MODULI AND CONSTANTS

...what a show!

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April 14, 2006

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0 Introduction

The classical modulus of convexity introduced by J.A. Clarkson in 1936 to define uniformly convex spaces is at the origin of a great number of moduli defined since then [Alonso-Ullán 88].

Indeed, there are a lot of quantitative descriptions of geometrical properties of Banach spaces. The most common way for creating these descriptions, is to define a real function (a "modulus") depending on the Banach space under consideration, and from this a suitable constant or coefficient closely related with this function. The moduli and/or the constants are attempts in order to get a better understanding about two facts:

- The shape of the unit ball of a space, and
- The hidden relations between weak and strong convergence of sequences.

One might well ask: Are there too many moduli for these purposes? Maybe! In part this is because many of these moduli involve very difficult computations, and, often there are intricate links between them. Moreover, it is not unusual to find some moduli defined in (seemingly) different ways, depending on the preferences of the writer.

The aim of this survey is only to give a brief summary of a few of those properties which are in some ways related to Metric Fixed Point Theory. It is not a chronological comprehensive list of the geometrical properties of Banach spaces which has been described using moduli and/or constants. Among other things, such a list would be too long, and have unavoidable overlaps.

Further, similar coefficients, for example those given in ([Maluta 84]) have not been included as independent items. We have only concerned ourselves with coefficients and moduli defined for general Banach spaces. For example, we have not listed those defined in [Khamsi 87] for Banach spaces with Schauder's finite dimensional decomposition. For the sake of brevity, we have also considered only those constants depending on the weak topology, and of course, on the norm topology of the space. The author apologizes in advance for any omissions.

We shall assume throughout this paper that $(X, \|\cdot\|)$ is a Banach space with dimension at least two. We will use B_X, S_X to stand for the unit ball and the unit sphere of Banach , respectively. Wherever possible we have retained the original notation used for the various moduli and constants, however a few exceptions were necessary to avoid confusion.

Since this collection of moduli and constants was posted at the web site
http://www.uv.es/ llorens/
several colleagues sent me interesting remarks and/or reprints of their papers. I would like to express my gratitude to all of them, in particular to the professors Stan Prus, Javier Alonso and Fenghui Wang. This gratitude includes also several authors who have cited this "on line paper".

Valencia, April 14, 2006.

PART I. MODULI AND RELATED PROPERTIES

1 Clarkson's Modulus of Convexity, [Clarkson 36].

It is the function $\delta_X : [0,2] \longrightarrow [0,1]$ given by

$$\delta_X(\varepsilon) := \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}$$

Related coefficient

The *characteristic of convexity of* $(X, \|.\|)$,

$$\varepsilon_0(X) := \sup\{\varepsilon \in [0,2] : \delta_X(\varepsilon) = 0\}$$

was defined in [Goebel 70].

Geometrical properties in terms of this modulus and/or this coefficient

1. **Definition 1.1** *The Banach space* $(X, \|\cdot\|)$ *is* **uniformly convex** *(UC for short) whenever* $\delta_X(\varepsilon) > 0$ *for* $0 < \varepsilon \leq 2$*, or equivalently if* $\varepsilon_0(X) = 0$*.*

2. The Banach space $(X, \|\cdot\|)$ is *strictly convex* (SC) if and only if $\delta_X(2) = 1$.

3. The Banach space $(X, \|\cdot\|)$ is *uniformly non square* if and only if $\varepsilon_0(X) < 2$.

4. [James 72], [Enflo 72].

   For a Banach space $(X, \|\cdot\|)$ *the following statements are equivalent*

   a) X *is superreflexive.*

   b) X *has an equivalent uniformly nonsquare norm.*

   c) X *has an equivalent uniformly convex norm.*

5. A classical result in metric fixed point theory is that the uniformly convex Banach spaces have the fixed point property for nonexpansive mappings (FPP). It was published in 1965. It is due (independently) to F. Browder, D. Göhde, and (in a more general form) to W.A. Kirk.

6. [Goebel 70]. If $\delta_X(1) > 0$ (that is, if $\varepsilon_0(X) < 1$), then $(X, \|\cdot\|)$ (is super-reflexive) and has (uniform) normal structure. In fact, see [Turret 82] both X and X^* have the fixed point property for nonexpansive mappings.

7. [Gao-Lau 91]. If $\delta_X\left(\frac{3}{2}\right) > \frac{1}{4}$ then $(X, \|\cdot\|)$ has uniform normal structure.

8. [Gao-Lau 91]. If $\delta_X(\varepsilon) \geq \frac{1}{6}\varepsilon$ for some $\varepsilon \in \left(0, \frac{3}{2}\right]$ then $(X, \|\cdot\|)$ has uniformly normal structure.

9. [Prus 91]. If $\delta_X(\varepsilon) > \delta_h(\varepsilon) := \max\left\{0, \frac{\varepsilon-1}{2}\right\}$ for some $\varepsilon \in (0,2]$, then $(X, \|\cdot\|)$ has uniform normal structure.

10. [Gao 03] $\delta_X(1+\varepsilon) > \varepsilon$ for any $\varepsilon \in [0,1]$ implies $(X, \|\cdot\|)$ has uniformly normal structure.

11. Ph. D. E. Mazcuñán 2003: $\varepsilon_0(X) < 2$ implies that $(X, \|\cdot\|)$ has the FPP.

EXAMPLES

Example 1 If $(H, \|.\|)$ is a Hilbert space,

$$\delta_H(\varepsilon) := \frac{2-\sqrt{4-\varepsilon^2}}{2}.$$

Example 2 *[Hanner 56]. See also [Prus 96].*

Let (Ω, μ) be a measure space such that μ takes at least two different values. For $p \geq 2$,

$$\delta_{L^p(\Omega,\mu)}(\varepsilon) = 1 - (1-(\varepsilon/2)^p)^{\frac{1}{p}}.$$

For $1 < p < 2$ then

$$\left(1 - \delta_{L^p(\Omega,\mu)}(\varepsilon) + \frac{\varepsilon}{2}\right)^p + \left|1 - \delta_{L^p(\Omega,\mu)}(\varepsilon) - \frac{\varepsilon}{2}\right|^p = 2.$$

This shows in particular that that the spaces $L^p(\Omega, \mu)$ are (UC) whenever $1 < p < \infty$. Clearly this covers the case of the spaces ℓ_p.

Example 3 Let $\lambda \geq 1$ and let X_λ denote the space obtained by renorming the Hilbert space $(\ell_2, \|\cdot\|)$ by means of

$$\|x\|_\lambda := \max\{\lambda^{-1}\|x\|, \|x\|_\infty\}.$$

Then

$$\varepsilon_0(X_\lambda) = \begin{cases} 2(\lambda^2-1)^{\frac{1}{2}} & \lambda \leq \sqrt{2} \\ 2 & \lambda \geq \sqrt{2}. \end{cases}$$

Example 4 For $x \in \ell_p$, $(1 \leq p < \infty)$, we denote by x^+ and x^- the vectors whose i component are respectively

$$x^+(i) := \max\{x(i), 0\} = \frac{x(i)+|x(i)|}{2}, x^-(i) := \max\{-x(i), 0\} = \frac{-x(i)+|x(i)|}{2}.$$

For any $q \in [1, \infty)$, and for $x \in \ell_p$ we denote

$$\|x\|_{p,q} := (\|x^+\|_p^q + \|x^-\|_p^q)^{\frac{1}{q}}$$
$$\|x\|_{p,\infty} := \max\{\|x^+\|_p, \|x^-\|_p\}.$$

It is easy to check that all these norms are equivalent to the usual norm in ℓ_p. The Banach spaces $\ell_{p,q} = (\ell_p, \|.\|_{p,q})$ where introduced by Bynum [Bynum 80].

[Goebel-Kirk 90] p. 93 reads:

> The spaces $\ell_{p,q}$ are uniformly convex if $p, q \in (1, \infty)$, while $\varepsilon_0(\ell_{p,\infty}) = 1$ and $\varepsilon_0(\ell_{q,1}) = 2^{\frac{1}{q}}$. The space $\ell_{p,\infty}$ fails to have normal structure. On the other hand, the space $\ell_{p,1}$ has normal structure in spite of the fact that $\varepsilon_0(\ell_{p,1}) = 2^{\frac{1}{p}} > 1$. Moreover the spaces $\ell_{q,1}$ and $\ell_{p,\infty}$ are dual to each other (whenever $p + q = pq$ and $p > 1$). This shows that normal structure is not a condition which is invariant under passing to dual spaces.

Example 5 *[Prus 96].*

Let E be the space $\mathbb{R}^2$ endowed with the norm $\|(x,y)\| := \max\{|x|,|y|,|x-y|\}$. We have

$$\delta_E(\varepsilon) = \max\left\{0, \frac{1}{2}(\varepsilon-1)\right\} \Rightarrow \varepsilon_0(E) = 1.$$

Example 6 *[Goebel-Kirk 90], p.59*

Let $\|.\|$ be the norm on $\mathbb{R}^2$ defined by

$$\|(x_1,x_2)\| := \begin{cases} (x_1^2+x_2^2)^{\frac{1}{2}} & x_1x_2 \geq 0 \\ |x_1|+|x_2| & x_1x_2 < 0 \end{cases}$$

One has that

$$\delta_{(\mathbb{R}^2,\|.\|)}(\varepsilon) = \begin{cases} 0 & 0 \leq \varepsilon \leq \sqrt{2} \\ \min\{1-(2-\varepsilon^2/2)^{1/2}, 1-\varepsilon^2/8)^{1/2}\} & \sqrt{2} < \varepsilon \leq 2 \end{cases}$$

which is a nonconvex function.

