OPERATOR VALUED BMO AND COMMUTATORS.

OSCAR BLASCO

ABSTRACT. If E is a Banach space, $b \in BMO(\mathbb{R}^n, \mathcal{L}(E))$ and T is a $\mathcal{L}(E)$-valued Calderón-Zygmund type operator with operator-valued kernel k, we show the boundedness of the commutator $T_b(f) = bT(f) - T(bf)$ on $L^p(\mathbb{R}^n, E)$ for $1 < p < \infty$ whenever b and k verify some commuting properties. Some endpoint estimates are also provided.

1. Introduction and notation

We shall work on $\mathbb{R}^n$ endowed with the Lebesgue measure dx and use the notation $|A| = \int_A dx$. Given a Banach space $(X, \|\cdot\|)$ and $1 \leq p < \infty$ we shall denote by $L^p(\mathbb{R}^n, X)$ the space of Bochner p-integrable functions endowed with the norm $\|f\|_{L^p(\mathbb{R}^n,X)} = (\int_{\mathbb{R}^n} \|f(x)\|^p dx)^{1/p}$, by $L^\infty_c(\mathbb{R}^n, X)$ the closure of the compactly supported functions in $L^\infty(\mathbb{R}^n, X)$ and by $L_{weak,\alpha}(\mathbb{R}^n, X)$ the space of measurable functions such that $|\{x \in \mathbb{R}^n : \|f(x)\| > \lambda\}| \leq \alpha(\lambda)$ where $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a non increasing function. We write $H^1(\mathbb{R}^n, X)$ for the Hardy space defined by X-valued atoms, that is the space of integrable functions $f = \sum_k \lambda_k a_k$ where $\lambda_k \in \mathbb{R}$, $\sum_k |\lambda_k| < \infty$ and a_k belong to $L^\infty_c(\mathbb{R}^n, X)$, $supp(a_k) \subset Q_k$ for some cube Q_k, $\int_{Q_k} a(x)dx = 0$ and $\|a(x)\| \leq \frac{1}{|Q_k|}$. We also write, for a positive function ϕ defined on $\mathbb{R}^+$, $BMO_\phi(\mathbb{R}^n, X)$ for the space of locally integrable functions such that there exists $C > 0$ such that for all cube Q

$$\frac{1}{|Q|} \int_Q \|f(x) - f_Q\| dx \leq C\phi(|Q|)$$

where $f_Q = \frac{1}{|Q|} \int_Q f(x)dx$. For $\phi(t) = 1$ we denote the space $BMO(\mathbb{R}^n, X)$ and the above condition is equivalent to

$$osc_p(f, Q) = (\frac{1}{|Q|} \int_Q \|f(x) - f_Q\|^p dx)^{1/p} < \infty$$

for each (equivalently for all) $1 \leq p < \infty$. The infimum of the constants satisfying the above inequalities define the "norm" in the space.

Let us denote by $f^\#$ and $M(f)$ the sharp and the Hardy-Littlewood maximal functions of f defined by

$$f^\#(x) = \sup_{x \in Q} osc_1(f, Q) \quad \text{and} \quad M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q \|f(x)\| dx.$$

We write $M_q(f) = M(\|f\|^q)^{1/q}$ for $1 \leq q < \infty$.

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It is well known that

$$f^{\#}(x) \approx \sup_{x\in Q} \inf_{c_Q \in X} \frac{1}{|Q|} \int_Q \|f(x) - c_Q\| dx \tag{1}$$

and that $f^{\#} \in L^p(\mathbb{R}^n)$ implies that $f \in L^p(\mathbb{R}^n, X)$ for $1 < p < \infty$.

Recall also that M_q maps $L^q(\mathbb{R}^n, X)$ into $L_{weak,1/t^q}$ and

$$M_q : L^p(\mathbb{R}^n, X) \to L^p(\mathbb{R}^n) \text{ is bounded for } q < p \leq \infty. \tag{2}$$

Throughout the paper E denotes a Banach space and $\mathcal{L}(E)$ denotes the space of bounded linear operators on E.

