of such functions by $\mathrm{L}^0_{\mathcal{B}}(X)$.*

Given a Banach space X there are many standard ways to find triples Y, Z and $\mathcal{B}$ where $\mathcal{B} : X \times Y \to Z$ becomes a bounded bilinear map.

The basic ones are:

$$\mathcal{B}_X = \mathcal{B} : X \times \mathbb{K} \to X, \qquad \mathcal{B}(x, \lambda) = \lambda x. \tag{1}$$

$$\mathcal{D}_X = \mathcal{D} : X \times X^* \to \mathbb{K}, \qquad \mathcal{D}(x, x^*) = \langle x, x^*\rangle. \tag{2}$$

Note that $\mathrm{L}^0_{\mathcal{B}}(X) = \mathrm{L}^0(X)$ and $\mathrm{L}^0_{\mathcal{D}}(X) = L^0_{\text{weak}}(X)$.
Natural generalizations of (1) and (2) are the following: For any other Banach space Y one has

$$\pi_Y : X \times Y \to X \hat{\otimes} Y, \qquad \pi_Y(x, y) = x \otimes y. \tag{3}$$

$$\widetilde{\mathcal{O}}_Y : X \times \mathrm{L}(X, Y) \to Y, \qquad \widetilde{\mathcal{O}}_Y(x, T) = T(x). \tag{4}$$

In the case of dual spaces X^* we have also

$$\mathcal{D}_{1,X} = \mathcal{D}_1 : X^* \times X \to \mathbb{K}, \qquad \mathcal{D}_1(x^*, x) = \langle x, x^*\rangle. \tag{5}$$

Note that $\mathrm{L}^0_{\mathcal{D}_1}(X^*) = \mathrm{L}^0_{\text{weak}*}(X)$.
Observe that (5) is a particular case of $X = \mathrm{L}(Y, Z)$, which plays an important role in what follows. In this case we denote by

$$\mathcal{O}_{Y,Z} : \mathrm{L}(Y, Z) \times Y \to Z, \qquad \mathcal{O}_{Y,Z}(T, y) = T(y). \tag{6}$$

If $X = \mathrm{L}(E, E)$ one can also consider,

$$\mathcal{C}_E : \mathrm{L}(E, E) \times \mathrm{L}(E, E) \to \mathrm{L}(E, E), \qquad \mathcal{C}_E(T, S) = TS. \tag{7}$$

Actually (7) is just the product on a Banach algebra A:

$$\mathcal{P}r : A \times A \to A, \qquad \mathcal{P}r(a, b) = ab. \tag{8}$$

Given a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, we can define the "adjoint" $\mathcal{B}^* : X \times Z^* \to Y^*$ by the formula

$$\langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle, \qquad \text{for every } x \in X,\ y \in Y \text{ and } z^* \in X^*.$$

Note that

$$\mathcal{B}^* = \mathcal{D},\ (\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*} \text{ and } (\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*).$$

Definition 2 *We write $\mathcal{L}^p_{\mathcal{B}}(X)$ for the space of functions f in $\mathrm{L}^0_{\mathcal{B}}(X)$ such that*

$$\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{\mathrm{L}^p(Z)} : \|y\| = 1\} < \infty.$$

Clearly for $f, g \in \mathcal{L}^p_{\mathcal{B}}(X)$ and $\lambda \in \mathbb{K}$ we have that

(1) $\|f + g\|_{\mathcal{L}^p_{\mathcal{B}}(X)} \le \|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} + \|g\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$,

(2) $\|\lambda f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = |\lambda|\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$,

(3) If $f \equiv 0$ then $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = 0$,

but in general the $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = 0$ does not imply $f = 0$ a.e. (It suffices to take $\mathcal{B}$ such that there exists $x \neq 0$ for which $\mathcal{B}(x, y) = 0$ for all $y \in Y$, and select $f = x\mathbf{1}_\Omega$).
Observe that $\mathrm{L}^p(X) \subset \mathcal{L}^p_{\mathcal{B}}(X)$ for any bounded bilinear map $\mathcal{B}$. Also one has

$$\mathcal{L}^p_{\mathcal{B}}(X) = \mathrm{L}^p(X), \qquad \mathcal{L}^p_{\mathcal{D}}(X) = \mathrm{L}^p_{\text{weak}}(X), \qquad \mathcal{L}^p_{\mathcal{D}_1}(X^*) = \mathrm{L}^p_{\text{weak}*}(X^*).$$

Remark 1 *Observe that simple functions, say $s = \sum_{k=1}^n x_k \mathbf{1}_{A_k}$, $x_k \in X$, and pairwise disjoint sets A_k, belong to $\mathcal{L}^p_{\mathcal{B}}(X)$. Actually*

$$\|s\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{(\sum_{k=1}^n \|\mathcal{B}(x_k, y)\|^p \mu(A_k))^{\frac{1}{p}} : \|y\| = 1\}$$

A simple duality argument gives

$$\|s\|_{\mathcal{L}^p_{\mathcal{B}}(X)} = \sup\{\|\sum_{k=1}^n \mathcal{B}^*(x_k, z_k^*)\mu(A_k)^{\frac{1}{p}}\| : (\sum_{k=1}^n \|z_k^*\|^{p'})^{\frac{1}{p'}} = 1\}.$$

Definition 3 *A function $f \in \mathcal{L}^p_{\mathcal{B}}(X)$ is said to belong to $\mathrm{L}^p_{\mathcal{B}}(X)$ if there exists a sequence of simple functions $(s_n)_n \in \mathcal{S}(X)$ such that*

(1) $(s_n)_n$ converges to f a.e.,

(2) $(s_n)_n$ converges to f in the norm $\|\cdot\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$.