Example 7 *[Hudzyk-Landes 92].*

The Musielak-Orlicz space $L^\phi = L^\phi(\mu)$ is defined to be the space of all (equivalence classes of) Σ-measurable functions $x : S \to \mathbb{R}$ such that

$$I_\phi(\lambda x) := \int_S \phi(s, \lambda x(s)) d\mu < \infty$$

for some λ depending on x. (Here (S,Σ,μ) stands for a non-atomic σ-finite measure space and ϕ denotes a Musielak-Orlicz function, i.e. a function from $S \times \mathbb{R}$ into $\mathbb{R}_+$ satisfying the Carathéodory conditions which means that $\phi(s,.)$ is convex, even, continuous and vanishing at 0 for μ-a.e. $s \in S$, and $\phi(.,u)$ is a Σ-measurable function for every $u \in \mathbb{R}$). This space is endowed with the Luxemburg norm,

$$\|x\| = \|x\|_\phi := \inf\left\{\lambda > 0 : I_\phi\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

and it is a Banach space. The Musielak-Orlicz function ϕ is said to satisfy the Δ_2-condition if there are a null set S_0, a positive constant K and a non-negative Σ-measurable function h with $I(h) < \infty$, such that

$$\phi(s, 2u) \leq K\phi(s,u)$$

for all $s \in S \backslash S_0$, $u \geq h(s)$.

Assume that $\phi(s,.)$ is a strictly convex function on $\mathbb{R}$ for μ-a.e. $s \in S$, then

$$\varepsilon_0(L^\phi) = \begin{cases} \frac{2(1-p(\phi))}{1+p(\phi)} & \text{if } \phi \text{ satisfies } \Delta_2 \text{ cond.} \\ 2 & \text{otherwise .} \end{cases}$$

where $p(\phi)$ is defined as follows:

For every $c, \sigma \in (0,1)$ and $s \in S$:

$$q(s,u,v) := \begin{cases} 0 & \text{if } \phi(s, \frac{1}{2}(u+v)) = 0 \\ \frac{2\phi(s,\frac{1}{2}(u+v))}{\phi(s,u)+\phi(s,v)} & \text{otherwise} \end{cases}$$

$$G(\phi) := \{g \in Meas(S, \mathbb{R}^+) : I(g) < \infty\}$$
$$A(c,\sigma,s) := \{u > 0 : q(s,u,cu) > 1-\sigma\}$$
$$h_{c,\sigma(s)} := \sup\{u > 0 : u \in A(c,\sigma,s)\}$$
$$p(\phi) := \sup\{c \in (0,1) : h_{c,\sigma} \in G(\phi) \text{ for some } \sigma \in (0,1)\}.$$

Other facts concerning this modulus and/or constant

1. In the above definition 'B_X' and '$\geq$' can be replaced respectively by 'S_X' and '$=$'. (For a proof see, for instance, [Danes 76]).

2. δ_X is a monotone increasing function on $[0,2]$. Even more.

3. ([Diestel 84], p.125.)
$$\frac{\delta_X(\varepsilon_1)}{\varepsilon_1} \leq \frac{\delta_X(\varepsilon_2)}{\varepsilon_2}$$
whenever $0 < \varepsilon_1 < \varepsilon_2 \leq 2$.

4. If Y is a closed subspace of Banach , $\delta_Y(\varepsilon) \geq \delta_X(\varepsilon)$ for $\varepsilon \in [0,2]$.

5. For $x, y, p \in X$, $R > 0$, and $r \in [0, 2R]$
$$\left.\begin{array}{l} \|x-p\| \leq R \\ \|y-p\| \leq R \\ \|x-y\| \geq r \end{array}\right\} \Rightarrow \left\|p - \frac{1}{2}(x+y)\right\| \leq \left(1 - \delta_X\left(\frac{r}{R}\right)\right) R.$$
In particular,
$$\left.\begin{array}{l} \|x\| \leq R \\ \|y\| \leq R \end{array}\right\} \Rightarrow \left\|\frac{1}{2}(x+y)\right\| \leq \left(1 - \delta_X\left(\frac{\|x-y\|}{R}\right)\right) R.$$

6. ([Goebel-Kirk 90], p. 111.) For all $u, v \in B_X$ and $c \in [0,1]$,
$$2\min\{c, 1-c\}\delta_X(\|u-v\|) \leq 1 - \|cu + (1-c)v\|.$$

7. [Nordlander 60]. The highest possible value of δ_X is attained in Hilbert spaces, that is, for any Banach space $(X, \|\cdot\|)$
$$\delta_X(\varepsilon) \leq \frac{2 - \sqrt{4-\varepsilon^2}}{2}.$$

8. ([Ullán 90] p. 23. See also [Gurarii 67].) For $0 \leq \varepsilon_2 < \varepsilon_1 \leq 2$,
$$\delta_X(\varepsilon_1) - \delta_X(\varepsilon_2) \leq \frac{\varepsilon_1 - \varepsilon_2}{2 - \varepsilon_2}.$$
Hence δ_X is continuous in $[0,2)$. (See [Goebel-Kirk 90] or [Diestel 84] for a different proof). (Also at $\varepsilon = 2$ if $(X, \|\cdot\|)$ is (UC), [Alonso-Ullán 88]).

   Example 8 *See [Ullán 90].* Let X be the space c_0 endowed with the norm
$$\|x\| = \|x\|_\infty + \left(\sum_{n=1}^{\infty} \frac{1}{4^n} x_n^2\right)^{\frac{1}{2}}.$$
One has that $\varepsilon_0(X) = 2$ while $\delta_X(2) = 1$. Thus, δ_X is discontinuous at $\varepsilon = 2$.

9. (See [Goebel-Kirk 90] or [Ullán 90]). For all $\varepsilon \in [\varepsilon_0(X), 2]$,
$$\delta_X\left(2(1 - \delta_X(\varepsilon))\right) = 1 - \frac{1}{2}\varepsilon_0(X).$$

10. [Goebel-Kirk 90].

$$\lim_{\varepsilon \to 2^-} \delta_X(\varepsilon) = 1 - \frac{1}{2}\varepsilon_0(X).$$

11. [Downing-Turret 83].

If μ is a measure,

$$\varepsilon_0\left(L^p(\mu, X)\right) = \varepsilon_0(X).$$

For more information about uniform convexity of direct sums of (uniformly convex) Banach spaces see [Dowling 03].

12. [Goebel et al. 74]. Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, and let C be a convex, bounded and closed subset of Banach . If $T : C \to C$ is uniformly lipschitzian with constant k and k is less than the (unique) solution of the equation

$$h\left(1 - \delta_X\left(\frac{1}{h}\right)\right) = 1,$$

then T has a fixed point in C.

13. [Downing-Turret 83]. Stability of condition $\varepsilon_0(X) < 1$.

Let X be a Banach space and let $X_1 := (X, \|.\|_1)$ and $X_2 := (X, \|.|_2)$, where $\|.\|_1$ and $\|.\|_2$ are two equivalent norms on X satisfying for $\alpha, \beta > 0$,

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$$

for all $x \in X$.

If $\varepsilon_0(X_1) < 1$ and if

$$k\left(1 - \delta_{X_1}\left(\frac{1}{k}\right)\right) < 1,$$

where $k := \frac{\beta}{\alpha}$, then $\varepsilon_0(X_2) < 1$.

Equivalent statement:

([Downing-Turret 83], Th. 6). Let $(X, \|\cdot\|)$ be a Banach space with $\varepsilon_0(X) < 1$. Let Y be a Banach space isomorphic to X. If k is the unique solution of the equation

$$k\left(1 - \delta_X\left(\frac{1}{k}\right)\right) = 1,$$

and $d(X, Y) < k$, then $\varepsilon_0(Y) < 1$.

14. [Kim-Kirk 95]. If $\varepsilon_0(X) = 1$ then every nonempty bounded closed convex subset of Banach has the fixed point property for *asymptotically* nonexpansive mappings.

15. [Lim 80] Nonexpansive multivalued mappings taking compact values have fixed points. With more precision:

If C is a closed convex bounded subset of the uniformly convex Banach space $(X, \|\cdot\|)$ and $T : C \to 2^C$ satisfies that

a) For each $x \in C$, $T(x)$ is a nonempty compact subset of C.

b) For $x, y \in C$, $H(T(x), T(x)) \leq \|x - y\|$, where $H(T(x), T(x))$ stands for the Hausdorff distance between the sets $T(x)$ and $T(y)$.

Then there exists $x \in C$ such that $x \in T(x)$.

16. [Nerven 05] The unit sphere of every infinite-dimensional uniformly convex Banach space $(X, \|\cdot\|)$ contains a $r := \left(1 + \frac{1}{2}\delta_X\left(\frac{2}{3}\right)\right)$-separated sequence (x_n), which we mean that $\|x_j - x_k\| \geq r$ for all $j \neq k$.

