REMARKS ON THE SEMIVARIATION OF VECTOR MEASURES WITH RESPECT TO BANACH SPACES.

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ABSTRACT. Let $L^q(\nu)\hat{\otimes}_{\gamma_q} Y = L^q(\nu, Y)$ and $X\hat{\otimes}_{\Delta_p} L^p(\mu) = L^p(\mu, X)$. It is shown that any $L^p(\mu)$-valued measure has finite $L^2(\nu)$-semivariation with respect to the tensor norm $L^2(\nu)\hat{\otimes}_{\Delta_p} L^p(\mu)$ for $1 \le p < \infty$ and finite $L^q(\nu)$-semivariation with respect to the tensor norm $L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu)$ whenever either $q = 2$ and $1 \le p \le 2$ or $q > \max\{p, 2\}$. However there exist measures with infinite L^q-semivariation with respect to the tensor norm $L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu)$ for any $1 \le q < 2$. It is also shown that the measure $m(A) = \chi_A$ has infinite L^q-semivariation with respect to the tensor norm $L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu)$ if $q < p$.

1. INTRODUCTION

Let Z be a Banach space and let $m : \Sigma \to Z$ be a vector measure defined on a σ-algebra Σ of subsets of Ω. We write $|m|$ for the variation of the measure

$$|m|(A) = \sup\{\sum_{j=1}^{k} \|m(A_j \cap A)\| : A_j \text{ pairwise disjoints }, k \in \mathbb{N}\}$$

and denote, for $1 \le p < \infty$, the p-variation of the measure

$$\|m\|_p = \sup\{(\sum_{j=1}^{k} \|m(A_j)\|^p)^{1/p} : A_j \text{ pairwise disjoints }, k \in \mathbb{N}\}.$$

We also write $\|m\| = \sup_{A\in\Sigma} \|m(A)\|$, which is equivalent to the semivariation of the vector measure m, that is

$$\|m\| \approx \sup\{|\langle z^*, m\rangle|(\Omega) : \|z^*\| = 1\}.$$

Let X, Y be Banach spaces and let τ be a norm on $X \otimes Y$ such that $\|x \otimes y\|_\tau \le C\|x\|\|y\|$ for $x \in X, y \in Y$ and denote $X\hat{\otimes}_\tau Y$ the completion under such a norm. Given a vector measure $m : \Sigma \to Y$ defined on a σ-algebra Σ of subsets of Ω, R. Bartle (see [2, 7]) introduced the notion of X-semivariation of m in $X \otimes_\tau Y$ given by

$$\beta_X(m, \tau, Y)(A) = \sup\{\| \sum_{j=1}^{k} x_j \otimes m(A \cap A_j)\|_\tau\}$$

for every $A \in \Sigma$ where the supremum is taken over $\|x_j\| \le 1$, A_j pairwise disjoints sets in Σ and $k \in \mathbb{N}$. We shall denote

$$\beta_X(m, \tau, Y) = \sup_{A\in\Sigma} \beta_X(m, \tau, Y)(A).$$

Key words and phrases. vector measures, semivariation, vector-valued Bochner spaces.
2000 Mathematical Subjects Classifications. Primary 28B05, 46G10, Secondary 46B42,47B65
Partially supported by Proyecto MTM 2005-08350.

It is clear that

$$\|m\| \le \beta_X(m, \tau, Y) \le \|m\|_1.$$

If $X\hat{\otimes}_\epsilon Y$ and $X\hat{\otimes}_\pi Y$ stands for the injective and projective tensor norms respectively, then one always has

$$\|m\| \le \beta_X(m, \epsilon, Y) \le \beta_X(m, \tau, Y) \le \beta_X(m, \pi, Y) \le \|m\|_1$$

It is well-known and easy to see that actually $\beta_X(m, \epsilon, Y) = \|m\|$.

In [7] B. Jefferies, and S. Okada developed a theory of integration of X-valued functions with respect to Y-valued measures of bounded X-semivariation in the case of completely separated tensor norms.

We shall be concerned with some interesting examples of norms coming from the theory of vector-valued functions: Throughout the paper $(\Omega_1, \Sigma_1, \mu)$ and $(\Omega_2, \Sigma_2, \nu)$ are finite measure spaces, $1 \le p, q < \infty$ and the Banach spaces will be either $Y = L^p(\mu)$ or $X = L^q(\nu)$. We define γ_q and Δ_p the norms on $L^q(\nu) \otimes Y$ and $X \otimes L^p(\mu)$ identified as subspace of $L^q(\nu, Y)$ and $L^p(\mu, X)$, that is to say

$$L^q(\nu)\hat{\otimes}_{\gamma_q} Y = L^q(\nu, Y), \quad X\hat{\otimes}_{\Delta_p} L^p(\mu) = L^p(\mu, X).$$

In the case $p = q$ the $L^p(\nu)$-semivariation of $L^p(\mu)$-valued measures with respect to the topology τ_p such that $L^p(\mu)\hat{\otimes}_{\tau_p} L^p(\nu)$ becomes $L^p(\mu \times \nu)$ for the product measure was studied in [9] and [10].

