p-variation of vector measures with respect to bilinear maps.

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Abstract

We introduce the spaces $\mathrm{V}^p_{\mathcal{B}}(X)$ (resp. $\mathcal{V}^p_{\mathcal{B}}(X)$) of the vector measures $\mathcal{F}:\Sigma\to X$ of bounded $(p,\mathcal{B})$-variation (resp. of bounded $(p,\mathcal{B})$-semivariation) with respect to a bounded bilinear map $\mathcal{B}: X\times Y\to Z$ and show that the spaces $L^p_{\mathcal{B}}(X)$ consisting in functions which are p-integrable with respect to $\mathcal{B}$, defined in [4], are isometrically embedded into $\mathrm{V}^p_{\mathcal{B}}(X)$. We characterize $\mathcal{V}^p_{\mathcal{B}}(X)$ in terms of bilinear maps from $L^{p'}\times Y$ into Z and $\mathrm{V}^p_{\mathcal{B}}(X)$ as a subspace of operators from $L^{p'}(Z^*)$ into Y^*. Also we define the notion of cone absolutely summing bilinear maps in order to describe the $(p,\mathcal{B})$-variation of a measure in terms of the cone-absolutely summing norm of the corresponding bilinear map from $L^{p'}\times Y$ into Z.

1 Notation and preliminaries.

Throughout the paper X denotes a Banach space, (Ω,Σ,μ) a positive finite measure space, $\mathcal{D}_E$ the set of all partitions of $E\in\Sigma$ into a finite number of pairwise disjoint elements of Σ of positive measure and $\mathcal{S}_\Sigma(X)$ the space of simple functions, $\mathbf{s}=\sum_{k=1}^n x_k\mathbf{1}_{A_k}$, where $x_k\in X$, $(A_k)_k\in\mathcal{D}_\Omega$ and $\mathbf{1}_A$ denotes the characteristic function of the set $A\in\Sigma$. Also Y and Z denote Banach spaces over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\mathcal{B}: X\times Y\to Z$ a bounded bilinear map. We use the notation B_X for the closed unit ball of X, $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X to Y and $X^*=\mathcal{L}(X,\mathbb{K})$.

For a vector measure $\mathcal{F}:\Sigma\to X$ we use the notation $|\mathcal{F}|$ and $\|\mathcal{F}\|$ for the non negative set functions $|\mathcal{F}|:\Sigma\to\mathbb{R}^+$ and $\|\mathcal{F}\|:\Sigma\to\mathbb{R}^+$ given by

$$|\mathcal{F}|(E)=\sup\{\sum_{A\in\pi}\|\mathcal{F}(A)\|_X : \pi\in\mathcal{D}_E\}$$

and

$$\|\mathcal{F}\|(E)=\sup\{|\langle\mathcal{F},x^*\rangle|(E): x^*\in\mathrm{B}_{X^*}\}$$

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respectively. In the case of operator-valued measures $\mathcal{F} : \Sigma \to \mathcal{L}(Y, Z)$ we use $|||\mathcal{F}|||$ for the strong-variation defined by

$$|||\mathcal{F}|||(E) = \sup\{\sum_{A\in\pi} \|\mathcal{F}(A)y\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_E\}.$$

Given a norm τ defined on the space $Y \otimes X$ satisfying $\|y \otimes x\|_\tau \leq C\|y\|\cdot\|x\|$ we write $Y\hat{\otimes}X$ for its completion. In [1] R. Bartle introduced the notion of Y-semivariation of a vector measure $\mathcal{F} : \Sigma \to X$ with respect to τ by the formula

$$\beta_Y(\mathcal{F}, \tau)(E) = \sup\{\| \sum_{A\in\pi} y_A \otimes \mathcal{F}(A)\|_\tau : y_A \in \mathrm{B}_Y, \pi \in \mathcal{D}_E\}$$

for every $E \in \Sigma$. This is an intermediate notion between the variation and semivariation, since for every $E \in \Sigma$ we clearly have

$$\|\mathcal{F}\|(E) \leq \beta_Y(\mathcal{F}, \tau)(E) \leq |\mathcal{F}|(E).$$

If $Y\hat{\otimes}_\epsilon X$ and $Y\hat{\otimes}_\pi X$ stand for the injective and projective tensor norms respectively, then we actually have

$$\|\mathcal{F}\|(E) = \beta_Y(\mathcal{F}, \epsilon)(E) \leq \beta_Y(\mathcal{F}, \tau)(E) \leq \beta_Y(\mathcal{F}, \pi)(E) \leq |\mathcal{F}|(E).$$

We refer the reader to [10] for a theory of integration of Y-valued functions with respect to X-valued measures of bounded Y-semivariation iniciated by B. Jefferies and S. Okada and to [3] for the study of this notion in the particular cases $X = L^p(\mu)$, $Y = L^q(\nu))$ and τ the norm in the space of vector-valued functions $L^p(\mu, L^q(\nu))$.

We are going to use notions of $\mathcal{B}$-variation (or $\mathcal{B}$-semivariation) which allow to obtain all the previous cases for particular instances of bilinear maps.

Recall that, for $1 < p < \infty$, the p-variation and p-semivariation of a vector measure $\mathcal{F}$ are defined by

$$|\mathcal{F}|_p(E) = \sup\{\big(\sum_{A\in\pi} \frac{\|\mathcal{F}(A)\|_X^p}{\mu(A)^{p-1}}\big)^{1/p} : \pi \in \mathcal{D}_E\} \tag{1}$$

and

$$\|\mathcal{F}\|_p(E) = \sup\{\big(\sum_{A\in\pi} \frac{|\langle\mathcal{F}(A), x^*\rangle|^p}{\mu(A)^{p-1}}\big)^{1/p} : x^* \in \mathrm{B}_{X^*}, \pi \in \mathcal{D}_E\}. \tag{2}$$

We denote $\mathrm{V}^p(X)$ and $\mathcal{V}^p(X)$ the Banach spaces of vector-measures for which $|\mathcal{F}|_p(\Omega) < \infty$ and $\|\mathcal{F}\|_p(\Omega) < \infty$ respectively.

The limiting case $p = 1$ corresponds to $\|\mathcal{F}\|_1(E) = |\mathcal{F}|(E)$ and $\|\mathcal{F}\|_1(E) = \|\mathcal{F}\|(E)$. For $p = \infty$ $\mathrm{V}^\infty(X) = \mathcal{V}^\infty(X)$ is given by vector-measures satisfying that there exists $C > 0$ such that $\|\mathcal{F}(A)\| \leq C\mu(A)$ for any $A \in \Sigma$ and the ∞-variation of a measure is defined by

$$\|\mathcal{F}\|_\infty(E) = \sup\{\frac{\|\mathcal{F}(A)\|_X}{\mu(A)} : A \in \Sigma, A \subset E, \mu(A) > 0\}. \tag{3}$$

We denote by $L^0(X)$ and $L^0_{\text{weak}}(X)$ the spaces of strongly and weakly measurable functions with values in X and write $L^p(X)$ and $L^p_{\text{weak}}(X)$ for the space of functions in $L^0(X)$ and $L^0_{\text{weak}}(X)$ such that $\|f\| \in L^p$ and $\langle f, x^*\rangle \in L^p$ for every $x^* \in X^*$ respectively. As usual for $1 \leq p \leq \infty$ the conjugate index is denoted by p', i.e. $1/p + 1/p' = 1$.

For each $f \in L^p(X)$, $1 \leq p \leq \infty$, one can define a vector measure

$$\mathcal{F}_f(E) = \int_E f d\mu, \quad E \in \Sigma$$

which is of bounded p-variation and $|\mathcal{F}_f|_p(\Omega) = \|f\|_{L^p(X)}$. On the other hand the converse depends on the Radon-Nikodým property, that is, given $1 < p \leq \infty$, X has the RNP if and only if for any X-valued measure $\mathcal{F}$ of bounded p-variation there exists $f \in L^p(X)$ such that $\mathcal{F} = \mathcal{F}_f$.

For general Banach spaces X, $\mathrm{V}^\infty(X)$ can be identified with the space of operators $\mathcal{L}(L^1, X)$ by means of the map $\mathcal{F} \to T_\mathcal{F}$ where

$$T_\mathcal{F}(\mathbf{1}_E) = \mathcal{F}(E), \quad E \in \Sigma,$$

and for $1 < p < \infty$ the space $\mathrm{V}^p(X)$ can be identified (isometrically) with the space $\Lambda(L^{p'}, X)$, formed by the cone absolutely summing operators from $L^{p'}$ into X with the π_1^+ norm (see [13, 2]). We refer the reader to [8, 6, 10, 13] for the notions appearing in the paper and the basic concepts about vector measures and their variations.