2 V.L. Smulian modulus of weak uniform rotundity

For $x^* \in S_{X^*}$, the **modulus of convexity of Banach with respect to** $x*$, is the function given by
$\delta_X(x^*, .) : [0, 2] \longrightarrow [0, 1]$

$$\delta_X(x^*, \varepsilon) := \inf\left(\{1\} \cup \left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in S_X, |x^*(x-y)| \geq \varepsilon\right\}\right).$$

The reason for specifically including 1 in the set whose infimum defines this modulus is to avoid the following particular situation. When x^* is a non-norm attaining functional there are no points x, y in S_X such that $|x^*(x-y)| \geq 2$. So $\delta_X(x^*, 2)$ would not be well defined.

Geometrical property in terms of this modulus and/or this coefficient

Definition 2.1 *Smulian, 1939. The Banach space* $(X, \|\cdot\|)$ *is* **weakly uniformly rotund** *(WUR) whenever* $\delta_X(x^*, \varepsilon) > 0$ *for all* $x^* \in S_X$ *and* $\varepsilon \in (0, 2]$.

An equivalent definition is (see [Ullán 90] for a proof):

$$\|x_n\| \to 1, \|y_n\| \to 1, \|x_n + y_n\| \to 2 \Rightarrow x_n - y_n \rightharpoonup 0$$

where the symbol $\rightharpoonup$ stands for the weak convergence.

Separation of this property

1. Every (UC) Banach space is (WUR). See, for example, the book of R.E. Megginson, p. 465, [Megginson 98].

   The converse is not true: The space $(\ell_2, \|.\|_w)$ given in [Smith 78] is (WUR) but not (UC).

2. (WUR) Banach spaces are strictly convex. See again the book of R.E. Megginson, p.465, [Megginson 98] for a proof.

   The converse is not true.

   Example 9 *[Smith 78].*

   Recall the equivalent norm defined on c_0 by M.M. Day: For u in c_0 enumerate the support of u as (n_k) in such a way that $|u(n_k)| \geq |u_{n_{k+1}}|$. Define $Du \in \ell_2$ by

   $$Du(n) := \begin{cases} \frac{u(n_k)}{2^k} & n = n_k \\ 0 & \text{otherwise} \end{cases}$$

   and define $|||u||| = \|Du\|_2$. For $x \in \ell_2$ let

   $$\|x\|_L := |||(\frac{1}{2}\|x\|_2, x_1, x_2, x_2, ..., \overbrace{x_j, x_j, ...x_j}^{j}, ...)|||$$

   Then $(\ell_2, \|.\|_L)$ is (LUR) (see below) and hence rotund, but not (WUR).

Other facts concerning this modulus and/or constant

1. [Ullán 90]. The function $\delta_X(x^*, .)$ is continuous in $[0, 2)$. Moreover for $0 \leq \varepsilon_1 \leq \varepsilon_2 < 2$,

   $$\delta_X(x^*, \varepsilon_2) - \delta_X(x^*, \varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{2 - \varepsilon_1}.$$

2. [Ullán 90]. The function $\varepsilon \mapsto \frac{\delta_X(x^*, \varepsilon)}{\varepsilon}$ is increasing in $(0, 2]$.

3 V.L. Smulian modulus of weak* uniform rotundity

For $x \in S_X$, the **modulus of convexity of X^* with respect to x**, is the function $\delta_X(x,.) : [0,2] \longrightarrow [0,1]$ given by the formula

$$\delta_{X^*}(x,\varepsilon) := \inf\left\{1 - \left\|\frac{1}{2}(x^* + y^*)\right\| : x^*, y^* \in S_{X^*}, |(x^* - y^*)(x)| \geq \varepsilon\right\}$$

Geometrical property in terms of this modulus and/or this coefficient

1. **Definition 3.1** *V.L. Šmulian, 1939. The Banach space $(X^*, \|.\|)$ is* **weak* uniformly rotund** *(W* UR) whenever $\delta_{X^*}(x,\varepsilon) > 0$ for all $x \in S_X$ and $\varepsilon \in (0,2]$.*

   Megginson's book [Megginson 98], on page 466 reads

   > *It should be noted that some sources say that if X is a normed space such that X^* satisfies the above definition of weak* uniform rotundity, then it is X instead of X^* which is called weak* uniformly rotund.*

2. ([Megginson 98], p. 466.)

   For dual Banach spaces, (WUR)⇒(W*UR)⇒ (SC).

4 Lovaglia local modulus of convexity, [Lovaglia 55]

For $x \in S_X$, the **modulus of convexity of Banach at the point** x, is the function

$$\delta_X(x,\varepsilon) := \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : y \in B_X, \|x-y\| \geq \varepsilon\right\}.$$

Related coefficient with this modulus

Characteristic of convexity of $(X, \|\cdot\|)$ **at** $x \in S_X$

$$\varepsilon_0(X,x) := \sup\{\varepsilon \in [0,2] : \delta_X(x,\varepsilon) = 0\}.$$

Geometrical property in terms of this modulus and/or this coefficient

Definition 4.1 *Lovaglia [Lovaglia 55]. The Banach space* $(X, \|\cdot\|)$ *is* **locally uniformly rotund** *(LUR) whenever* $\delta_X(x,\varepsilon) > 0$ *for all* $x \in S_X$ *and* $\varepsilon \in (0,2]$.

Separation of this property

1. Obviously, every (UC) Banach space is (LUR). The converse is not true:

   Example 10 *M.A. Smith, 1978. [Smith 78]*

   Let ℓ_1 be endowed with the norm

   $$\|x\| := (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}.$$

   Then for all $x \in \ell_1$

   $$\|x\|_1 \leq \|x\| \leq \sqrt{2}\|x\|_1$$

   and $(\ell_1, \|.\|)$ is (LUR) but it does not admit a uniformly convex norm (otherwise it would be a reflexive Banach space).

2. Neither of the conditions (LUR) and (WUR) implies the other. (See [Smith 78]).

3. (LUR) Banach spaces are strictly convex. The converse is not true again.

   Example 11 *M.A. Smith, 1978. [Smith 78]*

   Let ℓ_2 be endowed with the norm

   $$\|x\|_w := (\|x\|_S^2 + \|Tx\|_2^2)^{\frac{1}{2}}.$$

   where $\|x\|_S := \max\{|x_1|, \|(0, x_2, \ldots)\|_2\}$, and

   $$T(x_1, x_2, x_3, ...) := (x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, ...).$$

   Then the norm $\|.\|_w$ is equivalent to $\|.\|_2$ but $(\ell_2, \|.\|_w)$ is reflexive (SC) but not (LUR).

4. **Example 12** *See [Nelson et al. 87].* Consider the classical space c_0 endowed with Day's norm

$$\|x\| := \sup \left[\sum_1^\infty \frac{1}{2^j} (x(\alpha_j))^2 \right]^{\frac{1}{2}},$$

where the supremum is taken over all permutations α of the positive integers.

One has that $Z := (c_0, \|.\|)$ is (LUR) and hence (SC). Nevertheless, the well known mapping $T : B_Z^+ \to B_Z^+$ given by

$$T(x_1, x_2, ...) := (1, x_1, x_2, ...)$$

is $\|.\|$-nonexpansive and lacks fixed points. Thus, (LUR)$\not\Rightarrow$(FPP).

EXAMPLES

Example 13 *[Ullán 90].*
Let $X = (\mathbb{R}^2, \|.\|_\infty)$ and $x = (1, 0)$. Then

$$\delta_X(x, \varepsilon) = \begin{cases} 0 & 0 \le \varepsilon \le 1 \\ \frac{1}{2} & 1 < \varepsilon \le 2 \end{cases}$$

Other facts concerning this modulus and/or constant

1. [Danes 76]

$$\delta_X(x, \varepsilon) = \inf \left\{ 1 - \left\| \frac{1}{2}(x + y) \right\| : x, y \in B_X, \|x - y\| = \varepsilon \right\}.$$

5 Lovagia modulus of weak local uniform rotundity

For $x \in S_X$ and $x \in S_{X^*}$, the **modulus of convexity of Banach at x with respect to $x*$**, is the function $\delta_X(x, x^*, .) : [0,2] \longrightarrow [0,1]$

$$\delta_X(x, x^*, \varepsilon) := \inf \left(\{1\} \cup \left\{ 1 - \left\| \frac{1}{2}(x+y) \right\| : y \in S_X, |x^*(x-y)| \geq \varepsilon \right\} \right)$$

Geometrical property in terms of this modulus and/or this coefficient

Definition 5.1 *Lovaglia, 1955. The Banach space $(X, \|\cdot\|)$ is* **weakly locally uniformly rotund** *(WLUR) whenever $\delta_X(x, x^*, \varepsilon) > 0$ for all $x^* \in S_X$ and $\varepsilon \in (0,2]$.*

Separation of this property

1. (LUR)⇒ (WLUR). (See [Megginson 98])- The converse is not true: The space $(\ell_2, \|.\|_w)$ given in [Smith 78] is (WLUR) but not (LUR).
2. (WUR)⇒ (WLUR)⇒ (SC). None of the implications of this statement is reversible. M.A. Smith's example $(\ell_2, \|.\|_L)$ is (WLUR) but not (WUR).