For $f \in \mathrm{L}^p_{\mathcal{B}}(X)$ we write $\|f\|_{\mathrm{L}^p_{\mathcal{B}}(X)}$ instead of $\|f\|_{\mathcal{L}^p_{\mathcal{B}}(X)}$. Clearly one has that

$$\|f\|_{\mathrm{L}^p_{\mathcal{B}}(X)} = \lim_{n\to\infty} \|s_n\|_{\mathrm{L}^p_{\mathcal{B}}(X)}.$$

Remark 2 *Let $\Omega = [0, 1]$ with the Lebesgue measure. Let $f = \sum_{k=1}^\infty 2^k x_k \mathbf{1}_{I_k}$ where $x_k \in X$ and $I_k = (2^{-k}, 2^{-k+1}]$ for $k \in \mathbb{N}$. It is elementary to see that*

$$f \in L^p_{\mathcal{B}}(X) \text{ if and only if } \sup_{\|y\|=1} \sum_{k=1}^\infty \|\mathcal{B}(x_k, y)\|^p < \infty.$$

From this it follows that if $\lim_{N\to\infty} \sup_{\|y\|=1} \sum_{k=N}^\infty \|\mathcal{B}(x_k, y)\|^p = 0$ then $f \in \mathrm{L}^p_{\mathcal{B}}(X)$.

Remark 3 *Observe that* $\mathrm{L}^p(X) \subseteq \mathrm{L}^p_{\mathcal{B}}(X)$ *for any* $\mathcal{B}$ *and* $\mathcal{L}^p_{\mathcal{B}}(X) = \mathrm{L}^p_{\mathcal{B}}(X) = \mathrm{L}^p(X)$.
Also $\mathrm{L}^p_{\mathcal{D}}(X) = P^p(X)$ *(see [10], page 54 for the case* $p = 1$*), which shows that* $\mathrm{L}^p_{\mathcal{B}}(X) \subsetneq \mathcal{L}^p_{\mathcal{B}}(X)$ *(see [6] page 53, for the case* $\mathcal{B} = \mathcal{D}$*).*

As expected the bilinear map $\mathcal{B}$ defines the smallest space in the scale $\{\mathrm{L}^p_{\mathcal{B}}(X) : \mathcal{B}$ bilinear and bounded $\}$. One might expect the space of Pettis p-integrable functions, $\mathrm{L}^p_{\mathcal{D}}(X)$, to be the biggest in the scale. We shall now see that the inclusion $\mathrm{L}^p_{\mathcal{B}}(X) \subset P^p(X)$ holds true only among certain class of bilinear maps.

Given $x \in X$ and $y \in Y$ we shall be denoting by $\mathcal{B}_x \in \mathrm{L}(Y,Z)$ and $\mathcal{B}^y \in \mathrm{L}(X,Z)$ the corresponding linear operators

$$\mathcal{B}_x(y) = \mathcal{B}(x,y) \text{ and } \mathcal{B}^y(x) = \mathcal{B}(x,y).$$

Definition 4 *Let* Y *and* Z *be Banach spaces and* $\mathcal{B} : X \times Y \to Z$ *be a bounded bilinear map. We shall say that the triple* $(Y,Z,\mathcal{B})$ *is admissible for* X *if the map* $x \to \mathcal{B}_x$ *is injective from* $X \to \mathrm{L}(Y,Z)$*, i.e.* $\mathcal{B}(x,y) = 0$ *for all* $y \in Y$ *implies* $x = 0$.

Notice that if $(Y,Z,\mathcal{B})$ is admissible for X if and only if $(Z^*,Y^*,\mathcal{B}^*)$ is.

It is elementary to see that examples in (1)-(7) are admissible triples. In the example (8) the admissibility condition becomes "no zero divisors" and holds true for Banach algebras with identity or with bounded approximation of the identity.

Definition 5 *Let* Y *and* Z *be Banach spaces and let* $\mathcal{B} : X \times Y \to Z$ *be a bounded bilinear map.* X *is said to be* $(Y,Z,\mathcal{B})$*-normed (or normed by* $\mathcal{B}$*) if there exists* $C > 0$ *such that for all* $x \in X$

$$\|x\| \leq C\|\mathcal{B}_x\|.$$

This simply means X *can be understood as a subspace of* $\mathrm{L}(Y,Z)$ *and that* $|||x||| = \|\mathcal{B}_x\|$ *defines an equivalent norm on* X.