In particular, if both $X = L^q(\nu)$ and $Y = L^p(\mu)$ then $L^q(\nu)\hat{\otimes}_{\Delta_p} L^p(\mu)$ and $L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu)$ coincide with the spaces of measurable functions $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ such that

$$(\int_{\Omega_1} \int_{\Omega_2} |f(x, y)|^q d\nu(y))^{p/q} d\mu(x))^{1/p} < \infty\}$$

and

$$(\int_{\Omega_2} (\int_{\Omega_1} |f(x, y)|^p d\mu(x))^{q/p} d\nu(x))^{1/q} < \infty\}.$$

In this paper we shall try to understand better the difference between the classical semivariation or variation of a $L^p(\mu)$-valued measure m and the $L^q(\nu)$-semivariation with respect to the norms Δ_p, γ_q and π.

Let us establish the main results of the paper. Our first result establishes the following descriptions L^q-semivariation of L^p-valued measures with respect respect the proyective tensor norm, where we denote $L^p = L^p([0, 1])$ for $1 \le p \le \infty$.

Theorem 1.1. *Let $1 \le p, q \le \infty$ and let $m : \Sigma \to L^p([0, 1])$ be a vector measure. Then*

(i) $\beta_{L^{p'}}(m, \pi, L^p) \approx \|m\|_1 \qquad 1 \le p \le \infty.$
(ii) $\beta_{L^2}(m, \pi, L^p) \approx \|m\|_1, \qquad 1 < p < \infty.$
(iii) $\beta_{L^2}(m, \pi, L^1) \approx \|m\|.$

This result shows that L^2-valued measures are of finite L^2-semivariation on $L^2 \otimes_\pi L^2$ if and only if they are of finite variation.

It was noticed in [9] that any L^2-valued measure is of bounded L^2-semivariation with respect to $L^2([0, 1])\hat{\otimes}_{\tau_2} L^2([0, 1])$, in other words $\beta_{L^2}(m, \Delta_2, L^2) \approx \|m\|$.

On the other hand $\beta_{L^q}(m, \pi, L^1) = \beta_{L^q}(m, \Delta_1, L^1)$. Hence Theorem 1.1 shows that $\beta_{L^2}(m, \Delta_1, L^1) = \|m\|$.

Let us just point out that this implies

$$\beta_{L^2}(m, \Delta_p, L^p) \approx \|m\|, 1 \le p \le 2 \tag{1}$$

due to the simple observation

$$\beta_{L^q(\nu)}(m, \Delta_{p_1}, L^{p_1}(\mu)) \leq C\beta_{L^q(\nu)}(m, \Delta_{p_2}, L^{p_2}(\mu)) \qquad p_1 \leq p_2. \tag{2}$$

We shall present another alternative proof that cover all the cases and gives an alternative proof of the known case $p = q = 2$ and extend (1) as follows.

Theorem 1.2. *Let $1 \leq p < \infty$ and let $m : \Sigma \to L^p([0,1])$ be a vector measure. Then*

$$\beta_{L^2}(m, \Delta_p, L^p) \approx \|m\|.$$

The question which now arises is whether or not there exist L^p-valued measures with $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ if $q \neq 2$. In [7] examples of $L^p([0,1])$-valued measures of infinite $L^p([0,1])$-semivariation in $L^p([0,1])\hat{\otimes}_{\tau_p} L^p([0,1]$ were obtained for the values $p \neq 2$. For $1 \leq p < 2$ the approach was much simpler than for $p > 2$ and the example in this case relies on the existence of a non absolutely summing operator from $\ell^1 \to \ell^p$ for $p > 2$ (see [9, 10]).

We shall use the relationship between the tensor norms γ_q and Δ_p to get other examples. Recall that Minkowski's inequality gives $L^p(\mu, L^q(\nu)) \subseteq L^q(\nu, L^p(\mu))$ for $p \leq q$ and $L^q(\nu, L^p(\mu)) \subseteq L^p(\mu, L^q(\nu))$ for $q \leq p$. Hence

$$\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) \leq \beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)), \quad p \leq q, \tag{3}$$

$$\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) \leq \beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)), \quad q \leq p. \tag{4}$$

Also using general techniques, similar to those used in [9] one can show that for $1 \leq p \leq \infty$ and $1 \leq q < 2$ there exist $L^p(\mu)$-valued measures m such that $\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \infty$. This, in particular, using the estimate (3), shows the existence of measures for which $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ if $1 \leq q < 2, p \leq q$, completing and extending the case $p = q$.