Quite recently the authors started studying the spaces of X-valued functions which are p-integrable with respect to a bounded bilinear map $\mathcal{B} : X \times Y \to Z$, that is to say functions f satisfying the condition $\mathcal{B}(f, y) \in L^p(Z)$ for all $y \in Y$. Some basic theory was developed and applied to different examples (see [4, **?**, 5]). Note that the use of certain bilinear maps, such as

$$\mathcal{B} : X \times \mathbb{K} \to X, \quad \text{given by} \quad \mathcal{B}(x, \lambda) = \lambda x, \tag{4}$$

$$\mathcal{D} : X \times X^* \to \mathbb{K}, \quad \text{given by} \quad \mathcal{D}(x, x^*) = \langle x, x^* \rangle, \tag{5}$$

$$\mathcal{D}_1 : X^* \times X \to \mathbb{K}, \quad \text{given by} \quad \mathcal{D}_1(x^*, x) = \langle x, x^* \rangle, \tag{6}$$

$$\pi_Y : X \times Y \to X \hat{\otimes} Y, \quad \text{given by} \quad \pi_Y(x, y) = x \otimes y, \tag{7}$$

$$\tilde{\mathcal{O}}_Y : X \times \mathcal{L}(X, Y) \to Y, \quad \text{given by} \quad \tilde{\mathcal{O}}_Y(x, T) = T(x), \tag{8}$$

$$\mathcal{O}_{Y,Z} : \mathcal{L}(Y, Z) \times Y \to Z, \quad \text{given by} \quad \mathcal{O}_{Y,Z}(T, y) = T(y) \tag{9}$$

have been around for many years and have been used in different aspects of vector-valued functions, but a systematic study for general bilinear maps was started in [4] and used, among other things, to extend the results on boundedness from $L^p(Y)$ to $L^p(Z)$ of operator-valued kernels by M. Girardi and L. Weiss [9] to the case where $K : \Omega \times \Omega' \to X$ is measurable and the integral operators are defined by

$$T_K(f)(w) = \int_{\Omega'} \mathcal{B}(K(w, w'), f(w')) d\mu'(w').$$

The reader is also referred [5] for some versions of Hölder's inequality in this setting.

We shall need some notations and definitions from the previous papers. We write $\Phi_\mathcal{B} : X \to \mathcal{L}(Y, Z)$ and $\Psi_\mathcal{B} : Y \to \mathcal{L}(X, Z)$ for the bounded linear operators defined by $\Phi_\mathcal{B}(x) = \mathcal{B}_x$ and $\Psi_\mathcal{B}(y) = \mathcal{B}^y$ where $\mathcal{B}_x$ and $\mathcal{B}^y$ are given by $\mathcal{B}_x(y) = \mathcal{B}^y(x) = \mathcal{B}(x, y)$.

A bounded bilinear map $\mathcal{B} : X \times Y \to Z$ is called admissible (see [4]) if $\Phi_\mathcal{B}$ is injective. Throughout the paper $\mathcal{B}$ will be always assumed to be admissible. However a stronger condition will be also needed for some results: A Banach space X is said to be $(Y, Z, \mathcal{B})$-normed if there exists $k > 0$

$$\|x\|_X \leq k \|\mathcal{B}_x\|_{\mathcal{L}(Y,Z)}, \quad x \in X.$$

The bounded bilinear maps (4)-(9) provide examples of $\mathcal{B}$-normed spaces.

As in [4] we write $\mathcal{L}^p_\mathcal{B}(X)$ for the space of functions $f : \Omega \to X$ with $\mathcal{B}(f, y) \in L^0(Z)$ for any $y \in Y$ and such that

$$\|f\|_{\mathcal{L}^p_\mathcal{B}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^p(Z)} : y \in \mathrm{B}_Y\} < \infty,$$

and we use the notation $L^p_\mathcal{B}(X)$ for the space of functions $f \in \mathcal{L}^p_\mathcal{B}(X)$ for which there exists a sequence of simple functions $(\mathbf{s}_n)_n \in \mathcal{S}_\Sigma(X)$ such that $\mathbf{s}_n \to f$ a.e. and $\|\mathbf{s}_n - f\|_{\mathcal{L}^p_\mathcal{B}(X)} \to 0$. In such a case, we write $\|f\|_{L^p_\mathcal{B}(X)}$ instead of $\|f\|_{\mathcal{L}^p_\mathcal{B}(X)}$ and $\|f\|_{L^p_\mathcal{B}(X)} = \lim_{n\to\infty} \|\mathbf{s}_n\|_{L^p_\mathcal{B}(X)}$.

In particular, for the examples $\mathcal{B}$ and $\mathcal{D}$ we have that $\mathcal{L}^p_{\mathcal{B}}(X) = L^p(X)$ and $\mathcal{L}^p_{\mathcal{D}}(X) = L^p_{\text{weak}}(X)$. Also $L^p_{\mathcal{B}}(X) = L^p(X)$ and $L^p_{\mathcal{D}}(X)$ coincides with the space of Pettis p-integrable functions $\mathcal{P}^p(X)$ (see [12], page 54 for the case $p = 1$).

Observe that, for any $\mathcal{B}$, $L^p(X) \subseteq L^p_{\mathcal{B}}(X)$ and the inclusion can be strict (see [6] page 53, for the case $\mathcal{B} = \mathcal{D}$). Regarding the connection between $L^p_{\mathcal{B}}(X)$ and $L^p_{\text{weak}}(X)$ it was shown that X is $(Y, Z, \mathcal{B})$-normed if and only if $L^p_{\mathcal{B}}(X) \subseteq L^p_{\text{weak}}(X)$ with continuous inclusion.

Due to this fact, if $f \in L^1_{\mathcal{B}}(X)$ for some $(Y, Z, \mathcal{B})$-normed space X then for each $E \in \Sigma$ there exists a unique element of X, to be denoted by $\int_E^{\mathcal{B}} f d\mu$, verifying

$$\int_E \mathcal{B}(f, y) d\mu = \mathcal{B}(\int_E^{\mathcal{B}} f d\mu, y), \text{ for all } y \in Y.$$

This allows us to define the vector measure

$$\mathcal{F}_f^{\mathcal{B}}(E) = \int_E^{\mathcal{B}} f d\mu, \quad E \in \Sigma.$$

We shall consider the notion of $(p, \mathcal{B})$-variation which fits with the theory allowing to show that the $(p, \mathcal{B})$-variation of $\mathcal{F}_f$ coincides with its norm $\|f\|_{L^p_{\mathcal{B}}(X)}$.

This paper is divided into three sections. In the first one we introduce the notion of $\mathcal{B}$-variation, $\mathcal{B}$-semivariation of a vector measure and study their connection with the classical notions. We prove that for $(Y, Z, \mathcal{B})$-normed spaces the $\mathcal{B}$-semivariation is equivalent to the semivariation and that the Y-semivariation considered by Bartle coincides the $\mathcal{B}$-variation for a particular bilinear map $\mathcal{B}$. Particularly interesting is the observation that any vector-measure with values in $X = L^1(\mu)$ is of bounded $\mathcal{B}$-variation for every $\mathcal{B}$ whenever Z is a Hilbert space. We also show in this section that the measure $\mathcal{F}_f^{\mathcal{B}}$ is μ-continuous and $\|\mathcal{F}_f^{\mathcal{B}}\|_{\mathcal{B}}(\Omega) = \|f\|_{L^1_{\mathcal{B}}(X)}$. In the next section the natural notion of $(p, \mathcal{B})$-semivariation is introduced. Starting with the case $p = \infty$ we describe, for $1 < p \leq \infty$, the space of measures with bounded $(p, \mathcal{B})$-semivariation as bounded bilinear maps from $L^{p'} \times Y \to Z$. Last section deals with the notion of $(p, \mathcal{B})$-variation of a vector measure. Several characterizations are presented and the new notion of "cone absolutely summing bilinear map" from $L \times Y \to Z$, where L is a Banach lattice, is introduced. This allow us to describe the $(p, \mathcal{B})$-variation of a vector measure as the norm of the corresponding bilinear map from $L^{p'} \times Y$ into Z in this class.