Remark 4 *Observe that*
(i) If X *is* $(Y,Z,\mathcal{B})$*-normed then* $(Y,Z,\mathcal{B})$ *is an admissible triple.*
(ii) X *is* $(Y,Z,\mathcal{B})$*-normed if and only if it is* $(Z^*,Y^*,\mathcal{B}^*)$*-normed.*

Remark 5 *Let* X *be* $(Y,Z,\mathcal{B})$*-normed and* $f \in \mathrm{L}^p_{\mathcal{B}}(X)$*. If we consider the function*

$$\widetilde{f} : \Omega \to \mathrm{L}(Y,Z), \qquad \widetilde{f}(w) = \mathcal{B}_{f(w)}.$$

then $\widetilde{f}$ *belongs to* $\mathrm{L}^p_{\mathcal{O}_{Y,Z}}(\mathrm{L}(Y,Z))$*. Moreover*

$$\|\widetilde{f}\|_{\mathrm{L}^p_{\mathcal{O}_{Y,Z}}(\mathrm{L}(Y,Z))} = \|f\|_{\mathrm{L}^p_{\mathcal{B}}(X)}.$$

Proposition 1 Let X, Y and Z be Banach spaces and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. The following are equivalent:

(1) X is $(Y,Z,\mathcal{B})$-normed.

(2) There exists a constant $k > 0$ such that for each $x^* \in X^*$ there exists a functional $\varphi_{x^*} \in \mathrm{L}(Y,Z)^*$ verifying $\|\varphi_{x^*}\| \leq k\|x^*\|$ and

$$\langle x, x^* \rangle = \varphi_{x^*}(\mathcal{B}_x) \text{ for all } x \in X.$$

PROOF. Assume that X is $(Y, Z, \mathcal{B})$-normed and denote by $\hat{X} = \{\mathcal{B}_x : x \in X\} \subseteq \mathrm{L}(Y, Z)$. By assumption $\hat{X}$ is a closed subspace of $\mathrm{L}(Y, Z)$. Given $x^* \in X^*$ the map $B_x \to \langle x, x^* \rangle$ defines bounded functional in $(\hat{X})^*$. Now, by the Hahn-Banach theorem there is an extension φ_{x^*} to $(\mathrm{L}(Y, Z))^*$ such that $\|\varphi_{x^*}\| = \|x^*\|$. The converse is immediate.

□

Of course, given a Banach space X there are many triples $(Y, Z, \mathcal{B})$ for which X is $(Y, Z, \mathcal{B})$-normed. In particular the ones considered in the examples (1)-(7).

However it is also easy to produce examples of admissible triples which are not $(Y, Z, \mathcal{B})$-normed:

Example 1 *Let $X = \ell_p$ for $1 \leq p < 2$, $Y = \ell_2$, $Z = \ell_1$ and $B : \ell_p \times \ell_2 \to \ell_1$ given by*

$$B((\alpha_n)_n, (\beta_n)_n) = (\alpha_n \beta_n)_n.$$

Then ℓ_p is not (Y, Z, B)-normed.

Theorem 1 Let X, Y and Z be Banach spaces and let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. The following are equivalent:

(1) X is $(Y, Z, \mathcal{B})$-normed.

(2) The inclusion $\mathrm{i} : \mathrm{L}^p_{\mathcal{B}}(X) \to P^p(X)$ is continuous for all $1 \leq p < \infty$.

(3) The inclusion $\mathrm{i} : \mathrm{L}^p_{\mathcal{B}}(X) \to P^p(X)$ is continuous for some $1 \leq p < \infty$.

PROOF.
(1)⇒(2) Let $1 \leq p < \infty$ and let $s = \sum_{k=1}^n x_k \mathbf{1}_{A_k} \in \mathcal{S}(X)$. Using that X is $(Y, Z, \mathcal{B})$-normed there exists a constant $C > 0$ such that $\|x\| \leq C \|\mathcal{B}_x\|$ for all $x \in X$. Let us write

$$\begin{aligned}
\|s\|_{P^p(X)} &= \sup\{(\sum_{k=1}^n |\langle x_k, x^* \rangle|^p \mu(A_k))^{\frac{1}{p}} : \|x^*\| = 1\} \\
&= \sup\{|\langle \sum_{k=1}^n x_k \mu(A_k)^{\frac{1}{p}} \alpha_k, x^* \rangle| : \|x^*\| = 1, \|\alpha\|_{\ell_{p'}} = 1\} \\
&\leq \sup\{\| \sum_{k=1}^n x_k \mu(A_k)^{\frac{1}{p}} \alpha_k \| \|x^*\| : \|x^*\| = 1, \|\alpha\|_{\ell_{p'}} = 1\} \\
&\leq C \sup\{\|\mathcal{B}_{\sum_{k=1}^n x_k \mu(A_k)^{\frac{1}{p}} \alpha_k}\| : \|\alpha\|_{\ell_{p'}} = 1\} \\
&= C \sup\{\|\mathcal{B}(\sum_{k=1}^n \alpha_k x_k \mu(A_k)^{\frac{1}{p}}, y)\| : \|\alpha\|_{\ell_{p'}} = 1, \|y\| = 1\} \\
&\leq C \sup\{\sum_{k=1}^n \|\mathcal{B}(x_k \mu(A_k)^{\frac{1}{p}}, y)\| \; |\alpha_k| : \|\alpha\|_{\ell_{p'}} = 1, \|y\| = 1\} \\
&= C \|s\|_{\mathrm{L}^p_{\mathcal{B}}(X)}.
\end{aligned}$$