Example 14 *M.A. Smith, [Smith 78].* For all x in ℓ_2 define

$$\|x\|_F := |x_1| + \|(0, x_2, x_3, ...)\|_2$$

and

$$\|x\|_A := \left(\|x\|_F^2 + \|(\frac{1}{2}x_2, \frac{1}{3}x_3, ...)\|_2^2 \right)^{\frac{1}{2}}.$$

Then $(\ell_2, \|.\|_A)$ is a reflexive (SC) not (WLUR) Banach space.

6 Lindenstrauss Modulus of Smoothness, [Lindenstrauss 63].

It is the function $\rho_X : [0, \infty) \longrightarrow \mathbb{R}$ given by

$$\rho_X(t) := \sup \left\{ \frac{1}{2}(\|x + ty\| + \|x - ty\|) - 1 : x, y \in B_X \right\}.$$

Related coefficient

$$\rho_0(X) := \lim_{t \to 0^+} \frac{\rho_X(t)}{t}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. **Definition 6.1** *The Banach* $(X, \|\cdot\|)$ *is* **uniformly smooth** *(US for short) whenever* $\rho_0(X) = 0$.
2. $\rho_0(X) < \frac{1}{2} \Rightarrow X$ is superrefexive and has uniformly normal structure. (See [Goebel-Kirk 90] p.70-71 for detailed references about this result).
3. (See[Kato et al. 01]). The following conditions are equivalent.
   a) $(X, \|\cdot\|)$ is uniformly non square.
   b) $\rho_X(\tau_0) < \tau_0$ for some $\tau_0 > 0$.
   c) $\rho_X(\tau) < \tau$ for all $\tau > 0$.
   d) $\lim_{t \to 0^+} \frac{\rho_X(t)}{t} < 1$.
   e) $\varepsilon_0(X^*) < 2$.
   d) $\lim_{t \to 0^+} \frac{\rho_{X^*}(t)}{t} < 1$.
4. [Gao t.a.1] If $\rho_X(\varepsilon) < \frac{\varepsilon}{2}$ for some $\varepsilon \in [0, 1]$ then $(X, \|\cdot\|)$ has uniformly normal structure.

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Example 15 .

For a non-trivial Hilbert space H $\rho_H(t) = \sqrt{1 + t^2} - 1$ for every $t \geq 0$. $\rho_X(t) \geq \rho_{\ell_2}(t)$ for $t \geq 0$.

Example 16 *See [Lindenstrauss 63]).*

Let (Ω, μ) be a measure space such that μ takes at least two different values. For $1 < p \leq 2$ then

$$\rho_{L^p(\Omega,\mu)}(t) = (1 + t^p)^{\frac{1}{p}} - 1,$$

and for $p > 2$, then

$$\rho_{L^p(\Omega,\mu)}(t) = \left(\frac{(1 + t)^p + |1 - t|^p}{2} \right)^{\frac{1}{p}} - 1$$

for every $t \geq 0$. Clearly this covers the case of the spaces ℓ_p.

Example 17 *[Ullán 90].*

$$\rho_{(\mathcal{C}([0,1]), \|.\|_\infty)}(t) = t$$

$(t \geq 0)$.

Other facts concerning this modulus and/or constant

1. For each positive t,
$$\max\{0, t-1\} \le \rho_X(t) \le t.$$
(See, for instance, [Megginson 98] for a proof).

2. $\rho_X(t) \ge \rho_{\ell_2}(t)$ for $t \ge 0$.

3. Alternative formulae for ρ_X:
$$\rho_X(t) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x \in B_X,\ \|y\| \le t\right\}.$$
$$\rho_X(t) := \sup\left\{\frac{1}{2}(\|x+ty\| + \|x-ty\|) - 1 : x, y \in S_X\right\}.$$

4. For every Banach space $(X, \|\cdot\|)$, the function ρ_X is increasing, continuous, convex on $[0,\infty)$, $\rho_X(0) = 0$ and $\rho_X(t) \le t$.

5. ([Figiel 76], Lemma 8).

   For all $\tau \ge 0$ one has $\rho_X(2\tau) \le 4(1 + \frac{1}{2}\tau)\rho_X(\tau)$.

6. ([Figiel 76], Lemma 9).

   If $u \ge 1$, $v \ge \frac{4}{3}$, then $\rho_X(uv) \le u^2\rho_X(v)$.

7. (See [Kato et al. 01]). The modulus of smoothness of any Banach space $(X, \|\cdot\|)$ satisfies either the equality $\rho_X(t) = t$ for all $t > 0$ of the strict inequality $\rho_X(t) < t$ for all $t > 0$.

8. Lindenstrauss formulae (see [Lindenstrauss 63]).
$$\rho_{X^*}(t) = \sup\left\{\frac{1}{2}\varepsilon t - \delta_X(\varepsilon) : 0 \le \varepsilon \le 2\right\}.$$
$$\rho_X(t) = \sup\left\{\frac{1}{2}\varepsilon t - \delta_{X^*}(\varepsilon) : 0 \le \varepsilon \le 2\right\}.$$
Thus, $\rho_0(X^*) = \frac{1}{2}\varepsilon_0(X)$ and $\rho_0(X) = \frac{1}{2}\varepsilon_0(X^*)$. Consequently, $(X, \|\cdot\|)$ is (UC) if and only if its dual space is (US) and $(X, \|\cdot\|)$ is (US) if and only if its dual space is (UC).

9. $\rho_0(X) = 0$ if and only if for each positive η there exits a positive ε (depending on η) such that, if $x \in S_X$, $y \in X$ and $\|x-y\| \le \varepsilon$, then
$$\|x+y\| \ge \|x\| + \|y\| - \eta\|x-y\|.$$
(See [Day 73], p. 147).

7 Directional modulus of rotundity

For a given non zero $z \in X$, the **modulus of convexity of Banach in the direction of** z, is the function $\delta_X(\to z, \cdot) : [0, 2] \longrightarrow [0, 1]$

$$\delta_X(\to z, \varepsilon) := \inf\left\{1 - \left\|\frac{1}{2}(x+y)\right\| : x, y \in S_X, \|x-y\| \geq \varepsilon,\ \exists \lambda \in \mathbb{R}\ s.t.\ x - y = \lambda z\right\}.$$

Related coefficient with this modulus

$$\varepsilon_{0,z}(X) := \sup\{\varepsilon : \delta_X(\to z, \varepsilon) = 0\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. **Definition 7.1** *The Banach space* $(X, \|\cdot\|)$ *is* **uniformly convex in every direction** *(UCED) whenever* $\delta_X(\to z, \varepsilon) > 0$ *for all* $z \in X \backslash \{0\}$ *and* $\varepsilon \in (0, 2]$.
2. ([Goebel 70].) If $\varepsilon_{0,z}(X) < 1$ for all $z \in X$, $z \neq 0$, then $(X, \|\cdot\|)$ has normal structure.
3. ([Day et al. 71], [Zidler 71]). $(UCED) \Rightarrow (WNS)$.

Separation of (UCED)

1. (WUR)⇒ (UCED)⇒ (SC).

   The converse implications are not true.

   The above example $(\ell_2, \|.\|_A)$ is a (UCED) Banach space which is not (WUR). On the other hand, $(\ell_2, \|.\|_L)$ is (SC) but not (UCED).

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Example 18 *(See [Prus 00]).*

For a Hilbert space H, $\delta_H(\to z, \varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2}$, for every $z \in S_H$ and $\varepsilon \in [0, 2]$.

Other facts concerning this modulus and/or constant

(See [Prus 00]).

1. $\delta_X(\to z, \cdot)$ need not be a convex function.
2. Every separable Banach space admits an equivalent norm with respect to which it is (UCED).

8 Gurarii modulus of convexity, [Gurarii 67].

It is defined by the formula

$$\beta_X(\varepsilon) := \inf\left\{1 - \inf_{t\in[0,1]} \|tx + (1-t)y\| : x, y \in S_X, \|x - y\| = \varepsilon\right\}.$$

(We follow [Sánchez-Ullán 98]).

Geometrical properties in terms of this modulus

1. $(X, \|\cdot\|)$ is (SC) if and only if $\beta_X(2) = 1$.
2. $(X, \|\cdot\|)$ is uniformly non square if and only if there exists $\varepsilon \in (0, 2)$ such that $\beta_X(\varepsilon) > 0$.
3. $(X, \|\cdot\|)$ is uniformly convex if and only $\beta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.
4. If $(X, \|\cdot\|)$ is strictly convex and β_X is continuous at $\varepsilon = 2$ then $(X, \|\cdot\|)$ is uniformly non square.
5. If the inequality $\beta_X(\varepsilon) \geq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ holds in a normed space $(X, \|\cdot\|)$ for every $\varepsilon \in (0, 2]$ then $(X, \|\cdot\|)$ is an inner product space.

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Example 19 .
If $(X, \|\cdot\|)$ is an inner product space, $\beta_X(\varepsilon) = \delta_X(\varepsilon)$.