Throughout the paper $\mathcal{F} : \Sigma \to X$ always denotes a vector measure, $\mathcal{B} : X \times Y \to Z$ is admissible and, for each $y \in Y$, $\mathcal{B}(\mathcal{F}, y)$ denotes the Z-valued measure $\mathcal{B}(\mathcal{F}, y)(E) = \mathcal{B}(\mathcal{F}(E), y)$.

2 Variation and semivariation with respect to bilinear maps.

Definition 1 *Let $E \in \Sigma$. We define the $\mathcal{B}$-variation of $\mathcal{F}$ on the set E by*

$$\begin{aligned} |\mathcal{F}|_{\mathcal{B}}(E) &= \sup\{|\mathcal{B}(\mathcal{F}, y)|(E) : y \in \mathrm{B}_Y\} \\ &= \sup\{\sum_{A \in \pi} \|\mathcal{B}(\mathcal{F}(A), y)\|_Z : \pi \in \mathcal{D}_E, y \in \mathrm{B}_Y\}. \end{aligned}$$

We say that $\mathcal{F}$ has bounded $\mathcal{B}$-variation if $|\mathcal{F}|_{\mathcal{B}}(\Omega) < \infty$.

Definition 2 *Let $E \in \Sigma$. We define the $\mathcal{B}$-semivariation of $\mathcal{F}$ on the set E by*

$$\begin{aligned} \|\mathcal{F}\|_{\mathcal{B}}(E) &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|(E) : y \in \mathrm{B}_Y\} \\ &= \sup\{|\langle \mathcal{B}(\mathcal{F}, y), z^* \rangle|(E) : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\} \\ &= \sup\{\sum_{A \in \pi} |\langle \mathcal{B}(\mathcal{F}(A), y), z^* \rangle| : \pi \in \mathcal{D}_E, y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\}. \end{aligned}$$

We say that $\mathcal{F}$ has bounded $\mathcal{B}$-semivariation if $\|\mathcal{F}\|_{\mathcal{B}}(\Omega) < \infty$.

Remark 1 *Let $\mathcal{F}$ be a vector measure and $E \in \Sigma$.*

(a) $|\mathcal{F}|_{\mathcal{B}}(E) \leq \|\mathcal{B}\| \cdot |\mathcal{F}|(E)$.

(b) $\|\mathcal{F}\|_{\mathcal{B}}(E) \leq \|\mathcal{B}\| \cdot \|\mathcal{F}\|(E)$.

(c) $\sup\{\|\mathcal{B}(\mathcal{F}(C), y)\| : y \in \mathrm{B}_Y, E \supseteq C \in \Sigma\} \approx \|\mathcal{F}\|_{\mathcal{B}}(E)$.

In particular any measure has bounded $\mathcal{B}$-semivariation for any $\mathcal{B}$.

We can easily describe the $\mathcal{B}$-variation and $\mathcal{B}$-semivariation of vector measures for the bilinear maps given in (1)-(8). The following results are elementary and left to the reader.

Proposition 1 *Let $\mathcal{F}$ be a vector measure and $E \in \Sigma$.*

(a) $|\mathcal{F}|_{\mathcal{B}}(E) = |\mathcal{F}|(E)$ *and* $\|\mathcal{F}\|_{\mathcal{B}}(E) = \|\mathcal{F}\|(E)$.

(b) $|\mathcal{F}|_{\mathcal{D}}(E) = \|\mathcal{F}\|_{\mathcal{D}}(E) = \|\mathcal{F}\|(E)$.

(c) $|\mathcal{F}|_{\mathcal{D}_1}(E) = \|\mathcal{F}\|_{\mathcal{D}_1}(E) = \|\mathcal{F}\|(E)$.

(d) $|\mathcal{F}|_{\pi_Y}(E) = |\mathcal{F}|(E)$ *and* $\|\mathcal{F}\|_{\pi_Y}(E) = \|\mathcal{F}\|(E)$ *(see Proposition 4).*

(e) $|\mathcal{F}|_{\widetilde{\mathcal{O}}_Y}(E) = \sup\{|T\mathcal{F}|(E) : T \in \mathrm{B}_{\mathcal{L}(Y,Z)}\}$ *and* $\|\mathcal{F}\|_{\widetilde{\mathcal{O}}_Y}(E) = \|\mathcal{F}\|(E)$.

(f) $|\mathcal{F}|_{\mathcal{O}_{Y,Z}}(E) = |\!|\!|\mathcal{F}|\!|\!|(E)$ *and* $\|\mathcal{F}\|_{\mathcal{O}_{Y,Z}}(E) = \|\mathcal{F}\|(E)$.

The notion of $\mathcal{B}$-normed space can be described in terms of vector measures.

Proposition 2 *Let $\mathcal{B} : X \times Y \to Z$ be an admissible bounded bilinear map. Then X is $(Y, Z, \mathcal{B})$-normed if and only if for any vector measure $\mathcal{F} : \Sigma \to X$ there exist $C_1, C_2 > 0$ such that*

$$C_1\|\mathcal{F}\|(E) \leq \|\mathcal{F}\|_{\mathcal{B}}(E) \leq C_2\|\mathcal{F}\|(E)$$

for all $E \in \Sigma$.

Proof. Obviously $\|\mathcal{F}\|_{\mathcal{B}}(E) \leq \|\mathcal{B}\| \cdot \|\mathcal{F}\|(E)$ for any $E \in \Sigma$. Assume X is $(Y, Z, \mathcal{B})$-normed. Then we have that

$$\begin{aligned}
\|\mathcal{F}\|(E) &= \sup\{\|\sum_{A\in\pi} \epsilon_A \mathcal{F}(A)\|_X : \pi \in \mathcal{D}_E,\ \epsilon_A \in \mathrm{B}_{\mathbb{K}}\} \\
&\leq k \sup\{\|\mathcal{B}_{\sum_{A\in\pi} \epsilon_A \mathcal{F}(A)}\|_{\mathcal{L}(Y,Z)} : \pi \in \mathcal{D}_E,\ \epsilon_A \in \mathrm{B}_{\mathbb{K}}\} \\
&= k \sup\{\|\sum_{A\in\pi} \epsilon_A \mathcal{B}(\mathcal{F}(A), y)\|_Z : \pi \in \mathcal{D}_E,\ \epsilon_A \in \mathrm{B}_{\mathbb{K}},\ y \in \mathrm{B}_Y\} \\
&= k\|\mathcal{F}\|_{\mathcal{B}}(E).
\end{aligned}$$

Conversely, for each $x \in X$ select the measure $\mathcal{F}_x(A) = x\mu(A)\mu(\Omega)^{-1}$ and observe that $\|\mathcal{F}_x\|(\Omega) = \|x\|$ and $\|\mathcal{F}_x\|_{\mathcal{B}}(\Omega) = \|\mathcal{B}_x\|$. ■

We use $\mathcal{B}^*$ for the "adjoint" bilinear map from $X \times Z^*$ to Y^*, i.e. $(\mathcal{B}^*)_x = (\mathcal{B}_x)^*$ or

$$\mathcal{B}^* : X \times Z^* \to Y^*, \quad \text{given by } \langle y, \mathcal{B}^*(x, z^*)\rangle = \langle \mathcal{B}(x, y), z^*\rangle.$$

Note that $\mathcal{B}^* = \mathcal{D}$, $\mathcal{D}_1^* = \mathcal{B}$, $(\pi_Y)^* = \widetilde{\mathcal{O}}_{Y^*}$ and $(\mathcal{O}_{Y,Z})^*(T, z^*) = \mathcal{O}_{Z^*,Y^*}(T^*, z^*)$.

Let us see that the $\mathcal{B}$-semivariation and the $\mathcal{B}^*$-semivariation always coincide.