Now if we take a function $f \in \mathrm{L}^p_{\mathcal{B}}(X)$ then there exists $(s_n)_n \in \mathcal{S}(X)$ convergent to f a.e and in the norm $\|\cdot\|_{\mathrm{L}^p_{\mathcal{B}}(X)}$. Since $(|\langle s_n, x^*\rangle|^p)_n$ converges to $(|\langle f, x^*\rangle|^p)$ a.e. , Fatou's Lemma implies that

$$\begin{aligned}
\|f\|^p_{P^p(X)} &= \sup\{\int_\Omega \lim_n |\langle s_n(w), x^*\rangle|^p d\mu : \|x^*\| = 1\} \\
&\leq \sup\{\liminf_n \int_\Omega |\langle s_n(w), x^*\rangle|^p d\mu : \|x^*\| = 1\} \\
&\leq \liminf_n \|s_n\|^p_{P^p(X)} \\
&\leq C^p \liminf_n \|s_n\|^p_{\mathrm{L}^p_{\mathcal{B}}(X)} \\
&\leq C^p \|f\|^p_{\mathrm{L}^p_{\mathcal{B}}(X)}.
\end{aligned}$$

(2)⇒(3) Obvious.
(3)⇒(1) Assume (3), fix $x \in X$ and consider the simple function

$$\begin{array}{rccl} f_x\colon & \Omega & \to & X \\ & w & \mapsto & x\mu(\Omega)^{-\frac{1}{p}} \mathbf{1}_\Omega(w) \end{array}$$

Since $\|f_x\|_{P^p(X)} = \|x\|$ and $\|f_x\|_{\mathrm{L}^p_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$ one gets (1).

□

Proposition 2 Let X be a $(Y, Z, \mathcal{B})$-normed space and $f \in \mathrm{L}^1_{\mathcal{B}}(X)$. For each $E \in \Sigma$ there exists a unique $x_E \in X$ such that for any $y \in Y$

$$\mathcal{B}(x_E, y) = \int_E \mathcal{B}(f(w), y) d\mu.$$

The value $x_E = (\mathcal{B}) \int_E f d\mu$ is called the $\mathcal{B}$-integral of f over E.

Proof. Note that the uniqueness follows from the bilinearity of $\mathcal{B}$ and the admissibility of the triple. To show the existence, observe that if $f \in \mathrm{L}^1(X)$ then x_E can be taken the Bochner integral of f over E, $\int_E f d\mu$, using that $\mathcal{B}^y \in \mathrm{L}(X, Z)$ and $B^y(x_E) = \int_E B^y(f) d\mu$ for any $y \in Y$.

Now, if $f \in \mathrm{L}^1_{\mathcal{B}}(X)$ and $(s_n)_n$ is the sequence of simple functions of the definition then we have

$$\int_E \mathcal{B}(f(w), y) d\mu = \lim_n \mathcal{B}(x_{n,E}, y),$$

for $E \in \Sigma$ and $y \in Y$ where $x_{n,E} = \int_E s_n d\mu$.

The fact that X is $(Y, Z, \mathcal{B})$-normed implies that there exists $\lim_n x_{n,E} \in X$, say x_E. Indeed,

$$\begin{aligned}
\|x_{n,E} - x_{m,E}\| &\leq C \sup\{\|\mathcal{B}_{x_{n,E} - x_{m,E}}(y)\| : \|y\| = 1\} \\
&\leq C \sup\{\|\mathcal{B}(s_n - s_m, y)\|_{\mathrm{L}^1(Z)} : \|y\| = 1\} \\
&\leq C \|s_n - s_m\|_{\mathrm{L}^1_{\mathcal{B}}(X)}.
\end{aligned}$$

Finally we have $\int_E \mathcal{B}(f(w), y) d\mu = \lim_n \mathcal{B}(x_{n,E}, y) = \mathcal{B}(\lim_n x_{n,E}, y) = \mathcal{B}(x_E, y)$.

□

Remark 6 *If X be (Y, Z, B)-normed space and $f \in \mathrm{L}^1_{\mathcal{B}}(X)$ then*

$$x_E = (\mathcal{B}) \int_E f d\mu = (P) \int_E f d\mu$$

for any $E \in \Sigma$ where $(P) \int_E f d\mu(w)$ stands for the Pettis integral over E.

We now will see more concrete examples of spaces and bilinear maps where the theory can give nice applications.

Example 2 (Hölder's bilinear map) *Let* (Ω_1, η) *be a* σ*-finite measure space, let* $1 \leq p_1, p_2, p_3 \leq \infty$ *and* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ *and consider*

$$\mathcal{H}_{p_1,p_2} : \mathrm{L}^{p_1}(\eta) \times \mathrm{L}^{p_2}(\eta) \to \mathrm{L}^{p_3}(\eta), \qquad (f,g) \to fg.$$

It is clear that $\mathrm{L}^{p_1}(\eta)$ is $(\mathrm{L}^{p_2}(\eta), \mathrm{L}^{p_3}(\eta), \mathcal{H}_{p_1,p_2})$-normed.