Definition 1.1. *We shall say that T is a $\mathcal{L}(E)$-Calderón-Zygmund type operator if the following properties are fulfilled:*

$$T : L^p(\mathbb{R}^n, E) \to L^p(\mathbb{R}^n, E) \text{ is bounded for some } 1 < p < \infty, \tag{3}$$

there exists a locally integrable function k from $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, x)\}$ into $\mathcal{L}(E)$ such that

$$Tf(x) = \int k(x, y)f(y)dy \tag{4}$$

for every E-valued bounded and compactly supported function f and $x \notin supp\ f$, and there exists $\varepsilon > 0$ such that

$$\|k(x, y) - k(x', y)\| \leq C \frac{|x - x'|^{\varepsilon}}{|x - y|^{n+\varepsilon}}, \quad |x - y| \geq 2|x - x'|. \tag{5}$$

Remark 1.2. *It is well known (see* [RRT] *or* [GR]*) that in such a case T is bounded on $L^q(\mathbb{R}^n, E)$ for any $1 < q < \infty$.*

Throughout the literature, after the result on commutators in [CRW], many results appeared in connection with the boundedness of commutators of Calderón-Zygmund type operators and multiplication by a function b given by $T_b(f) = bT(f) - T(bf)$ on many different function spaces and on their weighted and vector-valued versions (see [ST1, ST2, ST3, ST4, ST5]). Also endpoint estimates for the commutator was a topic that attracted several people on different directions (see [CP, HST, PP, P1, P2, PT1, PT2]).

We shall deal in this paper with the unweighted but operator-valued version of the commutators and will give some results about its boundedness on $L^p(\mathbb{R}^n, E)$ and produce some endpoint estimates.

The following result was shown by C. Segovia and J.L. Torrea (even with some weights and two different Banach spaces).

Theorem 1.3. *(*[ST1, Theorem 1]*) Let T be an $\mathcal{L}(E)$-valued Calderón-Zygmund type operator and let $\ell \to \tilde{\ell}$ be a correspondence from $\mathcal{L}(E)$ to $\mathcal{L}(E)$ such that*

$$\tilde{\ell}T(f)(x) = T(\ell f)(x) \tag{6}$$

and

$$k(x, y)\ell = \tilde{\ell}k(x, y). \tag{7}$$

If b is an $\mathcal{L}(E)$-valued function such that b and $\tilde{b}$ belong to $BMO(\mathbb{R}^n, \mathcal{L}(E))$ then

$$T_b(f) = bT(f) - T(bf)$$

is bounded from $L^p(\mathbb{R}^n, E) \to L^p(\mathbb{R}^n, E)$ for all $1 < p < \infty$.

The endpoint estimates of that result were later studied by E. Harboure, C. Segovia and J.L. Torrea (see Theorem A and Theorem 3.1 in [HST]) when b was assumed to be scalar-valued. From their results one concludes that non-constant scalar valued BMO functions do not define bounded commutators from $L^\infty(\mathbb{R}^n, E)$ to $BMO(\mathbb{R}^n, F)$ when kernel of the Calderón-Zygmund type operators are $\mathcal{L}(E, F)$-valued. Also it was shown that, in general, T_b does not map $H^1(\mathbb{R}^n, E)$ into $L^1(\mathbb{R}, F)$.

The aim of this note is to use the techniques developed in the papers [ST1, HST] to get some extensions for operator-valued BMO-functions having some commuting properties with the kernel. In particular we show that if $\|k(x, y)\| \leq \psi(|x-y|^n)$ for certain function ψ then the commutators of operator-valued BMO functions and operator-valued Calderón-Zygmund operators map $L^\infty_c(\mathbb{R}^n, E)$ into $BMO_\phi(\mathbb{R}^n, E)$ for a function ϕ depending on ψ. Also we shall see that the commutator is bounded from $H^1(\mathbb{R}^n, E)$ into $L_{weak,\alpha}(\mathbb{R}^n, E)$ for a suitable α defined from ψ.

Throughout the paper $b : \mathbb{R}^n \to \mathcal{L}(E)$ is locally integrable and T is a Calderón-Zygmund type operator defined on $L^p(\mathbb{R}^n, E)$ with a kernel k satisfying (3), (4) and (5). We write

$$T_b(f)(x) = b(x)(T(f)(x)) - T(bf)(x)$$

where we understand the product bf as the E-valued function $b(y)(f(y))$.