Proposition 3 *$\|\mathcal{F}\|_{\mathcal{B}}(E) = \|\mathcal{F}\|_{\mathcal{B}^*}(E)$ for all $E \in \Sigma$.*

Proof. Let us take $E \in \Sigma$. Then

$$\begin{aligned}
\|\mathcal{F}\|_{\mathcal{B}}(E) &= \sup\{\sum_{A\in\pi} |\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle| : \pi \in \mathcal{D}_E, y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\} \\
&= \sup\{\sum_{A\in\pi} |\langle y, \mathcal{B}^*(\mathcal{F}(A), z^*)\rangle| : \pi \in \mathcal{D}_E, y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\} \\
&= \sup\{|\sum_{A\in\pi} \epsilon_A \langle y, \mathcal{B}^*(\mathcal{F}(A), z^*)\rangle| : \pi \in \mathcal{D}_E, y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}, \epsilon_A \in \mathrm{B}_{\mathbb{K}}\} \\
&= \sup\{\|\sum_{A\in\pi} \epsilon_A \mathcal{B}^*(\mathcal{F}(A), z^*)\|_{Y^*} : \pi \in \mathcal{D}_E, z^* \in \mathrm{B}_{Z^*}, \epsilon_A \in \mathrm{B}_{\mathbb{K}}\} \\
&= \sup\{\sum_{A\in\pi} |\langle \mathcal{B}^*(\mathcal{F}(A), z^*), y^{**}\rangle| : \pi \in \mathcal{D}_E, y^{**} \in \mathrm{B}_{Y^{**}}, z^* \in \mathrm{B}_{Z^*}\} \\
&= \|\mathcal{F}\|_{\mathcal{B}^*}(E).
\end{aligned}$$

■

Proposition 4 *Let τ be a norm in $Y \otimes X$ with $\|y \otimes x\|_\tau = \|y\|\|x\|$ for all $y \in Y$ and $x \in X$. Define $\tau_Y : X \times Y \to Y\hat{\otimes}_\tau X$ given by $(x, y) \to y \otimes x$. Then, for each $E \in \Sigma$,*

$$\beta_Y(\mathcal{F}, \tau)(E) = |\mathcal{F}|_{(\tau_Y)^*}(E).$$

Proof. Taking into account that $Y\hat{\otimes}_\pi X \subseteq Y\hat{\otimes}_\tau X$, then $(Y\hat{\otimes}_\tau X)^*$ can be regarded as a subspace of the space of bounded operators $\mathcal{L}(Y, X^*)$. Moreover $\|T\| \le \|T\|_{(Y\hat{\otimes}_\tau X)^*}$ for any $T \in (Y\hat{\otimes}_\tau X)^*$, where the duality is given by

$$\langle T, \sum_{j=1}^k y_j \otimes x_j\rangle = \sum_{j=1}^k \langle x_j, T(y_j)\rangle.$$

From [3, Theorem 2.1]

$$\beta_Y(\mathcal{F}, \tau)(E) \approx \sup\{|T\mathcal{F}|(E) : T \in \mathcal{L}(Y, X^*), \|T\|_{(Y\hat{\otimes}_\tau X)^*} \le 1\}.$$

Hence

$$\beta_Y(\mathcal{F}, \tau)(E) \approx \sup\{|(\tau_Y)^*(\mathcal{F}, T)|(E) : T \in \mathcal{L}(Y, X^*), \|T\|_{(Y\hat{\otimes}_\tau X)^*} \le 1\}.$$

■

Of course vector measures need not be of bounded $\mathcal{B}$-variation for a general $\mathcal{B}$ (it suffices to take $\mathcal{B}$ such that $|\mathcal{F}|_{\mathcal{B}} = |\mathcal{F}|$), but there are cases where this happens to be true due to the geometrical properties of the spaces into consideration.

Proposition 5 *Let $X = L^1(\nu)$ for some σ-finite measure ν and let $Z = H$ be a Hilbert space. Then any vector measure $\mathcal{F} : \Sigma \to L^1(\nu)$ is of bounded $\mathcal{B}$-variation for any bounded bilinear map $\mathcal{B} : L^1(\nu) \times Y \to H$ and any Banach space Y.*

Proof. Recall first that Grothendieck theorem (see [7]) establishes that there exists a constant $\kappa_G > 0$ such that any operator from $L^1(\nu)$ to a Hilbert space H satisfies

$$\sum_{n=1}^N \|T(\phi_n)\|_H \le \kappa_G \|T\| \sup\left\{\big\|\sum_{n=1}^N \varepsilon_n \phi_n\big\|_{L^1(\nu)} : \varepsilon_n \in \mathcal{B}_{\mathbb{K}}\right\}$$

for any finite family of functions $(\phi_n)_n$ in $L^1(\nu)$.
If $\mathcal{F}:\Sigma\to L^1(\nu)$ is a vector measure and π a partition one has that $\|\sum_{A\in\pi}\epsilon_A\mathcal{F}(A)\|_{L^1(\nu)}\leq\|\mathcal{F}\|(\Omega)$. Hence $\mathcal{B}^y\in L(L^1(\nu)\to H$ for any $y\in Y$, one obtains

$$\sum_{A\in\pi}\|\mathcal{B}^y(\mathcal{F}(A))\|_Z\leq\kappa_G\cdot\|\mathcal{B}^y\|\cdot\|\mathcal{F}\|(\Omega).$$

Therefore one concludes $|\mathcal{F}|_{\mathcal{B}}(\Omega)\leq\kappa_G\cdot\|\mathcal{F}\|(\Omega)$.

■

Recall that a vector measure $\mathcal{F}:\Sigma\to X$ is called μ-continuous if $\lim_{\mu(E)\to 0}\|\mathcal{F}\|(E)=0$.

Theorem 1 *Let X be $(Y,Z,\mathcal{B})$-normed and $f\in L^1_{\mathcal{B}}(X)$. Then*

$$\mathcal{F}^{\mathcal{B}}_f:\Sigma\to X,\quad \textit{given by}\quad \mathcal{F}^{\mathcal{B}}_f(E)=\int_E^{\mathcal{B}}fd\mu \tag{10}$$

is a μ-continuous vector measure of bounded $\mathcal{B}$-variation. Moreover $|\mathcal{F}^{\mathcal{B}}_f|_{\mathcal{B}}(\Omega)=\|f\|_{L^1_{\mathcal{B}}(X)}$.

Proof. It was shown (see [4, Theorem 1]) that functions in $L^1_{\mathcal{B}}(X)$ are Pettis-integrable and $\int_E^{\mathcal{B}}fd\mu$ coincides with the Pettis-integral. Hence $\mathcal{F}^{\mathcal{B}}_f$ defines a vector measure.

Using now that, for each $y\in Y$, the vector measure $\mathcal{B}(\mathcal{F}^{\mathcal{B}}_f,y)$ has density $\mathcal{B}(f,y)$ which belongs to $L^1(Z)$ one gets that, for any $E\in\Sigma$,

$$|\mathcal{B}(\mathcal{F}_f,y)|(E)=\int_E\|\mathcal{B}(f,y)\|_Zd\mu.$$

Thus $|\mathcal{F}^{\mathcal{B}}_f|_{\mathcal{B}}(\Omega)=\|f\|_{L^1_{\mathcal{B}}(X)}$. It remains to show that $\mathcal{F}^{\mathcal{B}}_f$ is μ-continuous. Let us fix $\varepsilon>0$ and select, using that $f\in L^1_{\mathcal{B}}(X)$, a simple function s such that $\|f-s\|_{L^1_{\mathcal{B}}(X)}\leq\varepsilon$. Thus

$$\begin{aligned}
\|\mathcal{F}^{\mathcal{B}}_f(E)\|_X &\leq \|\int_E^{\mathcal{B}}(f-s)d\mu\|_X+\|\int_E^{\mathcal{B}}sd\mu\|_X\\
&= \|\int_E^{\mathcal{B}}(f-s)d\mu\|_X+\|\int_E sd\mu\|_X\\
&\leq k\|\mathcal{B}_{\int_E^{\mathcal{B}}(f-s)d\mu}\|_{\mathcal{L}(Y,Z)}+\|\int_E sd\mu\|_X\\
&\leq k\sup\{\int_E\|\mathcal{B}(f-s,y)\|_Zd\mu: y\in \mathrm{B}_Y\}+\|\int_E sd\mu\|_X\\
&\leq k\varepsilon+\|\int_E sd\mu\|_X.
\end{aligned}$$

We have the conclusion just taking limits when $\mu(E)\to 0$ and $\varepsilon\to 0^+$.

■

Corollary 1 *Let X is $(Y,Z,\mathcal{B})$-normed and $f\in L^1_{\mathcal{B}}(X)$. If $\int_E^{\mathcal{B}}fd\mu=0$ for all $E\in\Sigma$ then $f=0$ a.e. in Ω.*

3 Measures of bounded $(p, \mathcal{B})$-semivariation.

Extending the notion for $\mathcal{B} = \mathcal{B}$, we say that a vector measure $\mathcal{F} : \Sigma \to X$ is $(\mathcal{B}, \mu)$-continuous if $\lim_{\mu(E)\to 0} \|\mathcal{F}\|_{\mathcal{B}}(E) = 0$. Clearly both concepts coincide for $\mathcal{B}$-normed spaces.