In particular for $\Omega_1 = \mathbb{N}$ with the counting measure, one has for $p = p_3$:

Proposition 3 Let $1 \leq p_1 < \infty$, $1 \leq p_2 \leq \infty$, $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\mathcal{H}_{p_1,p_2} : \ell_{p_1} \times \ell_{p_2} \to \ell_{p_3}$. If $f = (f_n) \in \mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})$ then

$$\|f\|_{\mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})} = \|(f_n)\|_{\ell_{p_1}(\mathrm{L}^{p_3})}.$$

Proof. Note that

$$\begin{aligned}
\|f\|_{\mathcal{L}^{p_3}_{\mathcal{H}_{p_1,p_2}}(\ell_{p_1})} &= \sup\{(\int_\Omega \|(f_n(w)\beta_n)_n\|^{p_3}_{\ell_{p_3}} d\mu)^{\frac{1}{p_3}} : \|(\beta_n)_n\|_{\ell_{p_2}} = 1\} \\
&= \sup\{(\sum_{n=1}^{\infty}(\|f_n\|_{\mathrm{L}^{p_3}(\mu)}|\beta_n|)^{p_3})^{\frac{1}{p_3}} : \|(\beta_n)_n\|_{\ell_{p_2}} = 1\} \\
&= \|(\|f_n\|_{\mathrm{L}^{p_3}})_n\|_{\ell_{p_1}} = \|(f_n)_n\|_{\ell_{p_1}(\mathrm{L}^{p_3}(\mu))}
\end{aligned}$$

□

Example 3 (Young's bilinear map) *Let* G *be locally compact abelian group,* $1 \leq p_1, p_2 \leq \infty$ *and* $1/p_1 + 1/p_2 \geq 1$*. Let* $1 \leq p_3 \leq \infty$ *with* $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ *and consider*

$$\mathcal{Y}_{p_1,p_2} : \mathrm{L}^{p_1}(G) \times \mathrm{L}^{p_2}(G) \to \mathrm{L}^{p_3}(G), \qquad (f,g) \to f * g.$$

Proposition 4

(1) $\mathrm{L}^p(\mathbb{R})$ is $(\mathrm{L}^1(\mathbb{R}), \mathrm{L}^p(\mathbb{R}), \mathcal{Y}_{p,1})$-normed for any $1 \leq p < \infty$.

(2) $(\mathrm{L}^2(\mathbb{R}), \mathrm{L}^2(\mathbb{R}), \mathcal{Y}_{1,2})$ is an admissible triple for $\mathrm{L}^1(\mathbb{R})$ but $\mathrm{L}^1(\mathbb{R})$ is not $(\mathrm{L}^2(\mathbb{R}), \mathrm{L}^2(\mathbb{R}), \mathcal{Y}_{1,2})$-normed.

Proof.

(1) Since $\mathrm{L}^1(\mathbb{R})$ has a bounded approximation of the identity then

$$\|f\|_p = \sup\{\|f * g\|_p : \|g\|_1 = 1\} = \sup\{\|\mathcal{Y}_{p,1}(f,g)\|_p : \|g\|_1 = 1\}.$$

(2) Note that

$$\sup\{\|f * g\|_2 : \|g\|_2 = 1\} = \sup\{\|\mathcal{Y}_{1,2}(f,g)\|_p : \|g\|_2 = 1\} = \|\hat{f}\|_\infty$$

which is not equivalent to $\|f\|_1$.

□

In particular for $G = \mathbb{R}$ with the Lebesgue measure, the norm in the spaces $\mathcal{L}^p_{\mathcal{Y}_{p_1,p_2}}(\mathrm{L}^{p_1})$ can be easily described in some cases.

Proposition 5 Let $1 \leq p_1 < \infty$.

(1) $\mathcal{L}^p_{\mathcal{Y}_{p_1,1}}(\mathrm{L}^{p_1}(\mathbb{R})) = \mathrm{L}^p(\mathrm{L}^{p_1}(\mathbb{R}))$ for any $1 \le p < \infty$.
Moreover $\|f\|_{\mathrm{L}^p_{\mathcal{Y}_{p_1,1}}(\mathrm{L}^{p_1}(\mathbb{R}))} = \|f\|_{\mathrm{L}^p(\mathrm{L}^{p_1}(\mathbb{R}))}$.

(2) If $f \in \mathrm{L}^0(\mathrm{L}^1(\mathbb{R}))$ then

$$\|f\|_{\mathcal{L}^2_{\mathcal{Y}_{1,2}}(\mathrm{L}^1(\mathbb{R}))} = \sup_{x\in\mathbb{R}}(\int_\Omega |\hat{f}_w(x)|^2 d\mu)^{\frac{1}{2}}.$$

PROOF.