As usual, we shall use the notation Q for a cube in $\mathbb{R}^n$, x_Q for its center, $\ell(Q)$ for the side length, λQ for a cube centered at x_Q with side length $\lambda\ell(Q)$ and $Q^c = \mathbb{R}^n \setminus Q$. Finally, as usual, C stands for a constant that may vary from line to line.

2. The results

We improve Theorem 1.3 by realizing that conditions (6) and (7) are not of independent nature. Our basic assumptions throughout the paper are the following ones:

$$(\mathbf{A1}) \qquad b(z)k(x, y) = k(x, y)b(z), x, y, z \in \mathbb{R}^n, x \neq y.$$

$$(\mathbf{A2}) \qquad b_Q T(e\chi_A)(x) = T(b_Q e\chi_A)(x), \quad x \in Q, A \subseteq Q \text{ measurable }, e \in E.$$

We would like to point out that $(\mathbf{A1})$ produces the following cancelation property.

Lemma 2.1. *Let b satisfy ($\mathbf{A1}$), let Q, Q' be cubes in $\mathbb{R}^n$ and f_1 and f_2 be compactly supported E-valued with supp$f_1 \subset Q'$ and supp$f_2 \subset (Q')^c$. Then*

$$b_Q T(f_2)(x) = T(b_Q f_2)(x), \quad x \in Q'. \tag{8}$$

$$b_Q T(f_1)(x) = T(b_Q f_1)(x), \quad x \in (Q')^c. \tag{9}$$

Proof. Let us show (8). Recall that if $F \in L^1(\mathbb{R}^n, X)$ and $\Phi \in \mathcal{L}(X)$ for a given Banach space then $\Phi(\int F(x)dx) = \int \Phi F(x)dx$. Hence, considering $X = \mathcal{L}(E)$ and

$\Phi(T) = Tb_Q$ or $\Phi(T) = b_Q T$ one gets, for $x \in Q'$,

$$\begin{aligned} b_Q T(f_2)(x) &= b_Q(\int_{(Q')^c} k(x,y) f_2(y)dy \\ &= \int_{(Q')^c} b_Q k(x,y) f_2(y)dy \\ &= \int_{(Q')^c} (\frac{1}{|Q|}\int_Q b(z)dz) k(x,y) f_2(y)dy \\ &= \int_{(Q')^c} (\frac{1}{|Q|}\int_Q b(z)k(x,y)dz) f_2(y)dy \\ &= \int_{(Q')^c} k(x,y)(\frac{1}{|Q|}\int_Q b(z)dz) f_2(y)dy \\ &= T(b_Q f_2)(x). \end{aligned}$$

(9) follows similarly and it is left to the reader. ■

The assumptions (**A1**) and (**A2**) hold true, for instance, in the following cases.

Example 2.2. *Let T, S be operators in $\mathcal{L}(E)$ with $ST = TS$. Let $b(x) = b_0(x)T$ and $k(x,y) = k_0(x,y)S$ for scalar valued functions b_0 and k_0.*

Hence our results will apply whenever either b or k are scalar-valued.

Example 2.3. *Let E be a Banach space, $b_0(x) \in E^*$ and let $k(x,y)$ be scalar valued function such that T is bounded on $L^p(\mathbb{R}^n, E)$. The case $T_{b_0}(f) = \langle b_0, T(f)\rangle - T(\langle b_0, f\rangle)$ follows from the operator-valued case by selecting $e \in E$ and $b(x)(f) = \langle b_0(x), f\rangle e$ in $\mathcal{L}(E)$.*

We state here the results of the paper. The first one is just a modification of a similar result from [ST1] but stated here under slightly weaker assumptions.

Theorem 2.4. *Let $b \in BMO(\mathbb{R}^n, \mathcal{L}(E))$ and let T be a Calderón-Zygmund type operator defined on $L^p(\mathbb{R}^n, E)$ where the kernel and b satisfy (**A1**) and (**A2**). Then T_b is bounded on $L^p(\mathbb{R}^n, E)$ for any $1 < p < \infty$.*

Next we analyze the endpoint estimates. We construct a function ϕ for the commutator T_b to be bounded from $L^\infty_c(\mathbb{R}^n, E)$ into $BMO_\phi(\mathbb{R}^n, \mathcal{L}(E))$.