Definition 3 *We say that $\mathcal{F}$ has bounded $(\infty, \mathcal{B})$-semivariation if there exists $C > 0$ such that*

$$|\langle \mathcal{B}(\mathcal{F}(A), y), z^* \rangle| \leq C \cdot \|y\| \cdot \|z^*\| \cdot \mu(A), \quad y \in Y,\ z^* \in Z^*,\ A \in \Sigma. \tag{11}$$

The space of such measures is denoted by $\mathcal{V}^{\infty}_{\mathcal{B}}(X)$ and we set

$$\begin{aligned} \|\mathcal{F}\|_{\mathcal{V}^{\infty}_{\mathcal{B}}(X)} &= \inf\{C : \textit{ satisfying } (11)\ \} \\ &= \sup\{\frac{|\langle \mathcal{B}(\mathcal{F}(A), y), z^* \rangle|}{\mu(A)} : y \in \mathrm{B}_Y,\ z^* \in \mathrm{B}_{Z^*},\ A \in \Sigma,\ \mu(A) > 0\}. \end{aligned}$$

Observe that every vector measure $\mathcal{F}$ belonging to $\mathcal{V}^{\infty}_{\mathcal{B}}(X)$ is $(\mathcal{B}, \mu)$-continuous and it has bounded $\mathcal{B}$-variation. Also note that $\mathcal{F}$ has bounded $(\infty, \mathcal{B})$-semivariation if and only if

$$\|\mathcal{B}(\mathcal{F}(A), y)\| \leq C\|y\|\mu(A), \quad y \in Y,\ A \in \Sigma,$$

or

$$\|\mathcal{F}\|_{\mathcal{B}}(A) \leq C\mu(A), \quad A \in \Sigma,$$

or

$$|\mathcal{F}|_{\mathcal{B}}(A) \leq C\mu(A), \quad A \in \Sigma.$$

It is elementary to see, due to the admissibility of $\mathcal{B}$, that $\|\mathcal{F}\|_{\mathcal{V}^{\infty}_{\mathcal{B}}(X)}$ is a norm

Of course

$$\begin{aligned} \|\mathcal{F}\|_{\mathcal{V}^{\infty}_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^{\infty}(Z)} : y \in \mathrm{B}_Y\} \\ &= \sup\left\{\frac{\|\mathcal{B}(\mathcal{F}(A), y)\|_Z}{\mu(A)} : y \in \mathrm{B}_Y, A \in \Sigma\right\} \\ &= \sup\left\{\frac{\|\mathcal{F}\|_{\mathcal{B}}(A)}{\mu(A)} : A \in \Sigma\right\} \\ &= \sup\left\{\frac{|\mathcal{F}|_{\mathcal{B}}(A)}{\mu(A)} : A \in \Sigma\right\}. \end{aligned}$$

Proposition 6 *$\mathcal{F} \in \mathcal{V}^{\infty}_{\mathcal{B}}(X)$ if and only if there exists a bounded bilinear map $\mathcal{B}_{\mathcal{F}} : L^1 \times Y \to Z$ such that*

$$\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y) = \mathcal{B}(\mathcal{F}(A), y), \quad A \in \Sigma, y \in Y.$$

Moreover $\|\mathcal{B}_{\mathcal{F}}\| = \|\mathcal{F}\|_{\mathcal{V}^{\infty}_{\mathcal{B}}(X)}$.

Proof. Assume $\mathcal{F} \in \mathcal{V}^{\infty}_{\mathcal{B}}(X)$. Define $\mathcal{B}_{\mathcal{F}}$ on simple functions by the formula

$$\mathcal{B}_{\mathcal{F}}(\sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}, y) = \sum_{k=1}^{n} \alpha_k \mathcal{B}(\mathcal{F}(A_k), y).$$

Observe that

$$\|\mathcal{B}_{\mathcal{F}}(\sum_{k=1}^{n} \alpha_k \mathbf{1}_{A_k}, y)\|_Z \leq \|\mathcal{F}\|_{\mathcal{V}^{\infty}_{\mathcal{B}}(X)} \|y\| \sum_{k=1}^{n} |\alpha_k| \mu(A_k).$$

This allows to extend the bilinear map to $L^1 \times Y \to Z$ with norm $\|\mathcal{B}_{\mathcal{F}}\| \le \|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$. Conversely one has

$$\|\mathcal{B}(\mathcal{F}(A), y)\|_Z \le \|\mathcal{B}_{\mathcal{F}}\| \cdot \|y\| \cdot \|\mathbf{1}_A\|_{L^1}$$

which gives $\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} \le \|\mathcal{B}_{\mathcal{F}}\|$.

■

We use the notation $\mathrm{Bil}(L^1 \times Y, Z)$ for the space of bounded bilinear maps with its natural norm.

Corollary 2 *$\mathcal{V}^\infty_{\mathcal{B}}(X)$ is isometrically embedded into $\mathrm{Bil}(L^1 \times Y, Z)$. In the case $\mathcal{B} = \mathcal{O}_{Y,Z}$ we have $\mathcal{V}^\infty_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y, Z)) = \mathrm{Bil}(L^1 \times Y, Z)$.*

Let $L^\infty_{\mathcal{B}}(X)$ stand for the space of measurable functions $f : \Omega \to X$ such that $\mathcal{B}(f, y) \in L^\infty(Z)$ for all $y \in Y$ and write

$$\|f\|_{L^\infty_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^\infty(Z)} : y \in \mathrm{B}_Y\}.$$

Note that $L^\infty_{\mathcal{B}}(X) \subseteq L^1_{\mathcal{B}}(X)$ and $|\mathcal{B}(\mathcal{F}^{\mathcal{B}}_f, y)|(A) = \int^{\mathcal{B}}_A \|\mathcal{B}(f, y)\| d\mu$ for any set $A \in \Sigma$. In particular if $f \in L^\infty_{\mathcal{B}}(X)$ then the measure $\mathcal{F}^{\mathcal{B}}_f \in \mathcal{V}^\infty_{\mathcal{B}}(X)$ and $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} = \|f\|_{L^\infty_{\mathcal{B}}(X)}$.

Proposition 7 *The following are equivalent:*

(a) *X is $(Y, Z, \mathcal{B})$-normed.*

(b) *$\mathcal{V}^\infty_{\mathcal{B}}(X) = V^\infty(X)$.*

(c) *There exists $k > 0$ such that $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^\infty(X)} \le k\|f\|_{L^\infty_{\mathcal{B}}(X)}$ for any $f \in L^\infty_{\mathcal{B}}(X)$.*

Proof. (a) $\Longrightarrow$ (b) Always $V^\infty(X) \subseteq \mathcal{V}^\infty_{\mathcal{B}}(X)$. Assume X is $(Y, Z, \mathcal{B})$-normed and $\mathcal{F} \in \mathcal{V}^\infty_{\mathcal{B}}(X)$. Note that

$$\|\mathcal{F}(A)\| \le k\|\mathcal{B}_{\mathcal{F}(A)}\| \le k\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}\mu(A).$$

(b) $\Longrightarrow$ (c) Let $f \in L^\infty_{\mathcal{B}}(X)$. Clearly

$$\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^\infty(X)} \le k\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)} = k\|f\|_{L^\infty_{\mathcal{B}}(X)}.$$

(c) $\Longrightarrow$ (a) Let us take $f_x = x\mathbf{1}_\Omega$ for a given $x \in X$ and observe that $\mathcal{F}^{\mathcal{B}}_{f_x}(A) = x\mu(A)$ for all $A \in \Sigma$. Note that $\|f_x\|_{L^\infty_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$ and $\|\mathcal{F}^{\mathcal{B}}_{f_x}\|_{\mathrm{V}^\infty(X)} = \|x\|$.

■

Definition 4 *Let $1 \le p < \infty$. We say that $\mathcal{F}$ has bounded $(p, \mathcal{B})$-semivariation if*

$$\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} = \sup\left\{\left(\sum_{A\in\pi} \frac{|\langle\mathcal{B}(\mathcal{F}(A), y), z^*\rangle|^p}{\mu(A)^{p-1}}\right)^{\frac{1}{p}} : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}, \pi \in \mathcal{D}_\Omega\right\} < \infty.$$

The space of such measures will be denoted by $\mathcal{V}^p_{\mathcal{B}}(X)$.