Example 20 .
For $p \geq 2$ and for all $\varepsilon \in [0, 2]$,

$$\beta_{\ell_p}(\varepsilon) = \beta_{L^p}(\varepsilon) = 1 - \left(1 - \left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}.$$

Other facts concerning this modulus and/or constant

1. For any $\varepsilon \in [0, 2]$, $\delta_X(\varepsilon) \leq \beta_X(\varepsilon) \leq 2\delta_X(\varepsilon)$.
2. $\beta_X(\varepsilon)$ is a continuous function at $[0, 2)$. Continuity may fail at $\varepsilon = 2$. Nevertheless, if $(X, \|\cdot\|)$ is a uniformly convex Banach space then β_X is continuous at $\varepsilon = 2$.
3. $\beta_X(\varepsilon) = \inf\left\{1 - \inf_{t\in[0,1]} \|tx + (1-t)y\| : x, y \in B_X, \|x - y\| = \varepsilon\right\}$.
4. β_X is nondecreasing on $[0, 2]$.
5. $\beta_X(\varepsilon) = \inf\left\{1 - \inf_{t\in[0,1]} \|tx + (1-t)y\| : x, y \in B_X, \|x - y\| \geq \varepsilon\right\}$.
6. For all $r \in [0, 1]$, $\beta_X(r\varepsilon) \leq r\beta_X(\varepsilon)$.
7. The function $\varepsilon \mapsto \frac{\beta_X(\varepsilon)}{\varepsilon}$ is non-decreasing in $(0, 2]$.
8. β_X is strictly increasing in $[\varepsilon_0(X), 2]$.
9. For $p \geq 1$ and $(X, \|\cdot\|)$ is the Banach space ℓ_p or L^p then for all $\varepsilon \in [0, 2]$,

$$\beta_X(\varepsilon) \leq 1 - \left(1 - \left(\frac{1}{2}\right)^p\right)^{\frac{1}{p}}.$$

9 Milman k-dimensional modulus of convexity [Milman 71]

For any normed space $(X, \|\cdot\|)$ with $\dim(X) \geq k$ and for $\varepsilon \geq 0$ one defines

$$\Delta_X^{(k)}(\varepsilon) := \inf_{x \in S_X} \inf_{E \in \mathcal{E}_k} \sup\left\{\|x + \varepsilon y\| - 1 : y \in S_E\right\}.$$

Geometrical property in terms of this modulus and/or this coefficient

Definition 9.1 *The Banach space $(X, \|\cdot\|)$ is k-**uniformly convex** (k-UC for short) whenever $\Delta_X^{(k)}(\varepsilon) > 0$ for $\varepsilon > 0$.*

Geremia and Sullivan for $k = 2$ and P.K. Lin ([Lin 88]) in the general case have shown that k-uniform convexity and k-uniform rotundity are equivalent properties.

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Example 21 .If $(X, \|\cdot\|)$ is an inner product space, and $\dim(X) > k$ then

$$\Delta_X^k(\varepsilon) = \sqrt{1 + \varepsilon^2} - 1.$$

$(\varepsilon \geq 0)$.

Example 22 .

$$\Delta^1_{(\mathbb{R}^2, \|.\|_\infty)}(\varepsilon) = \begin{cases} 0 & 0 \leq \varepsilon \leq 1 \\ \varepsilon - 1 & 1 \leq \varepsilon \end{cases}$$

Other facts concerning this modulus and/or constant

(See [Prus 00].)

1. $$\Delta_X^{(k)}(\varepsilon) = \inf_{x \in S_X} \inf_{E \in \mathcal{E}_k} \sup\left\{\|x + \varepsilon y\| - 1 : y \in B_E\right\}.$$

2. $$\Delta_X^1(\varepsilon) = \inf_{x, y \in S_X} \left\{\max\{\|x + \varepsilon y\|, \|x - \varepsilon y\|\} - 1\right\}.$$

3. $0 = \Delta_X^{(k)}(0) \leq \Delta_X^{(k)}(\varepsilon)$ for every $\varepsilon \geq 0$.

4. If Banach is infinite dimensional we have, for any $\varepsilon \geq 0$,

$$\Delta_X^{(k)}(\varepsilon) = \inf\{\Delta_E^{(k)}(\varepsilon)\}$$

where the infimum is taken over all subspaces E of X with $\dim(E) \geq k + 1$.

5. The function $\varepsilon \mapsto \frac{\Delta_X^{(k)}(\varepsilon)}{\varepsilon}$ is nondecreasing in $(0, \infty)$. (See [Ullán 90] for a proof).

6. ([Ullán 90]). For $0 \leq \varepsilon_1 < \varepsilon_2 \leq a$ one has

$$\Delta_X^{(k)}(\varepsilon_2) - \Delta_X^{(k)}(\varepsilon_1) \leq \frac{\varepsilon_2 - \varepsilon_1}{a - \varepsilon_1} a$$

7. The function $\Delta_X^{(k)}(.)$ is continuous in $[0, \infty)$.

8. $\Delta_X^{(k)}(t) \leq \Delta_X^{(k+1)}(t)$ for every $t \geq 0$. Consequently $(k - UC) \Rightarrow ((k+1) - UC)$.

9. ([Figiel 76]).
$$\Delta_X^1\left(\frac{\varepsilon}{2(1-\delta_X(\varepsilon))}\right) = \frac{\delta_X(\varepsilon)}{1-\delta_X(\varepsilon)}.$$

10. ([Ullán 90]). For any $\varepsilon \in (0,1]$
$$\Delta_X^1\left(\frac{1}{\varepsilon}\right) = \frac{\Delta_X^1(\varepsilon)+1}{\varepsilon} - 1.$$

11. ([Ullán 90]). For any normed space $(X, \|\cdot\|)$,
$$\Delta_X^1(\varepsilon) \leq \sqrt{1+\varepsilon^2} - 1$$
($\varepsilon \geq 0$).

10 Milman k-dimensional moduli of smoothness

Notation: $\mathcal{E}_k$ is the collection of all k-dimensional subspaces of X.

For each $x \in S_X$ and $t \geq 0$, the k-dimensional modulus of smoothness at x is given by

$$\beta_X(k,x,t) := \sup_{E \in \mathcal{E}_k} \left\{ \inf\{ \frac{\|x-ty\| + \|x+ty\|}{2} - 1 : y \in S_E \} \right\}.$$

Milman's k-dimensional modulus of smoothness is given by

$$\beta_X^{(k)}(t) := \sup\{\beta_X(k,x,t) : x \in S_X\}.$$

Geometrical property in terms of this modulus and/or this coefficient

Definition 10.1 *A Banach space $(X, \|\cdot\|)$ is called k-**uniformly smooth**, (k-US) for short, if*

$$\lim_{t \to 0^+} \frac{\beta_X^{(k)}(t)}{t} = 0.$$

Other facts concerning this modulus and/or constant

([Prus 00].)

1. $0 = \beta_X^{(k)}(0) \leq \beta_X^{(k)}(\varepsilon)$ for every $\varepsilon \geq 0$.
2. The function $\varepsilon \mapsto \frac{\beta_X^{(k)}(\varepsilon)}{\varepsilon}$ is nondecreasing in $(0, \infty)$.
3. $\beta_X^{(k+1)}(t) \leq \beta_X^{(k)}(t)$ for every $t \geq 0$. Consequently $(k - US) \Rightarrow ((k+1) - US)$.

11 Modulus of k-rotundity [Sullivan 79]

For $x_1, x_2, \dots, x_{k+1} \in X$ the value

$$V(x_1, x_2, \dots, x_{k+1}) := \sup \left\{ \det \begin{pmatrix} 1 & \dots & 1 \\ f_1(x_1) & \dots & f_1(x_{k+1}) \\ \vdots & \dots & \vdots \\ f_k(x_1) & \dots & f_k(x_{k+1}) \end{pmatrix} : f_1, \dots, f_k \in B_{X^*} \right\}$$

is called k**-volume** of the set $\mathrm{co}\{x_1, \dots, x_{k+1}\}$.

The function

$$\delta_X^{(k)}(\varepsilon) := \inf \left\{ 1 - \frac{1}{k+1} \|x_1 + \dots + x_{k+1}\| : x_1, \dots, x_{k+1} \in B_X,\ V(x_1, \dots, x_{k+1}) \geq \varepsilon \right\}$$

is called the **modulus of** k**-rotundity of** $(X, \|.\|)$**.** Clearly this function is defined on the interval $[0, \mu_k(X))$ where

$$\mu_k(X) := \sup\{V(x_1, \dots, x_{k+1}) : x_1, \dots, x_{k+1} \in B_X\}.$$

Related coefficient with this modulus

Characteristic of k**-convexity of** $(X, \|\cdot\|)$

$$\varepsilon_0^{(k)}(X) := \sup\{\varepsilon > 0 : \delta_X^{(k)}(\varepsilon) = 0\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. **Definition 11.1** *[Sullivan 79]. (See also [Geremia-Sullivan 81]). The Banach space* $(X, \|\cdot\|)$ *is* k**-uniformly rotund** *((k-UR) for short) whenever* $\delta_X^{(k)}(\varepsilon) > 0$ *for* $\varepsilon \in (0, \mu_k(X))$. *(The case* $k = 1$ *corresponds to Clarkson's uniform convexity.)*

2. ([Bernal-Sullivan 83]).If $\varepsilon_0^{(k)}(X) < 1$ then $(X, \|\cdot\|)$ has (UNS).

Separation of (k-UR)

In [Kutzarova 91] D. Kutzarova gave the following

Example 23 Let X be the ℓ_1 direct sum of the Banach spaces $Y = \mathbb{R}^1$ and $Z = \ell_2$. Then X is 2-uniformly rotund. But $\varepsilon_0(X) = 2$. Of course, X is not uniformly convex.