(1) Assume $f \in \mathcal{L}^0_{\mathcal{Y}_{p_1,1}}(\mathrm{L}^{p_1}(\mathbb{R}))$ then, Proposition 4 and Theorem 1 give that f is weakly measurable and, due to the separability of $\mathrm{L}^{p_1}(\mathbb{R})$, we conclude that $f \in \mathrm{L}^0(\mathrm{L}^{p_1}(\mathbb{R}))$. Assuming that $f : \Omega \to \mathrm{L}^{p_1}(\mathbb{R})$ is given by $w \mapsto f_w$ and taking a bounded approximation of the identity in $\mathrm{L}^1(\mathbb{R})$, say g_n, one has

$$\begin{aligned}
\|f\|_{\mathrm{L}^p(\mathrm{L}^{p_1}(\mathbb{R}))} &= (\int_\Omega \|f_w\|^p_{\mathrm{L}^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} \\
&= (\int_\Omega \lim_{n\to\infty} \|f_w * g_n\|^p_{\mathrm{L}^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} \\
&\le \sup\{(\int_\Omega \|f_w * g\|^p_{\mathrm{L}^{p_1}(\mathbb{R})} d\mu)^{\frac{1}{p}} : \|g\|_{\mathrm{L}^1(\mathbb{R})} = 1\} \\
&= \|f\|_{\mathcal{L}^p_{\mathcal{Y}_{p_1,1}}(\mathrm{L}^{p_1}(\mathbb{R}))}
\end{aligned}$$

The other inclusion and inequality of norms are always true.

(2) Now if $f \in \mathrm{L}^0(\mathrm{L}^1(\mathbb{R}))$ then $f : \Omega \to \mathrm{L}^1(\mathbb{R})$ given by $w \mapsto f_w$ and we have (using Plancherel's identity and Fubini's theorem) that

$$\begin{aligned}
\|f\|_{\mathcal{L}^2_{\mathcal{Y}_{1,2}}(\mathrm{L}^1(\mathbb{R}))} &= \sup\{(\int_\Omega \|f_w * g\|^2_{\mathrm{L}^2(\mathbb{R})} d\mu)^{\frac{1}{2}} \ : \ \|g\|_{\mathrm{L}^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\Omega \|\widehat{f_w * g}\|^2_{\mathrm{L}^2(\mathbb{R})} d\mu)^{\frac{1}{2}} \ : \ \|g\|_{\mathrm{L}^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\Omega \int_\mathbb{R} |\widehat{f_w}(x)\widehat{g}(x)|^2 dx d\mu)^{\frac{1}{2}} \ : \ \|\widehat{g}\|_{\mathrm{L}^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\mathbb{R} (\int_\Omega |\widehat{f_w}(x)|^2 d\mu)|\widehat{g}(x)|^2 dx)^{\frac{1}{2}} \ : \ \|\widehat{g}\|_{\mathrm{L}^2(\mathbb{R})} = 1\} \\
&= \sup\{(\int_\mathbb{R} (\int_\Omega |\widehat{f_w}(x)|^2 d\mu)\ |h(x)| dx)^{\frac{1}{2}} \ : \ \|h\|_{\mathrm{L}^1(\mathbb{R})} = 1\} \\
&= \sup_{x\in\mathbb{R}}(\int_\Omega |\hat{f}_w(x)|^2 d\mu)^{\frac{1}{2}}
\end{aligned}$$

□

3 Integral operators by means of bilinear maps.

Throughout this section $(\Omega, \Sigma, d\mu(w))$ and $(\Omega', \Sigma', d\mu'(w'))$ are σ-finite complete measure spaces, X is a Banach space and $k : \Omega \times \Omega' \to X$ belong to $\mathrm{L}^0_{\mathcal{B}}(\Omega \times \Omega', X)$ for some admissible triple $(Y, Z, \mathcal{B})$ for X. Our objective is to study the boundedness of the integral operator associated to $\mathcal{B}$ given by

$$\begin{aligned}
T^{\mathcal{B}}_k : \quad \mathrm{L}^p(\Omega', Y) &\to \mathrm{L}^p(\Omega, Z) \\
g &\mapsto T^{\mathcal{B}}_k(g)(w) = \int_{\Omega'} \mathcal{B}(k(w, w'), g(w')) d\mu'(w')
\end{aligned}$$

As usual, denote by

$$k_w = k(w,\cdot)\colon \begin{array}{ccc} \Omega' & \to & X \\ w' & \mapsto & k(w,w') \end{array} \qquad k^{w'} = k(\cdot,w')\colon \begin{array}{ccc} \Omega & \to & X \\ w & \mapsto & k(w,w'). \end{array}$$

We also write $\mathcal{K}(w) = k_w$ and $\mathcal{K}'(w') = k^{w'}$.

We now introduce similar conditions to the ones appearing in [9] in our more general setting.

Definition 6 *We say that $k : \Omega \times \Omega' \to X$ satisfies the condition $(\mathrm{C}_0^{\mathcal{B}})$ if*

(1) $k_w \in \mathrm{L}^1_{\mathcal{B}}(\Omega', X)$ a.e. in Ω, and

(2) for each $y \in Y$ and $E \in \Sigma'$ the function

$$T_k^{\mathcal{B}}(y,E)\colon \begin{array}{ccc} \Omega & \to & Z \\ w & \mapsto & \displaystyle\int_E \mathcal{B}(k(w,w'),y)d\mu'(w') \end{array}$$

belongs to $\mathrm{L}^0(\Omega, Z)$.