We have the equivalent formulation

$$\begin{aligned}\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathcal{V}^p(Z)} : y \in \mathrm{B}_Y\} \\ &= \sup\{\|\langle\mathcal{B}(\mathcal{F}, y), z^*\rangle\|_{\mathcal{V}^p} : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}\}.\end{aligned}$$

Let us start with the following description.

Proposition 8 *Let $1 < p < \infty$. Then $\mathcal{F} \in \mathcal{V}^p_{\mathcal{B}}(X)$ if and only if there exists a bounded bilinear map $\mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ such that*

$$\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y) = \mathcal{B}(\mathcal{F}(A), y), \quad A \in \Sigma, y \in Y.$$

Moreover $\|\mathcal{B}_{\mathcal{F}}\| = \|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}$.

Proof Assume $\mathcal{F} \in \mathcal{V}^p_{\mathcal{B}}(X)$. Define as above $\mathcal{B}_{\mathcal{F}}$ on simple functions by the formula

$$\mathcal{B}_{\mathcal{F}}(\sum_{k=1}^n \alpha_k \mathbf{1}_{A_k}, y) = \sum_{k=1}^n \alpha_k \mathcal{B}(\mathcal{F}(A_k), y).$$

We use that

$$\begin{aligned}
\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} &= \sup\left\{\left|\sum_{A\in\pi} \frac{\langle \mathcal{B}(\mathcal{F}(A), y), z^*\rangle \gamma_A}{\mu(A)^{1/p'}}\right| : y \in \mathrm{B}_Y, z^* \in \mathrm{B}_{Z^*}, \pi \in \mathcal{D}_\Omega,\ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\right\} \\
&= \sup\left\{\left\|\sum_{A\in\pi} \mathcal{B}(\mathcal{F}(A), y)\beta_A\right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\right\} \\
&= \sup\left\{\left\|\mathcal{B}_{\mathcal{F}}(\sum_{A\in\pi} \beta_A \mathbf{1}_A, y)\right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}}\right\}.
\end{aligned}$$

Hence using the density of simple functions we extend to $L^{p'}$ and $\|\mathcal{B}_{\mathcal{F}}\| \leq \|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}$. The converse follows also from the previous formula.

■

It is known that $\mathcal{V}^p(X) = \mathcal{L}(L^{p'}, X)$ (see [11]). Next result is the analogue in the bilinear setting.

Corollary 3 *Let $1 < p < \infty$. $\mathcal{V}^p_{\mathcal{B}}(X)$ is isometrically embedded into $\mathrm{Bil}(L^{p'} \times Y, Z)$. In the case $\mathcal{B} = \mathcal{O}_{Y,Z}$ we have $\mathcal{V}^p_{\mathcal{O}_{Y,Z}}(\mathcal{L}(Y, Z)) = \mathrm{Bil}(L^{p'} \times Y, Z)$.*

Proposition 9 *Let $\mathcal{B} : X \times Y \to Z$ be an admissible bounded bilinear map and $1 < p < \infty$. Then X is $(Y, Z, \mathcal{B})$-normed if and only if the space $\mathcal{V}^p_{\mathcal{B}}(X)$ is continuously contained into $\mathcal{V}^p(X)$.*

Proof. Assume X is $(Y, Z, \mathcal{B})$-normed.

$$\begin{aligned}
\|\mathcal{F}\|_{\mathcal{V}^p(X)} &= \sup\left\{\left|\sum_{A\in\pi} \frac{\langle \mathcal{F}(A), x^*\rangle \gamma_A}{\mu(A)^{1/p'}}\right| : x^* \in \mathrm{B}_{X^*}, \pi \in \mathcal{D}_\Omega,\ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\right\} \\
&= \sup\left\{\left\|\sum_{A\in\pi} \frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}\right\|_X : \pi \in \mathcal{D}_\Omega,\ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\right\} \\
&\leq k \sup\left\{\left\|\mathcal{B}_{\sum \frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}}\right\|_{\mathcal{L}(Y,Z)} : \pi \in \mathcal{D}_\Omega,\ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\right\} \\
&= k \sup\left\{\left\|\sum_{A\in\pi} \mathcal{B}(\frac{\mathcal{F}(A)\gamma_A}{\mu(A)^{1/p'}}, y)\right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega,\ (\gamma_A)_A \in \mathrm{B}_{\ell^{p'}}\right\} \\
&= k \sup\left\{\left(\sum_{A\in\pi} \frac{\|\mathcal{B}(\mathcal{F}(A), y)\|^p}{\mu(A)^{p-1}}\right)^{1/p} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega,\right\} \\
&= k\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)}.
\end{aligned}$$

For the converse consider the vector measure $\mathcal{F}_x : \Sigma \to X$ given by $\mathcal{F}_x(A) = x\mu(A)\mu(\Omega)^{-1}$ for each $x \in X$. Note that $\|\mathcal{F}_x\|_{\mathcal{V}^p(X)} = \|x\|$ and $\|\mathcal{F}_x\|_{\mathcal{V}^p_{\mathcal{B}}(X)} = \|\mathcal{B}_x\|$.

■

4 Measures of bounded $(p, \mathcal{B})$-variation.

Definition 5 *We say that $\mathcal{F}$ has bounded $(p, \mathcal{B})$-variation if*

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\left\{ \left(\sum_{A\in\pi} \frac{\|\mathcal{B}(\mathcal{F}(A), y)\|_Z^p}{\mu(A)^{p-1}} \right)^{\frac{1}{p}} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega \right\} < \infty.$$

The space of such measures will be denoted by $\mathrm{V}^p_{\mathcal{B}}(X)$.

It is clear that the norm in the vector space $\mathrm{V}^p_{\mathcal{B}}(X)$ is also given by the expressions

$$\begin{aligned} \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\{\|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^p(Z)} : y \in \mathrm{B}_Y\} \\ &= \sup\left\{ \left\| \sum_{A\in\pi} \frac{\mathcal{B}(\mathcal{F}(A), y)}{\mu(A)} \mathbf{1}_A \right\|_{L^p(Z)} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega \right\} \\ &= \sup\left\{ \left\| \sum_{A\in\pi} \frac{\mathcal{F}(A)}{\mu(A)} \mathbf{1}_A \right\|_{L^p_{\mathcal{B}}(X)} : \pi \in \mathcal{D}_\Omega \right\}. \end{aligned}$$

Remark 2 *For $p = 1$ and $p = \infty$ this corresponds to $|\mathcal{F}|_{\mathcal{B}}(\Omega)$ and $\|\mathcal{F}\|_{\mathcal{V}^\infty_{\mathcal{B}}(X)}$. Hence we define $\mathrm{V}^\infty(X) = \mathcal{V}^\infty(X)$.*

It is clear that $\mathrm{V}^p_{\mathcal{B}}(X) \subseteq \mathcal{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}\|_{\mathcal{V}^p_{\mathcal{B}}(X)} \le \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

On the other hand, since

$$|\mathcal{F}|_{\mathcal{B}}(E) \le \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \|\mathbf{1}_E\|_{L^{p'}}, \quad E \in \Sigma,$$

one sees that if $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ then $\mathcal{F}$ has bounded $\mathcal{B}$-variation and it is $(\mathcal{B}, \mu)$-continuous.

Remark 3 *Using the inclusions $L^q(X) \subseteq L^p(X)$ for $1 \le p \le q \le \infty$ one also has*

$$\mathrm{V}^\infty_{\mathcal{B}}(X) \subseteq \mathrm{V}^q_{\mathcal{B}}(X) \subseteq \mathrm{V}^p_{\mathcal{B}}(X)$$

and

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \le \mu(\Omega)^{1/q-1/p} \|\mathcal{F}\|_{\mathrm{V}^q_{\mathcal{B}}(X)} \le \mu(\Omega)^{1/q} \|\mathcal{F}\|_{\mathrm{V}^\infty_{\mathcal{B}}(X)}.$$

Let us find different equivalent formulations for the norm in $\mathrm{V}^p_{\mathcal{B}}(X)$.