Theorem 1.3. *Let $1 \leq p \leq \infty$ and let $m : \Sigma \to L^p([0,1])$ be a vector measure. Then*

(i)$\beta_{L^2}(m, \gamma_2, L^p) \approx \|m\|, \qquad 1 \leq p \leq 2.$
(ii) $\beta_{L^q}(m, \gamma_q, L^p) \approx \|m\|, \qquad \max\{p, 2\} < q.$

This gives that any measure has $\beta_{L^q}(m, \gamma_q, L^p) < \infty$ for $q > p \geq 2$. However in the last section it is shown that the $L^p([0,1])$-valued measure $m_p(A) = \chi_A$ has infinite $L^q([0,1])$-semivariation in $L^q([0,1])\hat{\otimes}_{\gamma_q} L^p([0,1]$ for $q < p$.

2. Bounded X-semivariation.

We start by the following characterization of the bounded X-semivariation .

Taking into account that $X\hat{\otimes}_\pi Y \subset X\hat{\otimes}_\tau Y$, then $(X\hat{\otimes}_\tau Y)^*$ can be regarded as a subspace of the space of bounded operators $\mathcal{L}(Y, X^*)$. Moreover $\|u\| \leq \|u\|_{(X\hat{\otimes}_\tau Y)^*}$ for any $u \in (X\hat{\otimes}_\tau Y)^*$, where the duality is given by

$$\langle u, \sum_{j=1}^k x_j \otimes y_j \rangle = \sum_{j=1}^k \langle u(y_j), x_j \rangle.$$

Theorem 2.1. *Let $m : \Sigma \to Y$ be a vector measure. Then*

$$\beta_X(m, \tau, Y) \approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\|_{(X\hat{\otimes}_\tau Y)^*} \leq 1\}.$$

PROOF. Let (x_j) be a bounded sequence in X and (A_j) be a sequence of pairwise disjoint sets in Σ. Consider, for $k \in \mathbb{N}$, the X-valued simple function $\phi = \sum_{j=1}^{k} x_j \chi_{A_j}$ and denote

$$\phi \otimes_\tau m(A) = \sum_{j=1}^{k} x_j \otimes m(A \cap A_j) \in X \otimes Y.$$

Clearly this defines a new $X\hat{\otimes}_\tau Y$-valued measure and one can rewrite

$$\beta_X(m, \tau, Y) = \sup\{\|\phi \otimes_\tau m\| : \phi \in \mathcal{S}(X), \|\phi\|_\infty \leq 1\}.$$

We now write the semivariation of $\phi \otimes_\tau m$ using duality, that is to say

$$\begin{aligned} \|\phi \otimes_\tau m\| &\approx \sup\{|\langle u, \phi \otimes m\rangle|(\Omega) : \|u\|_{(X\hat{\otimes}_\tau Y)^*} \leq 1\} \\ &= \sup\{\sum_{j=1}^{k} |\langle u \circ m(A_j), x_j\rangle| : (A_j) \text{ pairwise disjoint}, \|u\|_{(X\hat{\otimes}_\tau Y)^*} \leq 1\}, \end{aligned}$$

which, taking supremum over $\|x_j\| \leq 1$, gives

$$\begin{aligned} \beta_X(m, \tau, Y) &\approx \sup\{\sum_{j=1}^{k} \|u \circ m(A_j)\| : (A_j) \text{pairwise disjoint}, \|u\|_{(X\hat{\otimes}_\tau Y)^*} \leq 1\} \\ &\approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\|_{(X\hat{\otimes}_\tau Y)^*} \leq 1\}. \end{aligned}$$

□

Let us see the formulation of Theorem 2.1 in the case $\tau = \Delta_p$ or $\tau = \gamma_q$.