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Example 24 . *For a infinite-dimensional Hilbert space* H*,*

$$\delta_H^k(\varepsilon) = 1 - \left(1 - \frac{k}{(k+1)^{1+1/k}} \varepsilon^{2/k}\right)^{\frac{1}{2}}.$$

Other facts concerning this modulus and/or constant

(See [Prus 00].)

1. In the definition of $\delta_X^k(\varepsilon)$ '$\geq \varepsilon$' can be replaced by '$= \varepsilon'$.

2. The function $\delta_X^k(\cdot)$ is nondecreasing.

3. The function $\delta_X^k(\cdot)$ is continuous on $[0, \mu_k(X))$.

4. The function $\delta_X^k(\cdot)$ is lipschitzian on each interval $(a, b)$where $0 < a < b < \mu_k(X)$.

5. The function $\varepsilon \mapsto \frac{\delta_X^k(\varepsilon^k)}{\varepsilon}$ is nondecreasing.

6. (UC)$\Rightarrow$ (kUR) $\Rightarrow$ ($k+1$UR). In fact,

$$\delta_X(\varepsilon) = \delta_X^{(1)}(\varepsilon) \leq \delta_X^{(k)}(\varepsilon) \leq \delta_X^{(k+1)}(\varepsilon).$$

   for all positive integer k and all suitable ε.

7. If X is infinite dimensional, as well as the Hilbert space H,

$$\delta_X^k(\varepsilon) \leq \delta_H^k$$

8. ([Bernal-Sullivan 83]).Let $k \in \mathbb{N}$. If $\varepsilon_0^{(k)}(X) < 2^k$ then X is superreflexive.

9. ([Lin 88]).

$$\delta_X^k\left(\frac{(k+1)\varepsilon^k}{3^k(1+\Delta_X^k(\varepsilon))^k}\right) \leq \frac{\Delta_X^k(\varepsilon))}{1+\Delta_X^k(\varepsilon))}.$$

$$\Delta_X^k\left(\frac{\varepsilon}{(k+1)^{(k+1)}(1-\delta_X^k(\varepsilon))}\right) \leq \frac{\delta_X^k(\varepsilon))}{1+\delta_X^k(\varepsilon))}.$$

12 Partington modulus of "UKK-ness", [Partington 83].

It was defined by J.R. Partington in the proof of Theorem 1 of [Partington 83]. It is the function $P_X : [0, K(X)) \longrightarrow [0,1]$ given by

$$P_X(\varepsilon) := \inf\{1 - \|x\| : \exists x_n \in B_X \ (n = 1, \ldots), x_n \overset{w}{\rightharpoonup} x, \ \ \|x_m - x_n\| \geq \varepsilon, \ (m \neq n)\}$$

Here $K(X) = K(B_X)$ is the so called *Kottman constant of the space* $(X, \|\cdot\|)$, that is,

$$K(B_X) := \sup\left\{\{\inf \|x_m - x_n\| : m \neq n\} : x_n \in B_X \ (n = 1, ...)\right\}.$$

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Example 25 .

$$P_{\ell_p}(\varepsilon) = 1 - \left(1 - \frac{\varepsilon^p}{2}\right)^{\frac{1}{p}}$$

for $\varepsilon \in [0, 2^{\frac{1}{p}})$. (See [Domínguez-López 92], or [Banaś-Fraczek 93]).

13 Goebel-Sekowski modulus of noncompact convexity, [Goebel-Sekowski 84].

Let us suppose that $\dim(X) = \infty$. The modulus of noncompact convexity of a Banach space $(X, \|\cdot\|)$ is the function $\Delta_X : [0,2] \to [0,1]$ defined by

$$\Delta_X(\varepsilon) := \inf\{1 - \text{dist}\,(0, A) : A \subset B_X, A \neq \emptyset,\ A = \overline{\text{co}}(A)\ \alpha(A) \geq \varepsilon\}$$

Here $\alpha(A) := \inf\{r > 0 : A \subset B_1 \cup ... \cup B_k, \text{diam}(B_i) < r\}$ is the Kuratowski measure of noncompactness of the set $A \subset X$.

Related coefficient

$$\varepsilon_1(X) := \sup\{\varepsilon \in [0,2] : \Delta_X(\varepsilon) = 0\}.$$

is called **Characteristic of noncompact convexity of** $(X, \|\cdot\|)$.

Geometrical property in terms of this modulus and/or this coefficient

1. **Definition 13.1** *A space for which $\varepsilon_1(X) = 0$ is said to be Δ-uniformly convex.*

2. A Banach space is (NUC) if $\dim(X) < \infty$ or $\dim(X) = \infty$ and $\Delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$.

   Recall that a Banach space is said to be nearly uniformly covex (NUC) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any sequence (x_n) in B_X with $\text{sep}((x_n)) > \varepsilon$ one has that

   $$\text{dist}(0, \text{co}(\{x_n\})) < 1 - \delta.$$

   Here $\text{sep}((x_n)) := \inf\{\|x_n - x_m\| : m \neq n\}$.

Separation of this property

Example 26 Let ℓ_1^n (ℓ_∞^n) be the space $\mathbb{R}^n$ endowed with the norm $\|\cdot\|_1$, ($\|\cdot\|_\infty$). We consider the Day spaces

$$D_1 := (\ell_1^1 \times \ell_1^2 \times \ldots \times \ell_1^n \times \ldots)_{\ell_2}$$
$$D_\infty := (\ell_\infty^1 \times \ell_\infty^2 \times \ldots \times \ell_\infty^n \times \ldots)_{\ell_2}$$

Then one has (see [Goebel-Sekowski 84]) that for all $\varepsilon \in [0,2]$

$$\Delta_{D_1}(\varepsilon) = \Delta_{D_\infty}(\varepsilon) = \delta_H(\varepsilon) = 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{1}{2}}.$$

Thus D_1 and D_∞ are Δ-uniformly convex. However, the spaces D_1, D_∞ are not uniformly convex, nor even superreflexive. Thus $\delta_{D_1}(\varepsilon) = \delta_{D_\infty}(\varepsilon) \equiv 0$ even under equivalent renormings.

The spaces D_1 and D_∞ do not have uniform normal structure although they have normal structure.

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Example 27 .

For $1 < p < \infty$,

$$\Delta_{\ell_p}(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

One can see the proof in the joint work of Goebel and Sekowski [Goebel-Sekowski 84].

Example 28 .
$\Delta_{\ell_1}(\varepsilon) \equiv 0.$

Example 29 *(See [Sekowski 86]).*
The modulus of noncompact convexity of the space $\ell_{p,1}$ is

$$\Delta_{\ell_{p,1}}(\varepsilon) = 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

for any $\varepsilon \in [0, 2]$.

Example 30 *See [Banaś 90].*
If $2 \leq p < \infty$ and $\varepsilon \in [0, 2]$

$$\Delta_{\ell_p}(\varepsilon) = \Delta_{L_p([0,1])}(\varepsilon) = \delta_{L_p([0,1])}(\varepsilon).$$

However, for $1 < p < 2$ and $\varepsilon \in (0, 2]$,

$$\Delta_{\ell_p}(\varepsilon) > \Delta_{\ell_2}(\varepsilon) = \delta_H(\varepsilon) > \delta_{\ell_p}(\varepsilon).$$

Other facts about this modulus and/or this coefficient

1. $\Delta_X(\varepsilon) \geq \delta_X(\varepsilon)$ for every $\varepsilon \in [0, 2]$.
2. For any positive integer k, and $0 \leq t < \varepsilon \leq 2$,
$$\Delta_X(\varepsilon) \geq \delta_X^{(k)}\left(\left(\frac{t}{2}\right)^k\right).$$
See [Kirk 88] for a proof.
3. ([Banaś-Fraczek 93]).
$$P_X(\varepsilon) \geq \Delta_X\left(\frac{\varepsilon}{K_{B_X}}\right)$$
where $\Delta_X(.)$ is the modulus of noncompact convexity.
4. Although $\Delta_{\ell_1}(\varepsilon) \equiv 0$, note that $\lim_{p\downarrow 1} \Delta_{\ell_p}(\varepsilon) = \frac{\varepsilon}{2} \neq \Delta_{\ell_1}(\varepsilon)$.
5. (?)
$$\Delta_{\ell_p} = \Delta_X.$$
whenever X is the ℓ_p direct sum of a sequence (X_n) of finite dimensional Banach spaces.
6. The exact values of $\Delta_{L_p([0,1])}$ are not known if $1 < p < 2$, although they are different from those of Δ_{ℓ_p}. For more details see [Prus 94].
7. ([Goebel-Sekowski 84]). If $\varepsilon_1(X) < 1$ then $(X, \|\cdot\|)$ is reflexive. (Then, (NUC) Banach spaces are reflexive).
8. ([Goebel-Sekowski 84]). If $\varepsilon_1(X) < 1$ then $(X, \|\cdot\|)$ has normal structure.

9. ([Sekowski 86], [Goebel-Kirk 90]). Stability of condition $\varepsilon_1(X) < 1$.