Proposition 10

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\left\{ \sum_{A\in\pi} \|\mathcal{B}(\mathcal{F}(A), \beta_A y)\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \right\}. \tag{12}$$

$$\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\left\{ \left\| \sum_{A\in\pi} \mathcal{B}^*(\mathcal{F}(A), z_A^*) \right\|_{Y^*} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} z_A^* \mathbf{1}_A \in \mathrm{B}_{L^{p'}(Z^*)} \right\}. \tag{13}$$

PROOF. Given a partition $\pi \in \mathcal{D}_\Omega$, $\alpha_A \in \mathbb{R}$ and $\beta_A = \frac{\alpha_A}{\mu(A)^{1/p'}}$ one has that the simple function $g = \sum_{A\in\pi} \beta_A \mathbf{1}_A$ satisfies $\|g\|_{L^{p'}} = \|(\alpha_A)_{A\in\pi}\|_{\ell^{p'}}$. Therefore

$$
\begin{aligned}
\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\left\{ \left\| \left(\left\| \mathcal{B}\left(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}}, y \right) \right\|_Z \right)_{A\in\pi} \right\|_{\ell^p} : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega \right\} \\
&= \sup\left\{ \sum_{A\in\pi} \left\| \mathcal{B}\left(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}}, y \right) \right\|_Z |\alpha_A| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, (\alpha_A)_{A\in\pi} \in \mathrm{B}_{\ell^{p'}} \right\} \\
&= \sup\left\{ \sum_{A\in\pi} \left\| \mathcal{B}\left(\mathcal{F}(A) \frac{\alpha_A}{\mu(A)^{1/p'}}, y \right) \right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, (\alpha_A)_{A\in\pi} \in \mathrm{B}_{\ell^{p'}} \right\} \\
&= \sup\left\{ \sum_{A\in\pi} \left\| \mathcal{B}\left(\mathcal{F}(A), \beta_A y \right) \right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \right\}.
\end{aligned}
$$

We get (13) from the duality $(\ell^1(Z))^* = \ell^\infty(Z^*)$ and (12). Indeed,

$$
\begin{aligned}
\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} &= \sup\left\{ \sum_{A\in\pi} \left\| \mathcal{B}\left(\mathcal{F}(A), \beta_A y \right) \right\|_Z : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \right\} \\
&= \sup\left\{ \left| \sum_{A\in\pi} \left\langle \mathcal{B}\left(\mathcal{F}(A), \beta_A y \right), z^*_A \right\rangle \right| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, z^*_A \in \mathrm{B}_{Z^*}, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \right\} \\
&= \sup\left\{ \left| \sum_{A\in\pi} \left\langle y, \mathcal{B}^*\left(\mathcal{F}(A), \beta_A z^*_A \right) \right\rangle \right| : y \in \mathrm{B}_Y, \pi \in \mathcal{D}_\Omega, z^*_A \in \mathrm{B}_{Z^*}, \sum_{A\in\pi} \beta_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}} \right\} \\
&= \sup\left\{ \left\| \sum_{A\in\pi} \mathcal{B}^*\left(\mathcal{F}(A), z^*_A \right) \right\|_{Y^*} : \pi \in \mathcal{D}_\Omega, \sum_{A\in\pi} z^*_A \mathbf{1}_A \in \mathrm{B}_{L^{p'}(Z^*)} \right\}.
\end{aligned}
$$

■

Let us give a characterization of the vector measures in the space $\mathrm{V}^p_{\mathcal{B}}(X)$ using only scalar valued functions $\{\varphi_y : y \in \mathrm{B}_Y\} \subseteq L^p$.

Theorem 2 *$\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ if and only if there exist $0 \le \varphi_y \in L^p$ for each $y \in Y$ such that*

- (a) $\sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\} < \infty$ *and*
- (b) $\|\mathcal{B}(\mathcal{F}(E), y)\| \le \int_E \varphi_y d\mu$ *for every $y \in Y$ and $E \in \Sigma$.*

Moreover $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$.

PROOF. Let $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$. Then we have that $\mathcal{B}(\mathcal{F}, y) \in \mathrm{V}^p(Z)$ for all $y \in \mathrm{B}_Y$ and $|\mathcal{B}(\mathcal{F}, y)|$ is a non negative μ-continuous measure that has bounded variation. Using the Radon-Nikodým theorem there exists a non negative integrable function φ_y such that for all $E \in \Sigma$

$$|\mathcal{B}(\mathcal{F}, y)|(E) = \int_E \varphi_y d\mu. \tag{14}$$

In fact φ_y can be chosen belonging to L^p and verifying that $\|\varphi_y\|_{L^p} = \|\mathcal{B}(\mathcal{F}, y)\|_{\mathrm{V}^p(Z)}$.

Then for every $E \in \Sigma$ and $y \in \mathrm{B}_Y$

$$\|\mathcal{B}(\mathcal{F}(E), y)\| \leq |\mathcal{F}|_{\mathcal{B}}(E) = \sup\{|\mathcal{B}(\mathcal{F}, y)|(E) : y \in \mathrm{B}_Y\} = \sup\{\int_E \varphi_y d\mu : y \in \mathrm{B}_Y\}$$

and we obtain the result.

Conversely observe that using Hölder's inequality we have that

$$\|\mathcal{B}(\mathcal{F}(E), y)\| \leq \int_E \varphi_y d\mu \leq \big(\int_E \varphi_y^p d\mu\big)^{1/p} \mu(E)^{1/p'}$$

for all $E \in \Sigma$ and $y \in \mathrm{B}_Y$. Hence for every $\pi \in \mathcal{D}_\Omega$

$$\sum_{A \in \pi} \frac{\|\mathcal{B}(\mathcal{F}(A), y)\|^p}{\mu(A)^{p-1}} \leq \int_\Omega \varphi_y^p d\mu.$$

This shows that $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \leq \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$.

■

Let us now see the analogue to Theorem 1 in the cases $1 < p < \infty$.

Theorem 3 *Assume X is $(Y, Z, \mathcal{B})$-normed and $1 < p < \infty$. If $f \in L^p_{\mathcal{B}}(X)$ then $\mathcal{F}^{\mathcal{B}}_f \in \mathrm{V}^p_{\mathcal{B}}(X)$ and $\|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \|f\|_{L^p_{\mathcal{B}}(X)}$.*

Proof. Let us take $f \in L^p_{\mathcal{B}}(X)$. From Theorem 1 one knows that $\mathcal{F}^{\mathcal{B}}_f : \Sigma \to X$ is a vector measure of bounded variation. Now, for each $y \in Y$, $\mathcal{B}^y \mathcal{F}^{\mathcal{B}}_f : \Sigma \to Z$ is a vector measure verifying that

$$\mathcal{B}^y \mathcal{F}^{\mathcal{B}}_f(E) = \mathcal{B}(\mathcal{F}^{\mathcal{B}}_f(E), y) = \mathcal{B}(\int_E^{\mathcal{B}} f d\mu, y) = \int_E \mathcal{B}(f, y) d\mu, \quad E \in \Sigma.$$

Therefore we have

$$\|f\|_{L^p_{\mathcal{B}}(X)} = \sup\{\|\mathcal{B}(f, y)\|_{L^p(Z)} : y \in \mathrm{B}_Y\} = \sup\{\|\mathcal{B}(\mathcal{F}^{\mathcal{B}}_f, y)\|_{\mathrm{V}^p(Z)} : y \in \mathrm{B}_Y\} = \|\mathcal{F}^{\mathcal{B}}_f\|_{\mathrm{V}^p_{\mathcal{B}}(X)}.$$

■

Corollary 4 *If X is $(Y, Z, \mathcal{B})$-normed then $L^p_{\mathcal{B}}(X)$ is isometrically contained into $\mathrm{V}^p_{\mathcal{B}}(X)$.*

From the definition one clearly has the following interpretations of $\mathrm{V}^p_{\mathcal{B}}(X)$ as operators:

$\mathrm{V}^p_{\mathcal{B}}(X)$ *is isometrically embedded into $\mathcal{L}(Y, \mathrm{V}^p(Z))$ by composition, i.e. $\mathcal{F} \to \Phi_{\mathcal{F}}(y) = \mathcal{B}^y \mathcal{F}$.*

Let us see other processes that generate operators from vector measures: Given a vector measure $\mathcal{F} : \Sigma \to X$ and a bounded bilinear map $\mathcal{B} : X \times Y \to Z$ we can consider the operators $T^{\mathcal{B}}_{\mathcal{F}}$ (resp. $S^{\mathcal{B}}_{\mathcal{F}}$) defined on Y-valued simple functions $s = \sum_{k=1}^n y_k \mathbf{1}_{A_k}$ (resp. Z^*-valued simple functions $t = \sum_{k=1}^n z_k^* \mathbf{1}_{A_k}$) by

$$T^{\mathcal{B}}_{\mathcal{F}}(s) = \sum_{k=1}^n \mathcal{B}(\mathcal{F}(A_k), y_k)$$

and

$$S^{\mathcal{B}}_{\mathcal{F}}(t) = \sum_{k=1}^n \mathcal{B}^*(\mathcal{F}(A_k), z_k^*).$$

Observe that actually $S^{\mathcal{B}}_{\mathcal{F}} = T^{\mathcal{B}^*}_{\mathcal{F}}$.