It is well known that for $1 < p, q < \infty$ and $1/p' + 1/p = 1, 1/q + 1/q' = 1$ and for X, Y such that X^* and Y^* have the Radon-Nikodym property (see [6]) then

$$(L^q(\nu)\hat{\otimes}_{\gamma_q} Y)^* = L^{q'}(\nu)\hat{\otimes}_{\gamma_{q'}} Y^*$$

and

$$(X\hat{\otimes}_{\Delta_p} L^p(\mu))^* = X^*\hat{\otimes}_{\Delta_{p'}} L^{p'}(\mu).$$

Now for each $f \in L^{p'}(\mu, X^*)$ we can define the operators $u_f : L^p(\mu) \to X^*$ and $v_f : X \to L^{p'}(\mu)$ given by

$$\langle u_f(\phi), x\rangle = \int_\Omega \langle f(t), x\rangle \phi(t) d\mu(t)$$

and

$$v_f(x) = \langle f, x\rangle.$$

Of course $(v_f)^* = u_f$ and $(u_f)^* = v_f$ if X is reflexive.

Theorem 2.2. *Let $1 < p, q < \infty$, $X = L^q(\nu)$ and $Y = L^p(\mu)$. If $m : \Sigma \to L^p(\mu)$ is a vector measure then*

$$\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \sup\{\|u_f \circ m\|_1 : \|f\|_{L^{p'}(\mu, L^{q'}(\nu))} \leq 1\}, \tag{5}$$

$$\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \sup\{\|v_g \circ m\|_1 : \|g\|_{L^{q'}(\nu, L^{p'}(\mu))} \leq 1\}. \tag{6}$$

PROOF. In the case $Y = L^p(\mu)$ and $X = L^q(\nu)$ for $1 < q, p < \infty$ the elements $u : L^p(\mu) \to L^{q'}(\nu)$ such that $u \in (L^q(\nu)\hat{\otimes}_{\Delta_p} L^p(\mu))^*$ can be seen as $u = u_f$ for some $f \in L^{p'}(\mu, L^{q'}(\nu))$, that is $u : L^p(\mu) \to L^q(\nu)$ is given by

$$u(\phi)(y) = \int_{\Omega_1} f(x,y)\phi(x)d\mu(x).$$

Then (5) follows from Theorem 2.1 in this case.

Similarly the elements $u : L^p(\mu) \to L^{q'}(\nu)$ such that $u \in (L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu))^*$ can be seen as $u = v_g$ for some $g \in L^{q'}(\nu, L^{p'}(\mu))$ and now

$$u(\psi)(y) = \langle g, \psi \rangle = \int_{\Omega_1} g(y,x)\psi(x)d\mu(x).$$

Again (6) follows from Theorem 2.1. □

3. Proof of the main theorems

We use first the characterization in Theorem 2.1 to get the following corollaries.

Corollary 3.1. *Let $m : \Sigma \to Y$ be a vector measure and X a Banach space. Then*

$$\beta_X(m, \pi, Y) \approx \sup\{\|u \circ m\|_1 : u \in \mathcal{L}(Y, X^*), \|u\| \leq 1\}.$$

We use the notation $\Pi_p(X, Y)$ for the space of p-summing operators from X into Y and write $\pi_p(u)$ for the p-summing norm. The reader is referred to [5] for the basics in the theory of summing operators.

Corollary 3.2. *Let Y be a Grothendieck space, i.e. $\Pi_1(Y, H) = \mathcal{L}(Y, H)$ for any Hilbert space H. Then*

$$\beta_H(m, \pi, Y) \approx \|m\|. \tag{7}$$

PROOF. Note that $\sum m(A_j)$ is an unconditionally convergent series in Y for any sequence of pairwise disjoint sets A_j. Now for any operator from $u : Y \to H$ one has and then $\sum \|u(m(A_j))\| \leq K_G\|u\|\|m\|$, where K_G is the Grothendieck constant. Now use Corollary 3.1. □

Proof of Theorem 1.1

(i) Let $Y = L^p$ and $X = L^{p'}$ then choosing $u = Id : L^p \to (L^{p'})^*$, one concludes that $\|u \circ m\|_1 = \|m\|_1$. This shows $\beta_{L^{p'}}(m, \pi, L^p) = \|m\|_1$

(ii) follows from the following observation: If X^* is isomorphic to a complemented subspace of Y then $\beta_X(m, \pi, Y) \approx \|m\|_1$.

Indeed, assume $id : Y \to Y$ factors through X^* as $id = u_1 \circ u_2$ where $u_2 : Y \to X^*$ and $u_1 : X^* \to Y$ are bounded operators. Now observe that $\|m\|_1 \leq \|u_1\|\|u_2 \circ m\|_1$ and use Corollary 3.1.

Now use that the space *Rad* is complemented in $L^p([0,1])$ and isomorphic to ℓ^2 (see Thm 1.12 [5]) and therefore to L^2, to conclude that

$$\beta_{L^2}(m, \pi, L^p([0,1])) \approx \|m\|_1, 1 < p < \infty. \tag{8}$$

(iii) follows from Corollary 3.2. □

We now recall a lemma that we will need in the sequel.