   Let X be a Banach space and let $X_1 := (X, \|.\|_1)$ and $X_2 := (X, \|.|_2)$ where $\|.\|_1$ and $\|.\|_2$ are two equivalent norms on X satisfying for $\alpha, \beta > 0$,

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_2$$

   for all $x \in X$. If $k := \frac{\beta}{\alpha}$ then

$$\Delta_{X_2}(\varepsilon) \geq 1 - k\left(1 - \Delta_{X_1}\left(\frac{\varepsilon}{k}\right)\right).$$

   Consequently, if $\varepsilon_1(X_1) < 1$ and

$$k\left(1 - \Delta_{X_1}\left(\frac{1}{k}\right)\right) < 1$$

   then $\varepsilon_0(X_2) < 1$

14 Banas modulus of noncompact convexity, [Banaś 87].

Let us suppose that $\dim(X) = \infty$. The modulus of noncompact convexity with respect to the Hausdorff measure of noncompactness of a Banach space $(X, \|\cdot\|)$ is the function $\Delta_{X,\chi} : [0,1] \longrightarrow [0,1]$ defined by

$$\Delta_{X,\chi}(\varepsilon) := \inf\left\{1 - \text{dist}\,(0, A) : A \subset B_X, A \neq \emptyset,\ \text{closed, and convex}\ \chi(A) \geq \varepsilon\right\}$$

Here $\chi(A) := \inf\{r > 0 : A \subset B_1 \cup ... \cup B_k, B_i$ balls of radii smaller than $r\}$ is the *Hausdorff measure of noncompactness of the set* $A \subset X$.

Related coefficient

$$\varepsilon_\chi(X) := \sup\{\varepsilon \geq 0 : \Delta_{X,\chi}(\varepsilon) = 0\}.$$

is called **Characteristic of noncompact convexity** of $(X, \|\cdot\|)$ associated to the **Hausdorff** measure of noncompactness.

Geometrical properties in terms of this modulus and/or this coefficient

1. A space for which $\varepsilon_\chi(X) = 0$ is said to be Δ_χ-uniformly convex.
2. ([Banaś 87]). $(X, \|\cdot\|)$ is (NUC) if and only if $\varepsilon_\chi(X) = 0$.
3. ([Banaś 87]). If $\varepsilon_\chi(X) < \frac{1}{2}$ then $(X, \|\cdot\|)$ (is reflexive and) has normal structure.
4. ([García et al. 94]). Banach spaces with nonstrict Opial condition and with $\varepsilon_\chi(X) < 1$ have weakly normal structure.

   Recall that $(X, \|\cdot\|)$ satisfies the nonstrict Opial condition provided that if a sequence in Banach is weakly convergent to $x \in X$ then

   $$\liminf_n \|x_n - x\| \leq \liminf_n \|x_n - y\|$$

   for every $y \in X$.

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Example 31 ([Banaś 90]).

$$\Delta_{\ell_p,\chi}(\varepsilon) = 1 - (1 - \varepsilon^p)^{\frac{1}{p}}.$$

Other facts concerning this modulus and/or constant

1. ([Banaś 87]). $\Delta_X(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon) \leq \Delta_X(2\varepsilon)$ for every $\varepsilon \in [0,1]$.
2. ([Banaś 87]). The function $\Delta_{X,\chi}(\cdot)$ is continuous on the interval $[0,1)$.
3. ([Banaś 87]). If X is a reflexive Banach space then
   (a) $\Delta_{X,\chi}$ is a subhomogeneous function, that is $\Delta_{X,\chi}(k\varepsilon) \leq k\Delta_{X,\chi}(\varepsilon)$ for any $k, \varepsilon \in [0,1]$.
   (b) For any $\varepsilon \in [0,1]$, $\Delta_{X,\chi}(\varepsilon) \leq \varepsilon$.
   (c) The function $\Delta_{X,\chi}(\cdot)$ is strictly increasing on the interval $[\varepsilon_\chi(X), 1]$. The function $\varepsilon \mapsto \Delta_{X,\chi}(\varepsilon)\varepsilon$ is nondecreasing on $(0,1]$ and $\Delta_{X,\chi}(\varepsilon_1 + \varepsilon_2) \geq \Delta_{X,\chi}(\varepsilon_1) + \Delta_{X,\chi}(\varepsilon_2)$ whenever $0 \leq \varepsilon_1 + \varepsilon_2 \leq 1$.

4. ([Ayerbe et al. 97 (b)], p. 89). If $\varepsilon_\chi(X) < 1$ then X is reflexive.

5. ([Banaś 87]). Stability of condition $\varepsilon_\chi(X) < \frac{1}{2}$.

   Let X be a Banach space with $\varepsilon_\chi(X) < \frac{1}{2}$. Let $B > 1$ be such that

$$1 - \frac{1}{B} = \Delta_\chi\left(\frac{1}{2B}\right).$$

   (which exists in view of the continuity of the function Δ_χ). If Y is another Banach space with $d(X,Y) < B$, then $\Delta_\chi(Y) < \frac{1}{2}$.

6. ([Domínguez-Japón 01]). Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty weakly compact subset of X and $T : C \to C$ and asymptotically regular mapping. Let $h := \sup\left\{t \geq 1 : \frac{1}{t}\Delta_{X,\chi}\left(\frac{1}{t}\right) \geq 1\right\}$. If $\liminf_n |T^n| < h$, then T has a fixed point. (Here $|T|$ denotes the (exact) Lipschitz constant of T on C).

15 Domínguez-López modulus of noncompact convexity, [Domínguez-López 92].

Let us suppose that $\dim(X) = \infty$. The modulus of noncompact convexity with respect to the Istrațescu measure (or separation measure) of noncompactness of a Banach space $(X, \|\cdot\|)$ is the function $\Delta_{X,\beta} : [0, \beta(B_X)] \longrightarrow [0,1]$ defined by

$$\Delta_{X,\beta}(\varepsilon) := \inf \{1 - \text{dist }(0, A) : A \subset B_X, A \neq \emptyset, \text{ closed, and convex } \beta(A) \geq \varepsilon\}.$$

Here $\beta(A) := \sup\{r > 0 : A \text{ has an infinite } r \text{ separation}\}$, where a r separation of A is a nonempty subset $S \subset A$ such that $\|x - y\| \geq r$ for all $x, y \in S$, $x \neq y$.

Related coefficient

$$\varepsilon_\beta(X) := \sup\{\varepsilon \geq 0 : \Delta_{X,\beta}(\varepsilon) = 0\}.$$

is called **characteristic of non-compact convexity** of $(X, \|\cdot\|)$ associated to the **separation** measure of noncompactness.

Geometrical properties in terms of this modulus and/or this coefficient

1. A space for which $\varepsilon_\beta(X) = 0$ is said to be Δ_β-uniformly convex.
2. ([Ayerbe et al. 97 (b)].) $(X, \|\cdot\|)$ is (NUC) if and only if $\varepsilon_\beta(X) = 0$.
3. ([Domínguez-López 92].) If $\varepsilon_\beta(X) < 1$ then $(X, \|\cdot\|)$ has normal structure.
4. ([García et al. 94].) If $\Delta_{X,\beta}(1) \neq 0$ then $(X, \|\cdot\|)$ has weak normal structure.

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Example 32 ([Domínguez-López 92]). For $1 < p < \infty$,

$$\Delta_{\ell_p,\beta}(\varepsilon) = 1 - \left(1 - \frac{\varepsilon^p}{2}\right)^{\frac{1}{p}}.$$

Example 33 ([Ayerbe et al. 97 (b)], p. 96.).

$$\Delta_{D_1,\beta}(\varepsilon) = \Delta_{D_\infty,\beta}(\varepsilon) = 1 - \left(1 - \frac{\varepsilon^2}{2}\right)^{\frac{1}{2}}.$$

So the spaces D_1 and D_∞ are (NUC) but fail to be k-UC for any k.

Other facts concerning this modulus and/or constant

1. ([Ayerbe et al. 97 (b)], p.86.) $\delta_X(\varepsilon) \leq \Delta_X(\varepsilon) \leq \Delta_{X,\beta}(\varepsilon) \leq \Delta_{X,\chi}(\varepsilon)$ and consequently
$$\varepsilon_0(X) \geq \varepsilon_1(X) \geq \varepsilon_\beta(X) \geq \varepsilon_\chi(X).$$
2. ([Ayerbe et al. 97 (b)], p. 90.). If $\varepsilon_\beta(X) < 1$ then X is reflexive.
3. ([Ayerbe et al. 97 (b)], Remark 1.12, p. 93.) For reflexive spaces, Partington's modulus is identical to the modulus of noncompact convexity associated to β.

4. ([Domínguez-Japón 01]). Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty weakly compact subset of X and $T : C \to C$ and asymptotically regular mapping. If $\liminf_n |T^n| < 1/(1-\Delta_{X,\beta}(1^-)$ then T has a fixed point. (Here $|T|$ denotes the (exact) Lipschitz constant of T on C).

5. ([Domínguez-Lorenzo 04]). Let C be a nonempty closed bounded convex subset of a Banach space $(X, \|\cdot\|)$ such that $\varepsilon_\beta(X) < 1$, and T be a compact-convex (set-)valued nonexpansive mapping. Then T has a fixed point.

16 Opial's modulus

It was defined by S. Prus in [Prus 92]. See also (Lin-Tan-Xu, [Lin et al. 95]).