Theorem 4 *Let $1 < p < \infty$. $\mathrm{V}^p_{\mathcal{B}}(X)$ is continuously contained into $\mathcal{L}(L^{p'}\hat{\otimes}Y, Z)$.*

PROOF. Let $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$. Consider the linear operator $T^{\mathcal{B}}_{\mathcal{F}}$ defined on Y-valued simple functions and with values in Z. Note that for any partition π, $\phi = \sum_{A\in\pi} \alpha_A \mathbf{1}_A$ and $y \in Y$

$$\|T^{\mathcal{B}}_{\mathcal{F}}(\phi \otimes y)\|_Z \leq \sum_{A\in\pi} \|\mathcal{B}(\mathcal{F}(A), \alpha_A y)\|_Z.$$

Using (12) and the definition of projective tensor product one gets $\|T^{\mathcal{B}}_{\mathcal{F}}\| \leq \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

■

Theorem 5 *Let $1 < p < \infty$. $\mathrm{V}^p_{\mathcal{B}}(X)$ is isometrically embedded into $\mathcal{L}(L^{p'}(Z^*), Y^*)$.*

PROOF. Let $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$. Consider the linear operator $S^{\mathcal{B}}_{\mathcal{F}}$ from the space of Z^*-valued simple functions into Y^*. Note that for any partition π

$$\left\| S^{\mathcal{B}}_{\mathcal{F}}(\sum_{A\in\pi} z^*_A \mathbf{1}_A) \right\|_{Y^*} = \left\| \sum_{A\in\pi} \mathcal{B}^*(\mathcal{F}(A), z^*_A) \right\|_{Y^*}.$$

Using (13) and the density of simple functions in $L^{p'}(Z^*)$ one gets $\|S^{\mathcal{B}}_{\mathcal{F}}\| = \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

■

Note that $\mathrm{V}^p_{\mathcal{B}}(X) \subseteq \mathcal{V}^p_{\mathcal{B}}(X)$ and, from Corollary 3, $\mathcal{V}^p_{\mathcal{B}}(X)$ is embedded into $\mathrm{Bil}(L^{p'} \times Y, Z)$. Hence $\mathrm{V}^p_{\mathcal{B}}(X)$ is continuously contained into $\mathrm{Bil}(L^{p'} \times Y, Z)$ by means of the mapping $\mathcal{F} \to \mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ given by

$$\mathcal{B}_{\mathcal{F}}(s, y) = \sum_{k=1}^{n} \mathcal{B}(\mathcal{F}(A_k), \alpha_k y)$$

where $s = \sum_{k=1}^n \alpha_k \mathbf{1}_{A_k}$. Let us find out which special class of bilinear maps represent elements in $\mathrm{V}^p_{\mathcal{B}}(X)$.

In the case of $Y = \mathbb{K}$ the corresponding operators would correspond to the class of cone absolutely summing ones.

Definition 6 *Let L be a Banach lattice, Y and Z be Banach spaces and $\mathcal{U} : L \times Y \to Z$ be a bounded bilinear map. We say that $\mathcal{U}$ is cone absolutely summing if there exists $C > 0$ such that*

$$\sup\{\sum_{n=1}^{N} \|\mathcal{U}(\varphi_n, y)\|_Z : y \in \mathrm{B}_Y\} \leq C \sup\{\sum_{n=1}^{N} |\langle \varphi_n, \psi\rangle| : \psi \in \mathrm{B}_{L^*}\}$$

for any finite family $(\varphi_n)_n$ of positive elements in L.

We denote by $\Lambda(L \times Y, Z)$ the space of such bilinear maps and we endow the space with the norm $\pi^+(\mathcal{U})$ given by the infimum of the constants satisfying the above inequality.

Theorem 6 *If $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ then $\mathcal{B}_{\mathcal{F}} \in \Lambda(L^{p'} \times Y, Z)$ and $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \pi^+(\mathcal{B}_{\mathcal{F}})$.*

PROOF. Given $\mathcal{F} \in \mathrm{V}^p_{\mathcal{B}}(X)$ then $\mathcal{B}_{\mathcal{F}} : L^{p'} \times Y \to Z$ is bounded. Let us show that $\mathcal{B}_{\mathcal{F}} \in \Lambda(L^{p'} \times Y, Z)$ and $\pi^+(\mathcal{B}_{\mathcal{F}}) = \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

From Theorem 2 there exists $0 \leq \varphi_y \in L^p$ such that $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} = \sup\{\|\varphi_y\|_{L^p} : y \in \mathrm{B}_Y\}$ and

$$\|\mathcal{B}_{\mathcal{F}}(\mathbf{1}_A, y)\| \leq \int_\Omega \mathbf{1}_A \varphi_y d\mu, \quad A \in \Sigma.$$

Using linearity and density of simple functions one also extends to

$$\|\mathcal{B}_{\mathcal{F}}(\psi, y)\| \leq \int_\Omega \psi\varphi_y d\mu,$$

for any $0 \leq \psi \in L^{p'}$ and $y \in Y$.

Now, given a finite family $0 \leq \psi_n \in L^{p'}$ and $y \in Y$, we can write

$$\begin{aligned}
\sum_{n=1}^N \|\mathcal{B}_{\mathcal{F}}(\psi_n, y)\| &\leq \sum_{n=1}^N \int_\Omega \psi_n\varphi_y d\mu \\
&= \sum_{n=1}^N \|\varphi_y\|_{L^p}\langle\psi_n, \frac{\varphi_y}{\|\varphi_y\|_{L^p}}\rangle d\mu \\
&\leq \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \sup\{\sum_{n=1}^N |\langle\psi_n, \varphi\rangle| : \varphi \in \mathrm{B}_{L^p}\}.
\end{aligned}$$

This shows $\pi^+(\mathcal{B}_{\mathcal{F}}) \leq \|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)}$.

On the other hand, given a partition π, a sequence $(\alpha_A)_A \in \ell^{p'}$ and denoting $\psi_A = \frac{|\alpha_A|}{\mu(A)^{1/p'}}\mathbf{1}_A$ one can apply the condition of cone absolutely summing bilinear map to get

$$\begin{aligned}
\sum_{A\in\pi} \|\mathcal{B}(\frac{\mathcal{F}(A)}{\mu(A)^{1/p'}}, \alpha_A y)\|_Z &= \sum_{A\in\pi} \|\mathcal{B}_{\mathcal{F}}(\psi_A, y)\|_Z \\
&\leq \pi^+(\mathcal{B}_{\mathcal{F}})\|y\| \sup\{\sum_{A\in\pi} \int_\Omega \psi_A|\varphi| d\mu : \varphi \in \mathrm{B}_{L^p}\} \\
&= \pi^+(\mathcal{B}_{\mathcal{F}})\|y\| \sup\{\sum_{A\in\pi} \frac{|\alpha_A|}{\mu(A)^{1/p'}} \int_A |\varphi| d\mu : \varphi \in \mathrm{B}_{L^p}\} \\
&\leq \pi^+(\mathcal{B}_{\mathcal{F}})\|y\| \sup\{\sum_{A\in\pi} |\alpha_A|(\int_A |\varphi|^p)^{1/p} d\mu : \varphi \in \mathrm{B}_{L^p}\} \\
&\leq \pi^+(\mathcal{B}_{\mathcal{F}}) \cdot \|y\| \cdot \|(\alpha_A)_A\|_{\ell^{p'}}.
\end{aligned}$$

Now (12) allows to conclude that $\|\mathcal{F}\|_{\mathrm{V}^p_{\mathcal{B}}(X)} \leq \pi^+(\mathcal{B}_{\mathcal{F}})$.

■

Corollary 5 $\mathrm{V}^p_{\mathcal{B}}(X)$ *is isometrically embedded into* $\Lambda(L^{p'} \times Y, Z)$.

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