Lemma 3.3. *(i) Let $1 < q < \infty$ and let Y be a Banach space such that $Y^* \in RNP$. If $u : Y \to L^{q'}(\nu)$ belongs to $(L^q(\nu)\hat{\otimes}_{\gamma_q} Y)^*$ then $\pi_{q'}(u) \le \|u\|_{(L^q(\nu)\hat{\otimes}_{\gamma_q} Y)^*}$.*

(ii) Let $1 < p < \infty$ and let X be a Banach space such that $X^* \in RNP$. If $u : L^p(\mu) \to X^*$ belongs to $(X\hat{\otimes}_{\Delta_p} L^p(\mu))^*$ then $\pi_{p'}(u^*) \le \|u^*\|_{(X\hat{\otimes}_{\Delta_p} L^p(\mu))^*}$.

PROOF. (i) It is well known (see Example 2.11, [5]) that if $g \in L^{q'}(\nu, Y^*)$ then $v_g : Y \to L^{q'}(\nu)$ given by $v_g(y) = \langle g, y\rangle$ is q'-summing and $\pi_{q'}(v_g) \le \|g\|_{L^{q'}(\nu,Y)}$. Now use that, under the assumptions, $(L^q(\nu)\hat{\otimes}_{\gamma_q} Y)^* = L^{q'}(\nu, Y^*)$ and $u = v_g$ for certain $g \in L^{q'}(\nu, Y^*)$.

(ii) Note that $u = u_f$ for some $f \in L^{p'}(\mu, X^*)$. Hence $v_f = u^* : X^{**} \to L^p(\mu)$ is p'-summing and $\pi_{p'}(u^*) \le \|f\|_{L^{p'}(\mu,X^*)} = \|u\|_{(L^q(\nu)\hat{\otimes}_{\gamma_q} Y)^*}$. □

Proof of Theorem 1.2

The case $p = 1$ is included in (iii) Theorem 1.1.

Assume now $1 < p < \infty$ and let $m : \Sigma \to L^p$ be a vector measure. Given $u : L^p \to L^2$ with $u \in (L^2\hat{\otimes}_{\Delta_p} L^p)^*$ we can use (ii) in Lemma 3.3 to conclude that there exist $f \in L^{p'}([0, 1], L^2)$ such that $v_f : L^2 \to L^{p'}$ given by $\phi \to \int_0^1 \phi(y)f(x, y)dy$ is p'-summing and $u = u_f = (v_f)^*$. Hence, using Theorem 2.21 in [5], one has that $(v_f)^* = u : L^p \to L^2$ is 1-summing. Therefore

$$\|u_f \circ m\|_1 \le C\|u_f\|\|m\| \le C\|f\|_{L^{p'}([0,1],L^2)}\|m\|. \square$$

Let us mention another useful lemma.

Lemma 3.4. *(Prop. 6, [1]) Let Y be a Banach space of finite cotype r and let $\sum_j y_j$ be an unconditionally convergent series in Y.*

(i) If $r = 2$ then there exist $(\alpha_j) \in \ell^2$ and a sequence in $(y'_j) \subset Y$ such that $y_j = \alpha_j y'_j$ and

$$\sum_j |\alpha_j|^2 \le \sup_{\|y^*\|=1} \sum_j |\langle y_j, y^*\rangle|,$$

$$\sup_{\|y^*\|=1} \sum_j |\langle y'_j, y^*\rangle|^2 \le \sup_{\|y^*\|=1} \sum_j |\langle y_j, y^*\rangle|.$$

(ii) If $r > 2$ then for any $q > r$ there exist $(\alpha_j) \in \ell^q$ and a sequence in $(y'_j) \subset Y$ such that $y_j = \alpha_j y'_j$ and

$$(\sum_j |\alpha_j|^q)^{1/q} \le (\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^*\rangle|)^{1/q}.$$

$$(\sup_{\|y^*\|=1} \sum_j |\langle y'_j, y^*\rangle|^{q'})^{1/q'} \le (\sup_{\|y^*\|=1} \sum_j |\langle y_j, y^*\rangle|)^{1/q'}.$$

PROOF. (i) Let $T : c_0 \to Y$ such that $T(e_j) = y_j$. Note that $\mathcal{L}(c_0, Y) = \Pi_2(c_0, Y)$ for any cotype 2 space Y. Now apply Lemma 2.23 in [5] to the sequence (e_j) which satisfies $\sup\{\sum_j |\langle e_j, z\rangle| : \|z\|_{\ell^1} = 1\}$ to conclude that $T(e_j) = y_j = \alpha_j y'_j$ with the desired properties.