Theorem 2.5. *Let T be a Calderón-Zygmund type operator with operator-valued kernel k and assume that*

$$\|k(x,y)\| \leq \psi(|x-y|^n), x \neq y \tag{10}$$

for some $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\int_s^\infty \psi(u)du = \phi(s) < \infty$ for all $s > 0$.

*If $b \in BMO(\mathbb{R}^n, \mathcal{L}(E))$ satisfies (**A1**) and that T_b is bounded on some $L^p(\mathbb{R}^n, E)$ then T_b is bounded from $L^\infty_c(\mathbb{R}^n, E)$ into $BMO_{1+\phi}(\mathbb{R}^n, E)$.*

We also discover a the function α such that the commutator of a function b in $BMO(\mathbb{R}^n, \mathcal{L}(E))$ with a a Calderón-Zygmund type operator T_b maps the space $H^1(\mathbb{R}^n, E)$ into $L_{weak,\alpha}(\mathbb{R}`n, E)$.

Theorem 2.6. *Let T be a Calderón-Zygmund type operator with operator-valued kernel k. Assume that*

$$\|k(x,y)\| \leq \gamma(|x-y|^n), \quad x \neq y \tag{11}$$

for some decreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *and*

$$\|k(x,y) - k(x,y')\| \leq C\frac{|y-y'|^{\varepsilon}}{|x-y|^{n+\varepsilon}}, \quad |x-y| \geq 2|y-y'|. \tag{12}$$

If $b \in BMO(\mathbb{R}^n, \mathcal{L}(E))$ *satisfies* (**A1**)*and* T_b *is bounded on some* $L^p(\mathbb{R}^n, E)$ *then* T_b *is bounded from* $H^1(\mathbb{R}^n, E)$ *into* $L^1_{weak,\alpha}(\mathbb{R}^n, E)$ *for* $\alpha(\lambda) = \gamma^{-1}(\|b\|^{-1}_{BMO}\lambda)$.

As corollaries of these results one obtains the following applications.

Corollary 2.7. *(see* [ST1]*) Let* H *be the Hilbert transform*

$$H(f)(x) = p.v. \int \frac{f(y)}{x-y}dy,$$

and E *be a UMD space (see* [GR]*). If* $b \in BMO(\mathbb{R}, \mathcal{L}(E))$ *then*
(i) H_b *maps* $L^p(\mathbb{R}, E)$ *to* $L^p(\mathbb{R}, E)$ *for* $1 < p < \infty$ *and*
(ii) H_b *maps* $H^1(\mathbb{R}, E)$ *to* $L_{weak,1/t}(\mathbb{R}, E)$.

Although our results are stated in $\mathbb{R}$, similar ones work in $\mathbb{T}$. In this case we can obtain

Corollary 2.8. *(see* [HST]*) Let* $\tilde{H}$ *be the conjugate function in the torus*

$$\tilde{H}(f)(x) = p.v.\frac{1}{2\pi} \int cot(\frac{x-y}{2})f(y)dy, x \in [-\pi, \pi]$$

and E *be a UMD space. If* $b \in BMO(\mathbb{R}, \mathcal{L}(E))$ *then*
(i) H_b *maps* $L^p(\mathbb{T}, E)$ *to* $L^p(\mathbb{T}, E)$ *for* $1 < p < \infty$,
(ii) H_b *maps* $H^1(\mathbb{T}, E)$ *to* $L_{weak,1/t}(\mathbb{T}, E)$ *and*
(ii) H_b *maps* $L^\infty(\mathbb{T}, E)$ *to* $BMO_{|\log t|^{-1}}(\mathbb{T}, E)$.

3. Proof of the results

Let us start by showing some consequences from (**A1**) and (**A2**).

Lemma 3.1. *Let* b *satisfy (***A1***) and (***A2***), let* Q *be cube in* $\mathbb{R}^n$ *and* f *be simple* E*-valued. Then*

$$b_Q T(f)(x) = T(b_Q f)(x) \, x \in Q. \tag{13}$$

Proof. Take $f_1 = f\chi_Q$ and $f_2 = f - f_1$. Using Lemma 2.1 one obtains $b_Q T(f_2)\chi_Q = T(b_Q f_2)\chi_Q$ and (**A2**) shows that $b_Q T(f_1)\chi_Q = T(b_Q f_1)\chi_Q$. ■

The following useful lemma is essentially included in [HST].