Remark 7 *Given $k : \Omega \times \Omega' \to X$ one can define*

$$k_{\mathcal{B}} : \Omega' \times \Omega \to \mathrm{L}(Y,Z), \quad \text{given by} \quad k_{\mathcal{B}}(w',w) = \mathcal{B}(k(w,w'),\cdot).$$

It is easy to see that if k verifies condition $(\mathrm{C}_0^{\mathcal{B}})$ then $k_{\mathcal{B}}$ verifies the condition (C_0) in [9]. Therefore our conditions are stronger than those appearing by Girardi and Weis, however the conditions can be easily formulated in terms of the new spaces

In particular if $\mathcal{K} \in \mathrm{L}^0(\Omega, \mathrm{L}^1_{\mathcal{B}}(\Omega', X))$ then the operator

$$T_k^{\mathcal{B}}\colon \begin{array}{ccc} \mathcal{S}(\Omega', Y) & \to & \mathrm{L}^0(\Omega, Z) \\ g & \mapsto & T_k^{\mathcal{B}}(g)(w) = \displaystyle\int_{\Omega'} \mathcal{B}(k(w,w'),g(w'))d\mu'(w') \end{array}$$

is well defined.

To study the boundedness of the operator from $\mathrm{L}^p(\Omega', Y)$ to $\mathrm{L}^p(\Omega, Z)$ some extra conditions are needed. We consider the following definition (see [9] for the case $p = 1$)

Definition 7 *Let $1 \le p < \infty$. We say that $k : \Omega \times \Omega' \to X$ satisfies $(\mathrm{C}_{\mathrm{p}}^{\mathcal{B}})$ if*

(1) $k^{\omega'} \in \mathrm{L}^p_{\mathcal{B}}(\Omega, X)$ a.e. in Ω',

(2) $w' \to \|k^{w'}\|_{\mathrm{L}^p_{\mathcal{B}}(\Omega,X)}$ belongs to $\mathrm{L}^{p'}(\Omega')$.

Remark 8 *If $\mathcal{K}' \in \mathrm{L}^{p'}(\Omega', \mathrm{L}^p_{\mathcal{B}}(\Omega, X))$ then k satisfies $(\mathrm{C}_{\mathrm{p}}^{\mathcal{B}})$.*

Proposition 6 Let $1 < p < \infty$ and $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $k : \Omega \times \Omega' \to X$ is a kernel satisfying $(\mathrm{C}_0^{\mathcal{B}})$ and $(\mathrm{C}_{\mathrm{p}}^{\mathcal{B}})$ then the integral operator

$$T_k^{\mathcal{B}}\colon \begin{array}{ccc} \mathcal{S}(\Omega', Y) & \to & \mathrm{L}^p(\Omega, Z) \\ g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \displaystyle\int_{\Omega'} \mathcal{B}(k(\omega,\omega'),g(\omega'))d\mu'(w') \end{array}$$

can be continuously extended to $\mathrm{L}^p(\Omega', Y)$.

PROOF. Let $g=\sum_{k=1}^n y_k \mathbf{1}_{E_k}$ and $T_k^{\mathcal{B}}(g)(w)=\int_{\Omega'}\mathcal{B}(k(\omega,\omega'),g(w'))d\mu'(w')$. Using Minkowski's inequality one gets Therefore

$$\begin{aligned}(\int_\Omega \|T_k^{\mathcal{B}}(g)(w)\|^p d\mu(w))^{\frac{1}{p}} &\le \int_{\Omega'}(\int_\Omega \|\mathcal{B}(k(\omega,\omega'),g(w'))\|^p d\mu(w))^{\frac{1}{p}} d\mu'(w')\\ &\le \int_{\Omega'}\|k^{\omega'}\|_{\mathrm{L}^p_{\mathcal{B}}(\Omega,X)}\|g(w')\|d\mu'(w')\\ &\le (\int_{\Omega'}\|k_{w'}\|^{p'}_{\mathrm{L}^p_{\mathcal{B}}(\Omega,X)}d\mu(w'))^{\frac{1}{p'}}\|g\|_{\mathrm{L}^p(\Omega',Y)}\end{aligned}$$

Now extend by the density of the simple functions on $\mathrm{L}^p(\Omega',Y)$. □

Recall that $\mathcal{B}^*$ denotes the adjoint $\mathcal{B}^*: X\times Z^*\to Y^*$ given by $\langle y,\mathcal{B}^*(x,z^*)\rangle=\langle\mathcal{B}(x,y),z^*\rangle$. We write $\widetilde{k}:\Omega'\times\Omega\to X$ for the map $\widetilde{k}(w',w)=k(w,w')$.

Proposition 7 Let $\mathcal{B}: X\times Y\to Z$ bounded bilinear map. If $k:\Omega\times\Omega'\to X$ satisfies $(\mathrm{C}_0^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(\mathrm{C}_1^{\mathcal{B}^*})$ then the integral operator

$$\begin{aligned}T_k^{\mathcal{B}}:\quad \mathcal{S}(\Omega',Y) &\to \mathrm{L}^\infty(\Omega,Z)\\ g &\mapsto T_k^{\mathcal{B}}(g)(\omega)=\int_{\Omega'}\mathcal{B}(k(\omega,\omega'),g(\omega'))d\mu'(w')\end{aligned}$$

can be continuously extended to $\overline{\mathcal{S}(\Omega',Y)}^{\mathrm{L}^\infty(\Omega',Y)}$ with norm bounded by $\mathrm{C}_1^{\mathcal{B}^*}$.