(ii) Repeat the proof using now $L(c_0, Y) = \Pi_q(c_0, Y)$ for any $q > r$ (see Theorem 11.14 [5]). □

Proof of Theorem 1.3

Note Theorem 1.2 and (4) give

$$\beta_{L^2}(m,\gamma_2,L^p)\approx\|m\|,\qquad 1\le p\le 2. \tag{9}$$

To obtain (ii) we simply use the following more general result.

Theorem 3.5. *If Y has cotype $r<\infty$ and Y^* has the RNP then*

$$\beta_{L^2(\nu)}(m,\gamma_2,Y)\approx\|m\|,\quad r=2. \tag{10}$$

$$\beta_{L^q(\nu)}(m,\gamma_q,Y)\approx\|m\|,\quad q>r>2. \tag{11}$$

PROOF. We only prove (11). The other is exactly the same.

Let (A_j) be a sequence of pairwise disjoint sets. Since $m(A_j)$ is unconditionally convergent in Y, Lemma 3.4 implies that there exist $(\alpha_j)\in\ell^q$ and a sequence in $(y_j)\subset Y$ with $m(A_j)=\alpha_j y_j$ and

$$(\sum_j|\alpha_j|^q)^{1/q}\le(\sup_{\|y^*\|=1}\sum_j|\langle m(A_j),y^*\rangle|)^{1/q}.$$

$$(\sup_{\|y^*\|=1}\sum_j|\langle y_j,y^*\rangle|^{q'})^{1/q'}\le(\sup_{\|y^*\|=1}\sum_j|\langle m(A_j),y^*\rangle|)^{1/q'}.$$

On the other hand if $u\in(L^q(\nu)\hat{\otimes}Y)^*$, using (i) in Lemma 3.3, one has $u\in\Pi_{q'}(Y,L^{q'})$. Therefore

$$\begin{aligned}\sum_j\|u(m(A_j))\| &= \sum_j|\alpha_j|\|u(y_j)\|\\ &\le (\sum_j|\alpha_j|^q)^{1/q}(\sum_j\|u(y_j)\|^{q'})^{1/q'}\\ &\le \pi_{q'}(u)(\sum_j|\alpha_j|^q)^{1/q}(\sup_{\|y^*\|=1}\sum_j|\langle y_j,y^*\rangle|^{q'})^{1/q'}\\ &\le C\|u\|_{(L^q(\nu)\hat{\otimes}Y)^*}\|m\|.\end{aligned}$$

□

4. Measures of infinite X-semivariation

We shall present now some necessary conditions to have bounded X-semivariation.

Proposition 4.1. *(i) Assume that $X\hat{\otimes}_\tau Y$ is of finite cotype q. If $m:\Sigma\to Y$ be a vector measure then*

$$\|m\|_q\le C_q\beta_X(m,\tau,Y)$$

for some constant C_q independent of m.

In particular, if X has finite cotype q and $1\le p<\infty$ then

$$\|m\|_{\max\{q,2,p\}}\le C\beta_X(m,\Delta_p,L^p(\mu)).$$

(ii) Let $1\le q<\infty$, let ν be a finite measure for which there exists a sequence of pairwise disjoint sets with $\nu(B_j)>0$ and let $m:\Sigma\to Y$ be a vector measure. Then

$$\|m\|_q\le C_q\beta_{L^q(\nu)}(m,\gamma_q,Y)$$

PROOF. (i) Let (x_j) be a sequence in the unit ball of X and a sequence of pairwise disjoint sets A_j. Hence , for $0 \leq t \leq 1$, one has

$$\|\sum_{j=1}^{k} r_j(t)x_k \otimes m(A_j)\|_{X\hat{\otimes}_\tau Y} \leq \beta_X(m,\tau,Y)$$

where r_j stands for the Rademacher sequence. Now integrate over $[0,1]$ and use the cotype estimate to get

$$(\sum_{j=1}^{k} \|x_k\|^q \|m(A_j)\|^q)^{1/q} \leq C_q \beta_X(m,\tau,Y).$$

Taking the sup over (x_j) and (A_j) one obtains the desired result.

Note that $L^p(\mu, X)$ has cotype equals $\max\{p,q,2\}$.