Lemma 3.2. *let* Q *be a cube, denote* $Q_j = 2^j Q$ *and let* f *be compactly supported* E*-valued with* $suppf \subset (2Q)^c$. *Then there exists* $C > 0$ *such that*

$$\|T(f)(x) - T(f)(x')\| \leq C\frac{|x-x'|^{\varepsilon}}{\ell(Q)^{\varepsilon}} \sum_{j=2}^{\infty} \frac{2^{-j\varepsilon}}{|Q_j|} \int_{Q_j} \|f(y)\|dy, x, x' \in Q. \tag{14}$$

Proof. Using (4) and (5) one has

$$\begin{aligned}
\|T(f)(x)-T(f)(x')\| &\leq \int_{(2Q)^c} \|k(x,y)-k(x',y)\|\|f(y)\|dy \\
&\leq C|x-x'|^{\varepsilon}\int_{(2Q)^c}\frac{\|f(y)\|}{|x-y|^{n+\varepsilon}}dy \\
&\leq C|x-x'|^{\varepsilon}\sum_{j=1}^{\infty}\int_{Q_{j+1}-Q_j}\frac{\|f(y)\|}{|x-y|^{n+\varepsilon}}dy \\
&\leq C|x-x'|^{\varepsilon}\sum_{j=2}^{\infty}\frac{1}{\ell(Q_j)^{n+\varepsilon}}\int_{Q_j}\|f(y)\|dy \\
&\leq C\frac{|x-x'|^{\varepsilon}}{\ell(Q)^{\varepsilon}}\sum_{j=2}^{\infty}2^{-j\varepsilon}\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|dy.
\end{aligned}$$

■

Proof of Theorem 2.4

Let f be a simple E-valued function. Let Q be a cube, $f_1 = f\chi_{2Q}$ and $f_2 = f - f_1$. Put $c_Q = T((b_Q - b)f_2)(x_Q)$.

For each $x \in Q$ one has, applying Lemma 3.1,

$$T_b f(x) - c_Q = \sum_{i=1}^{3}\sigma_i(x)$$

where

$$\sigma_1(x) = (b - b_Q)Tf(x),$$

$$\sigma_2(x) = T((b_Q - b)f_1)(x)$$

and

$$\sigma_3(x) = T((b_Q - b)f_2)(x) - T((b_Q - b)f_2)(x_Q).$$

Observe that for $1 < q < \infty$ and $1/q + 1/q' = 1$ we can write

$$\frac{1}{|Q|}\int_Q \|\sigma_1(x)\|dx \leq osc_{q'}(b,Q)(\frac{1}{|Q|}\int_Q \|Tf(x)\|^q dx)^{1/q}.$$

For any $q > q_1 > 1$ one can use Remark 1.2, for $1/r + 1/q = 1/q_1$, to obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \|\sigma_2(x)\|dx &\leq (\frac{1}{|Q|}\int_Q \|T(b_Q - b)f_1(x)\|^{q_1}dx)^{1/q_1} \\
&\leq C\|T\|_{\mathcal{L}(L^{q_1}(\mathbb{R}^n,E))}(\frac{1}{|Q|}\int_Q \|(b-b_Q)f_1(x)\|^{q_1}dx)^{1/q_1} \\
&\leq C\|T\|_{\mathcal{L}(L^{q_1}(\mathbb{R}^n,E))}osc_r(b.Q)(\frac{1}{|Q|}\int_Q \|f(x)\|^q dx)^{1/q}.
\end{aligned}$$