PROOF. Take $g\in\mathcal{S}(\Omega',Y)$. The condition $(\mathrm{C}^0_{\mathcal{B}})$ provides the measurability of the function $T_k^{\mathcal{B}}(g):\Omega\to Z$. Then, for those $w\in\Omega$ for which $k_w\in\mathrm{L}^1_{\mathcal{B}}(\Omega',X)$, we have that

$$\begin{aligned}\|T_k^{\mathcal{B}}(g)(w)\| &= \sup\{|\int_{\Omega'}\langle\mathcal{B}(k(w,w'),g(w')),z^*\rangle d\mu'(w')|:\|z^*\|=1\}\\ &= \sup\{|\int_{\Omega'}\langle g(w'),\mathcal{B}^*(k(w,w'),z^*)\rangle d\mu'(w')|:\|z^*\|=1\}\\ &\le \|g\|_{\mathrm{L}^\infty(\Omega',Y)}\|k_w\|_{\mathrm{L}^1_{\mathcal{B}^*}(\Omega',X)}.\end{aligned}$$

Hence $\|T_k^{\mathcal{B}}(g)\|_{\mathrm{L}^\infty(\Omega',Y)}\le\mathrm{C}_1^{\mathcal{B}^*}\|g\|_{\mathrm{L}^\infty(\Omega',Y)}$. □

The boundedness of the operator in the case $1<p<\infty$ can also be deduced now of the previous propositions by means of interpolation.

Lemma 1 (see [9], page 198).Let $1<p<\infty$ and let $T:\mathcal{S}(\Omega',Y)\to\mathrm{L}^1(\Omega,Z)+\mathrm{L}^\infty(\Omega,Z)$ be a linear map and there exist $c_1,c_2>0$ such that

$$\|T(g)\|_{\mathrm{L}^1(\Omega',Y)}\le c_1\|g\|_{\mathrm{L}^1(\Omega',Y)}\qquad\text{and}\qquad\|T(g)\|_{\mathrm{L}^\infty(\Omega',Y)}\le c_\infty\|g\|_{\mathrm{L}^\infty(\Omega',Y)}$$

for all $g\in\mathcal{S}(\Omega',Y)$. Then there exists a linear extension $T:\mathrm{L}^p(\Omega',Y)\to\mathrm{L}^p(\Omega,Z)$ with norm bounded by $c_1^{\frac{1}{p}}\ c_\infty^{\frac{1}{p'}}$.

Theorem 2 Let $1<p<\infty$, let $\mathcal{B}:X\times Y\to Z$ be a bounded bilinear map. If $k:\Omega\times\Omega'\to X$ is a kernel satisfying $(\mathrm{C}_0^{\mathcal{B}})$, k satisfies $(\mathrm{C}_1^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(\mathrm{C}_1^{\mathcal{B}^*})$ then the integral operator

$$\begin{aligned}T_k^{\mathcal{B}}:\quad \mathcal{S}(\Omega',Y) &\to \mathrm{L}^\infty(\Omega,Z)\\ g &\mapsto T_k^{\mathcal{B}}(g)(\omega)=\int_{\Omega'}\mathcal{B}(k(\omega,\omega'),g(\omega'))d\mu'(w')\end{aligned}$$

can be continuously extended to $T_k^{\mathcal{B}}:\mathrm{L}^p(\Omega',Y)\to\mathrm{L}^p(\Omega,Z)$ with norm bounded by $(\mathrm{C}_1^{\mathcal{B}})^{\frac{1}{p}}\ (\mathrm{C}_1^{\mathcal{B}^*})^{\frac{1}{p'}}$.

We finish this section mentioning some results about the extension of the operator to $L^\infty(Y))$ whose proofs can be obtained from the obvious modifications in the operator-valued case (see [9]).

Theorem 3 Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. If $k : \Omega \times \Omega' \to X$ satisfies $(C_0^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(C_1^{\mathcal{B}^*})$ then the integral operator

$$\begin{array}{cccl} T_k^{\mathcal{B}}: & \mathcal{S}(\Omega', Y) & \to & L^\infty(\Omega, Z) \\ & g & \mapsto & T_k^{\mathcal{B}}(g)(\omega) = \int_{\Omega'} \mathcal{B}(k(\omega, \omega'), g(\omega')) d\mu'(w') \end{array}$$

can be continuously extended to $S_k^{\mathcal{B}} : L^\infty(\Omega', Y) \to L^\infty_{\text{weak}*}(\Omega, Z^{**})$ given by

$$\langle z^*, S_k^{\mathcal{B}}(g)(w) \rangle = \int_{\Omega'} \langle \mathcal{B}(k(w, w'), g(w')), z^* \rangle d\mu'(w')$$

for each $z^* \in Z^*$, $w \in \Omega$ and $g \in L^\infty(\Omega', Y)$ with norm bounded by $C_1^{\mathcal{B}^*}$.

Theorem 4 Let $\mathcal{B} : X \times Y \to Z$ be a bounded bilinear map. Assume that $k : \Omega \times \Omega' \to X$ satisfies $(C_0^{\mathcal{B}})$ and $\widetilde{k}$ satisfies $(C_1^{\mathcal{B}^*})$ and that Z does not contain a copy of c_0. Then $T_k^{\mathcal{B}}$ has a continuous extension to $T_k^{\mathcal{B}} : L^\infty(\Omega', Y) \to L^\infty(\Omega, Z)$.

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