(ii) Take $x_j = \frac{\chi_{B_j}}{\nu(B_j)^{1/q}}$, $\phi = \sum_{j=1}^k x_j \chi_{A_j}$ for some sequence of pairwise disjoint sets in Σ and notice that, for any $A \in \Sigma$,

$$\|\phi \otimes m(A)\|_{L^q(\nu,Y)} = (\sum_{j=1}^{k} \|m(A \cap A_j)\|^q)^{1/q}.$$

This gives the result □

Corollary 4.2. *Let Y be infinite dimensional Banach space, $1 \leq q < 2$ and ν be a finite measure for which there exists a sequence of pairwise disjoint sets with $\nu(E_n) > 0$.*

(i) There exist Y-valued measure such that $\beta_{L^q(\nu)}(m, \gamma_q, Y) = \infty$.

(ii) If $L^p(\mu)$ is infinite dimensional then there exist $L^p(\mu)$-valued measures m such that $\beta_{L^q(\nu)}(m, \Delta_p, L^p(\mu)) = \infty$ for $1 \leq q < 2$ and $q \geq p$.

PROOF. (i) Select an unconditionally convergent series (y_n) with $\sum_k \|y_k\|^q = \infty$ (this can be done for $1 \leq q < 2$, see, for instance [5])).

Now we define the measure over $\mathbb{N}$ given by $m(\{k\}) = y_k$. Clearly $\|m\|_q = \infty$ and therefore $\beta_{L^q(\nu)}(m, \gamma_q, Y) = \infty$ from (ii) in Proposition 4.1.

(ii) follows from (i) and the estimate (3). □.

A very important example to analyze is $m_p : \Sigma \to L^p(\mu)$ given by $m_p(A) = \chi_A$. We shall see that these measures are enough to produce examples with $\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \infty$ for $q < p$.

Theorem 4.3. *Let $\mu(\Omega_1) < \infty$, $\nu(\Omega_2) < \infty$, $X = L^q(\nu)$ and $Y = L^p(\mu)$. Then the $L^p(\mu)$-valued measure $m_p(A) = \chi_A$ has finite $L^q(\nu)$-semivariation in $L^q(\nu)\hat{\otimes}_{\gamma_q} L^p(\mu)$ if and only if $L^{q'}(\nu, L^{p'}(\mu)) \subseteq L^1(\mu, L^{q'}(\nu))$.*

PROOF. Let $g : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be such that

$$\|g\|_{L^{q'}(\nu,L^{p'}(\mu))} = \int_{\Omega_2} (\int_{\Omega_1} |g(y,x)|^{p'} d\mu(x)^{q'/p'} d\nu(y))^{1/q'} < \infty.$$

Note that the operator $v_g : L^p(\mu) \to L^{q'}(\nu)$ becomes

$$v_g(\psi)(y) = \int_{\Omega_1} g(y,x)\psi(x)d\mu(x),$$

hence, we have $v_g \circ m_p(A) = \int_A g(y,x)d\mu(x)$ for all $A \in \Sigma_1$. This shows that $v_g \circ m_p$ is the $L^{q'}(\nu)$-valued measure with Radon-Nikodym derivative $g(y,.)$. Therefore $\|v_g \circ m_p\|_1 = \int_{\Omega_1}(\int_{\Omega_2}|g(y,x)|^{q'}d\nu(y))^{1/q'}d\mu(x)$.

Now Theorem 2.2 shows that m_p is of bounded $L^q(\nu)$-semivariation in $L^q(\nu)\hat{\otimes}_{\gamma_q}L^p(\mu)$ if and only if there exists $C>0$ such that

$$\int_{\Omega_1}(\int_{\Omega_2}|g(y,x)|^{q'}d\nu(y))^{1/q'}d\mu(y) \le C\int_{\Omega_2}(\int_{\Omega_1}|g(y,x)|^{p'}d\mu(x)^{q'/p'}d\nu(x))^{1/q'}.$$

That is to say $L^{q'}(\nu, L^{p'}(\mu)) \subset L^1(\mu, L^{q'}(\nu))$. □

Corollary 4.4. *Let $1 \le p < \infty$ and $m_p : \Sigma \to L^p(\mu)$ given by $m_p(A) = \chi_A$. Then $\beta_{L^q(\nu)}(m_p, \gamma_q, L^p(\mu)) < \infty$ for $p \le q$.*

PROOF. Note that for $p \le q$ one obviously has

$$L^{q'}(\nu, L^{p'}(\mu)) \subset L^{q'}(\nu, L^{q'}(\mu)) = L^{q'}(\mu, L^{q'}(\nu)) \subset L^1(\mu, L^{q'}(\nu)).$$

Apply now Theorem 4.3. □

Actually the previous result is also a consequence of the following general fact.