Using Lemma 3.2, and taking into account that $\|b_Q - b_{2Q}\| \le C osc_{q_1}(b, 2Q)$, we also can estimate

$$\begin{aligned}
\|\sigma_3(x)\| &\le C\sum_{j=2}^{\infty} 2^{-j\varepsilon}\frac{1}{|Q_j|}\int_{Q_j}\|(b(y)-b_Q)f(y)\|dy \\
&\le C\sum_{j=2}^{\infty} 2^{-j\varepsilon}(\frac{1}{|Q_j|}\int_{Q_j}\|b(y)-b_Q\|^{q'}dy)^{1/q'}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^q dy)^{1/q} \\
&\le C\sum_{j=2}^{\infty} 2^{-j\varepsilon}(\sum_{k=2}^{j} osc_{q'}(b,Q_k))(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^q dy)^{1/q} \\
&\le C\sup_{j\ge 2}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^q dy)^{1/q}\Big(\sum_{j=2}^{\infty}2^{-j\varepsilon}(\sum_{k=2}^{j} osc_{q'}(b,Q_k))\Big) \\
&\le C\|b\|_{BMO}\sup_{j\ge 2}(\frac{1}{|Q_j|}\int_{Q_j}\|f(y)\|^q dy)^{1/q}\sum_j j2^{-j\varepsilon}.
\end{aligned}$$

Hence, combining the previous estimates, one obtains

$$T_b(f)^{\#}(x) \le C\|b\|_{BMO}(M_q(Tf)(x) + M_q(f)(x)).$$

Now, for a given $1 < p < \infty$, select $1 < q < p$ and apply (2), which, combined with the boundedness of T on $L^p(\mathbb{R}^n, E)$, shows that $\|T_b(f)^{\#}\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n,E)}$. Now use the vector-valued analogue of Fefferman-Stein's result (see [FS, RRT]) to obtain that $\|T_b(f)\|_{L^p(\mathbb{R}^n,E)} \le C\|f\|_{L^p(\mathbb{R}^n,E)}$. ■

Proof of Theorem 2.5

As in the previous theorem, let f be a simple E-valued function. Let Q be a cube, $f_1 = f\chi_{2Q}$, $f_2 = f - f_1$ and $c_Q = T((b_Q - b)f_2)(x_Q)$ Now, using Lemma 2.1, we write

$$T_b f(x) = T_b(f_1)(x) + (b(x) - b_Q)T(f_2)(x) + T((b_Q - b)f_2)(x).$$

Denote now

$$\sigma_1(x) = T_b(f_1)(x),$$

$$\sigma_2(x) = (b(x) - b_Q)T(f_2)(x),$$

$$\sigma_3(x) = T((b_Q - b)f_2)(x) - T((b_Q - b)f_2)(x_Q).$$

Hence $T_b f - c_Q = \sum_{i=1}^{3}\sigma_i$. Note that the boundedness of T_b on $L^p(\mathbb{R}^n, E)$ gives

$$\frac{1}{|Q|}\int_Q \|\sigma_1(x)\|dx \le C\|T_b\|_{\mathcal{L}(L^p)}(\frac{1}{|2Q|}\int_{2Q}\|f(x)\|^p dx)^{1/p} \le C\|f\|_\infty.$$

On the other hand

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \|\sigma_2(x)\|dx &\leq \frac{1}{|Q|}\int_Q \|b(x)-b_Q\|\|\int_{(2Q)^c} k(x,y)f(y)dy)\|dx \\
&\leq C\frac{1}{|Q|}\int_Q \|b(x)-b_Q\|(\int_{(2Q)^c}\psi(|x-y|^n)\|f(y)\|dy)dx \\
&\leq C\|f\|_\infty \frac{1}{|Q|}\int_Q \|b(x)-b_Q\|(\int_{|u|>\ell(Q)}\psi(|u|^n)du)dx \\
&\leq C\|f\|_\infty (\frac{1}{|Q|}\int_Q \|b(x)-b_Q\|dx)(\int_{\ell(Q)}^\infty r^{n-1}\psi(r^n)dr) \\
&\leq C\|f\|_\infty \|b\|_{BMO}(\int_{|Q|}^\infty \psi(t)dt).
\end{aligned}$$

Finally Lemma 3.2 gives immediately

$$\frac{1}{|Q|}\int_Q \|\sigma_3(x)\|dx \leq C\|b\|_{BMO}\|f\|_\infty.$$

This allows us to conclude the estimate

$$\frac{1}{|Q|}\int_Q \|T_bf(x)-c_Q\|dx \leq C\|f\|_\infty(1+\phi(|Q|)).$$

This shows that T_b maps $L^\infty_c(\mathbb{R}^n,E)$ into $BMO_{1+\phi}(\mathbb{R}^n,E)$. ■