Proposition 4.5. *Let $1 \le p < \infty$, X a Banach space and let $m : \Sigma \to L^p(\mu)$ be a positive vector measure, that is $m(A) \ge 0$ for all $A \in \Sigma$. Then*

$$\beta_X(m, \Delta_p, L^p(\mu)) = \|m\|.$$

In particular, if m is positive and $p \le q$ then

$$\beta_{L^q(\nu)}(m, \gamma_q, L^p(\mu)) = \|m\|.$$

PROOF. It is well-known that $(L^p(\mu, X))^* = (L^p(\mu)\hat{\otimes}X)^*$ can be identified with the space of X^*-valued measures in $V^{p'}(\mu, X^*)$ (see [4]). In particular, if $u \in (L^p(\mu)\hat{\otimes}X)^* \subset L(L^p(\mu), X^*)$ (see for instance [3]) there exists $\phi \in L^{p'}(\mu)$ such that $\|\phi\|_{p'} \le \|u\|_{(L^p(\mu)\hat{\otimes}X)^*}$ and satisfies that

$$\|u(\psi)\| \le \int_\Omega \phi(t)\psi(t)d\mu(t)$$

for any positive function $\psi \in L^p(\mu)$. Therefore, if $\|u\|_{(L^p(\mu)\hat{\otimes}X)^*} = 1$ then

$$\begin{aligned}\sum_{j=1}^k \|u(m(A_j))\| &\le \|\phi\|_{p'}\int_\Omega \sum_{j=1}^k \frac{|\phi(t)|}{\|\phi\|_{p'}} m(A_j)(t)d\mu(t) \\ &\le \sup\{\sum_{j=1}^k |\langle \phi', m(A_j)\rangle| : \|\phi'\|_{L^{p'}} = 1\}\end{aligned}$$

Hence $\|u_f \circ m\|_1 \le \|m\|$. Apply now Theorem 2.2.

In the case $X = L^q(\nu)$ and $p \le q$ (4) allows us to conclude the proof. □

We shall now see that the range of values in Theorem 4.3 is sharp.

Lemma 4.6. *If $p > q$ then there exists $f : [0,1]^2 \to \mathbb{R}^+$ such that*

$$\int_0^1(\int_0^1 f(x,y)^q dy)^{p/q}dx < \infty$$

and

$$\int_0^1 (\int_0^1 f(x,y)^p dx)^{1/p} dy = \infty.$$

PROOF. Denoting $\beta = p/q > 1$ and $g(x,y) = f(x,y)^q$ it suffices to find $g : [0,1]^2 \to \mathbb{R}^+$ such that

$$\int_0^1 (\int_0^1 g(x,y)dy)^\beta dx < \infty$$

and

$$\int_0^1 (\int_0^1 g(x,y)^\beta dx)^{1/p} dy = \infty.$$

Recall that the Hardy operator $T(\phi)(x) = \frac{1}{x}\int_0^x \phi(y)dy$ is bounded on $L^\beta([0,1]$ for $\beta > 1$ and define

$$g(x,y) = \frac{1}{x}\chi_{[0,x]}(y)\phi(y)$$

for a function $\phi \in L^\beta([0,1])$ to be chosen later.

Clearly

$$\begin{aligned} \int_0^1 (\int_0^1 g(x,y)dy)^\beta dx &= \|T(\phi)\|_\beta^\beta \\ &\leq \|T\|^\beta \|(\phi)\|_\beta^\beta \end{aligned}$$

On the other hand

$$\begin{aligned} \int_0^1 (\int_0^1 g(x,y)^\beta dx)^{1/p} dy &= \int_0^1 \phi(y)^{\beta/p} (\int_y^1 \frac{dx}{x^\beta})^{1/p} dy \\ &\geq C\int_0^1 \phi(y)^{\beta/p} \frac{1}{y^{(\beta-1)/p}} dy \\ &= C(\int_0^1 (\frac{\phi(y)}{y^{1/\beta'}})^{\beta/p} dy \\ &\geq C(\int_0^1 \frac{\phi(y)}{y^{1/\beta'}} dy)^{\beta/p}. \end{aligned}$$

Now select $\phi(y) = \frac{1}{y^{1/\beta} log(1/y)}$ to have $\phi \in L^\beta([0,1])$ and

$$\int_0^1 \frac{\phi(y)}{y^{1/\beta'}} dy = \int_0^1 \frac{dy}{y log(1/y)} = \infty.$$

□

Corollary 4.7. *For $q < p$ the $L^p([0,1])$-valued measure $m_p(A) = \chi_A$ has infinite $L^q([0,1])$-semivariation in $L^q([0,1])\hat{\otimes}_{\gamma_q} L^p([0,1]$.*

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