Proof of Theorem 2.6

Let a be an E-valued atom supported on Q. Using Lemma 2.1 we can write

$$T_ba(x) = \chi_{2Q}(x)T_b(a)(x)+\chi_{(2Q)^c}(x)(b(x)-b_Q)T(a)(x)+\chi_{(2Q)^c}(x)T((b_Q-b)a)(x).$$

Denote now

$$\sigma_1(x) = \chi_{2Q}(x)T_b(a)(x),$$

$$\sigma_2(x) = \chi_{(2Q)^c}(x)(b(x)-b_Q)T(a)(x),$$

$$\sigma_3(x) = \chi_{(2Q)^c}(x)T((b_Q-b)a)(x).$$

Now, using the boundedness of T_b on $L^p(\mathbb{R}^n,E)$,

$$\begin{aligned}
\int_{\mathbb{R}^n}\|\sigma_1(x)\|dx &\leq C|Q|^{1/p'}\|T_b(a)\|_{L^p(\mathbb{R}^n,E)} \\
&\leq C\|T_b\|_{\mathcal{L}(L^p)}|Q|(\frac{1}{|Q|}\int_Q \|a(x)\|^p dx)^{1/p} \\
&\leq C\|T_b\|_{\mathcal{L}(L^p)}.
\end{aligned}$$

Also we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \|\sigma_2(x)\|dx &\leq \int_{(2Q)^c} \|b(x)-b_Q\|\|\int_Q k(x,y)a(y)dy\|dx \\
&\leq \int_{(2Q)^c} \|b(x)-b_Q\|\|\int_Q (k(x,y)-k(x,x_Q))a(y)dy\|dx \\
&\leq C\int_{(2Q)^c} \|b(x)-b_Q\|(\int_Q \frac{|y-x_Q|^\varepsilon}{|x-y|^{n+\varepsilon}}\|a(y)\|dy)dx \\
&\leq C\frac{\ell(Q)^\varepsilon}{|Q|}\int_Q(\int_{(2Q)^c} \frac{\|b(x)-b_Q\|}{|x-y|^{n+\varepsilon}}dx)dy \\
&\leq C\frac{\ell(Q)^\varepsilon}{|Q|}\int_Q(\sum_{j=2}^\infty \frac{1}{\ell(Q_j)^{n+\varepsilon}}\int_{Q_j-Q_{j-1}} \|b(x)-b_Q\|dx)dy \\
&\leq C(\sum_{j=2}^\infty 2^{-j\varepsilon}\frac{1}{|Q_j|}\int_{Q_j}\|b(x)-b_Q\|dx) \leq C\|b\|_{BMO}.
\end{aligned}$$

Now decompose $\sigma_3 = \sigma_{3,1}+\sigma_{3,2}$ where

$$\sigma_{3,1}(x) = \chi_{(2Q)^c}(x)\int_Q (k(x,y)-k(x,x_Q))(b_Q-b(y))a(y)dy,$$

$$\sigma_{3,2}(x) = \chi_{(2Q)^c}(x)k(x,x_Q)\int_Q b(y)a(y)dy.$$

Therefore

$$\begin{aligned}
\int_{\mathbb{R}^n} \|\sigma_{3,1}(x)\|dx &\leq \int_{(2Q)^c}\int_Q \|k(x,y)-k(x,x_Q)\|\|b_Q-b(y)\|\|a(y)\|dydx \\
&\leq \int_{(2Q)^c} \frac{\ell(Q)^\varepsilon}{|Q|}(\int_Q \frac{\|b_Q-b(y)\|}{|x-y|^{n+\varepsilon}}dy)dx \\
&\leq \frac{\ell(Q)^\varepsilon}{|Q|}\int_Q \|b_Q-b(y)\|(\int_{(2Q)^c}\frac{dx}{|x-y|^{n+\varepsilon}})dy \\
&\leq \frac{\ell(Q)^\varepsilon}{|Q|}\int_Q \|b_Q-b(y)\|(\int_{|x|>\ell(Q)}\frac{dx}{|x|^{n+\varepsilon}})dy \leq C\|b\|_{BMO}.
\end{aligned}$$

Since $\|\int_Q b(y)a(y)dy\| \leq \frac{1}{|Q|}\int_Q \|b(y)-b_Q\|dy$ we can estimate

$$\begin{aligned}
\sigma_{3,2}(x) &\leq \chi_{(2Q)^c}(x)\|k(x,x_Q)\|\|b\|_{BMO} \\
&\leq \|b\|_{BMO}\chi_{(2Q)^c}(x)\gamma(|x-x_Q|^n)
\end{aligned}$$

Therefore one gets

$$\begin{aligned}
|\{x : \sigma_{3,2}(x) > \lambda\}| &\leq |\{x \in (2Q)^c : \gamma(|x-x_Q|^n) > \|b\|_{BMO}^{-1}\lambda\}| \\
&= |\{x \in (2Q)^c : |x-x_Q| < [\gamma^{-1}(\|b\|_{BMO}^{-1}\lambda)]^{1/n}\}|.
\end{aligned}$$

This gives the estimate $|\{x : \sigma_{3,2}(x) > \lambda\}| \leq \psi^{-1}(\|b\|_{BMO}^{-1}\lambda) = \alpha(\lambda)$.
The proof is then easily concluded. ■

References

[B] Bloom, S. *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. **292**, (1985), 103-122.

[CRW] Coifman, R.; Rochberg, R.; Weiss, G. *Factorization theorems for Hardy spaces in several variables.* Ann. of Math. **103** (1976), no. 2, 611–635.

[CP] Cruz-Uribe, D.; Pérez, C. *Two-weight, weak-type norm inequalities for fractional integrals, Calderón-Zygmund operators and commutators.* Indiana Univ. Math. J. **49** (2000), no. 2, 697–721.

[FS] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Math. **129** (1972), 137-193.

[GR] García-Cuerva, J.; Rubio de Francia, J.L. *Weighted norm inequalities and related topics*, North-Holland, Amsterdam, 1985.

[HST] Harboure, E., Segovia, C., Torrea, J.L. *Boundedness of commutators of fractional and singular integrals for the extreme values of p*, Illinois J. Math. **41**, no. 4 (1997), 676-700.

[P1] Pérez, C. *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128**, (1995), 163-185.

[P2] Pérez, C. *Sharp estimates for commutators of singular integrals via iterations of the Hardy-Littlewood maximal function.* J. Fourier Anal. Appl. **3** (1997), no. 6, 743–756.

[PP] Pérez, C.; Pradolini, G.*Sharp weighted endpoint estimates for commutators of singular integrals.* Michigan Math. J. **49** (2001), no. 1, 23–37.

[PT1] Pérez, C.; Trujillo-González, R. *Sharp weighted estimates for multilinear commutators.* J. London Math. Soc. (2) **65** (2002), no. 3, 672–692.

[PT2] Pérez, C.; Trujillo-González, R. *Sharp weighted estimates for vector-valued singular integral operators and commutators.* Tohoku Math. J. (2) **55** (2003), no. 1, 109–129.

[RRT] Rubio de Francia,J.L.; Ruiz,F.; Torrea,J.L. *Calderón-Zygmund theory for vector-valued functions*, Adv. in Math. **62** (1986), 7-48.

[ST1] Segovia, C.; Torrea, J.L. *Vector-valued commutators and applications*, Indiana Univ. Math. J. **38** (1989), no. 4, 959–971.

[ST2] Segovia, C,; Torrea, J. L. *A note on the commutator of the Hilbert transform.* Rev. Un. Mat. Argentina **35** (1989), 259–264.

[ST3] Segovia, C.; Torrea, J. L. *Weighted inequalities for commutators of fractional and singular integrals.* Conference on Mathematical Analysis (El Escorial, 1989). Publ. Mat. **35** (1991), no. 1, 209–235.

[ST4] Segovia, C.; Torrea, J. L. Commutators of Littlewood-Paley sums. Ark. Mat. **31** (1993), no. 1, 117–136.

[ST5] Segovia, C.; Torrea, J. L. *Higher order commutators for vector-valued Calderón-Zygmund operators.* Trans. Amer. Math. Soc. **336** (1993), no. 2, 537–556.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* , Princeton Univ. Press, [1971].

Department of Mathematics, Universitat de Valencia, Burjassot 46100 (Valencia) Spain

E-mail address: oscar.blasco@uv.es