It is the function $r_X : [0, \infty) \longrightarrow \mathbb{R}$ given by

$$r_X(c) := \inf\{\liminf \|x + x_n\| - 1 : \|x\| \geq c,\ x_n \overset{w}{\rightharpoonup} 0,\ \liminf_n \|x_n\| \geq 1\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

Definition 16.1 *A Banach space $(X, \|\cdot\|)$ is said to satisfy the* **uniform Opial property** *if for any $c > 0$ there exists an $r > 0$ such that*

$$1 + r \leq \liminf_{n\to\infty} \|x + x_n\|$$

for each $x \in X$ with $\|x\| \geq c$ and each sequence (x_n) in X such that $x_n \overset{w}{\rightharpoonup} 0$ and $\liminf \|x_n\| \geq 1$.

1. The space $(X, \|\cdot\|)$ satisfies the uniform Opial property if and only if $r_X(c) > 0$ for all $c > 0$.
2. ([Xu 96]).If $r_X(c) \geq 0$ for all $c \geq 0$ then $(X, \|\cdot\|)$ has the weak Opial property.
3. ([Xu 97]).If $r_X(c) > 0$ for some $c \in (0, 1)$, then $(X, \|\cdot\|)$ has weakly normal structure.

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Example 34 For $1 < p < \infty$

$$r_{\ell_p}(c) = (1 + c)^{\frac{1}{p}} - 1$$

$(c \geq 0)$. (See [Lin et al. 95]).

Example 35 (See [Xu 96].)
If $1 < p < \infty$ and $1 \leq q < \infty$ then for all $c \geq 0$,

$$r_{\ell_{p,q}}(c) = \min\left\{(1 + c^p)^{\frac{1}{p}} - 1, (1 + c^q)^{\frac{1}{q}} - 1\right\}.$$

Example 36 [Japón, 98].
Let X_p be ℓ_2 endowed with the norm

$$\|x\| := \left(|x_1|^p + \left(\sum_{n=2}^{\infty} |x_n|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}.$$

where $p > 2$. Then

$$r_{X_p}(c) = (c^p + 1)^{\frac{1}{p}} - 1.$$

Other facts concerning this modulus and/or constant

1. For all $c \geq 0$,
$$c - 1 \leq r_X(c) \leq c.$$
In particular, $r_X(c) > 0$ for all $c > 1$. (See [Lin et al. 95]).

2. The function r_X is continuous on $[0, \infty)$.

3. The function $c \mapsto \frac{1+r_X(c)}{c}$ is nondecreasing on $(0, \infty)$. In fact we have

$$r_X(c_1) - r_X(c_2) \leq (c_2 - c_1)\frac{1 + r_X(c_1)}{c_1}$$

   whenever $0 < c_1 \leq c_2$. (See [Lin et al. 95].)

4. ([Kuczumow 99]. See also [Kukzumow-Reich 97] for a generalization.) Let X be a Banach space with $r_X(1) > 0$ and with the nonstrict Opial property, C a nonempty weakly compact subset of X and $\mathcal{T} = \{T_t : t \in G\}$ an asymptotically regular semigroup with

$$\sigma(\mathcal{T}) = k < 1 + r_X(1).$$

   Then there exists $z \in C$ such that $T_t(z) = z$ for all $t \in G$. Here $\sigma(\mathcal{T}) := \liminf |T_t|$ and $k := \sup_t |T_t|$, where $|T_t|$ denotes the exact Lipschitz constant of T_t in C.

5. ([Lin et al. 95]). Suppose C is a weakly compact convex subset of X. If $(X, \|\cdot\|)$ satisfies the uniform Opial property and $T : C \to C$ is a mapping of asymptotically nonexpansive type ; that is, for every $x \in C$,

$$\limsup_n \left[\sup\{\|T^n(x) - T^n(y)\| - \|x - y\| : y \in C\}\right] \leq 0.$$

   If T^N is continuous for some integer $N \geq 1$, then T has a fixed point.

6. ([Domínguez-Japón 01]). Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty weakly compact subset of X and $T : C \to C$ and asymptotically regular mapping. If $\liminf_n |T^n| < 1 + r_X(1)$ then T has a fixed point.

7. See (12) for another result concerning the Opial modulus.

17 Six moduli for the property (β) of Rolewicz.

In a Banach space $(X, \|\cdot\|)$ the *drop* $D(x, B_X)$ defined by an element $x \in X \backslash B_X$ is the set $\mathrm{co}(\{x\} \cup B_X)$, and we write $R_x := D(x, B_X) \backslash B_X$.

Recall that $(X, \|\cdot\|)$ is said to have the property (β) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$1 < \|x\| < 1 + \delta \Rightarrow \mu(R_x) < \varepsilon$$

where μ is any of the three measures of noncompactness α, β, S. ($S(A)$ stands for the separation measure of the set A, that is $S(A) := \sup\{\varepsilon > 0 : \exists (x_n) \text{ in } A \text{ with } sep(x_n) > \varepsilon\}$.)

In [Ayerbe et al. 94] the authors defined three moduli as follows

$R_X : [0, 2] \longrightarrow [0, 1]$:

$$R_X(\varepsilon) := 1 - \sup\left\{ \inf\left\{ \frac{1}{2}\|x + x_n\| : n \in \mathbb{N} \right\} : \{x_n\} \subset B_X, x \in B_X, \alpha(\{x_n\}) \geq \varepsilon \right\},$$

$R'_X : [0, 2] \longrightarrow [0, 1]$:

$$R'_X(\varepsilon) := 1 - \sup\left\{ \inf\left\{ \frac{1}{2}\|x + x_n\| : n \in \mathbb{N} \right\} : \{x_n\} \subset B_X, x \in B_X, \beta(\{x_n\}) \geq \varepsilon \right\},$$

$R''_X : [0, a] \longrightarrow [0, 1]$:

$$R''_X(\varepsilon) := 1 - \sup\left\{ \inf\left\{ \frac{1}{2}\|x + x_n\| : n \in \mathbb{N} \right\} : \{x_n\} \subset B_X, x \in B_X, \mathrm{sep}\left(\{x_n\}\right) \geq \varepsilon \right\}$$

where a is a real number in the interval $[1, 2)$ depending on $(X, \|\cdot\|)$.

Moreover, the same authors in [Ayerbe et al. 97] defined

$P_{X,\mu} : [0, \mu(B_X)) \to [0, \infty)$ by

$$P_{X,\mu}(\varepsilon) := \inf\{\|x\| - 1 : x \in X,\ \|x\| > 1,\ \mu(R_x) \geq \varepsilon\}.$$

Related coefficients

$$\begin{aligned} R_0(X) &:= \sup\{\varepsilon \geq 0 : R_X(\varepsilon) = 0\}. \\ R'_0(X) &:= \sup\{\varepsilon \geq 0 : R'_X(\varepsilon) = 0\}. \\ R''_0(X) &:= \sup\{\varepsilon \geq 0 : R''_X(\varepsilon) = 0\}. \\ P_{0,\mu}(X) &:= \sup\{\varepsilon \geq 0 : P_{X,\mu}(\varepsilon) = 0\}. \end{aligned}$$

Geometrical properties in terms of these moduli

1. $(X, \|\cdot\|)$ has the property (β) of Rolewicz if and only if either $R_0(X) = 0$, $R'_0(X) = 0$, or $R''_0(X) = 0$.

2. $P_{0,\mu}(X) = 0$ if and only if $(X, \|\cdot\|)$ has property (β) of Rolewicz.

   It is easy to prove that (UC) implies property (β) and property (β) implies NUC. Thus, this property lies between uniform and near uniform convexity.

3. If any of the coefficients $R_0(X)$, $R'_0(X)$, $R''_0(X)$ is less than one, then both X and X^* are reflexive and have normal structure.

4. If $P_{0,\mu}(X) < \frac{1}{2}$, then the spaces X and X^* are reflexive and have normal structure.

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Example 37 .
If $1 < p < \infty$, then

$$R''_{\ell_p}(\varepsilon) = 1 - \frac{1}{2}\left\{\frac{\varepsilon^p}{2} + \left(\left(1 - \frac{\varepsilon^p}{2}\right)^{\frac{1}{p}} + 1\right)^p\right\}^{\frac{1}{p}}.$$

Example 38 .
If $1 < p < \infty$, then

$$R'_{\ell_p}(\varepsilon) = 1 - \frac{1}{2}\left\{\left(\frac{\varepsilon}{2}\right)^p + \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}} + 1\right)^p\right\}^{\frac{1}{p}}.$$

Example 39 .
$R''_{\ell_\infty}(\varepsilon) = R'_{\ell_\infty}(\varepsilon) = R_{\ell_\infty}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$.

Example 40 .
$R''_{\ell_1}(\varepsilon) = R'_{\ell_1}(\varepsilon) = R_{\ell_1}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$.

Example 41 .
If $1 < p < \infty$ and $0 \leq \varepsilon < 2^{\frac{1}{p}}$, then

$$P_{\ell_p,S}(\varepsilon) = \left(\frac{2}{2 - \varepsilon^p}\right)^{1-\frac{1}{p}} - 1.$$

Example 42 .
If $1 < p < \infty$ and $0 \leq \varepsilon < 2$, then

$$P_{\ell_p,\alpha}(\varepsilon) = P_{\ell_p,\beta} = \left(\frac{2^p}{2^p - \varepsilon^p}\right)^{1-\frac{1}{p}} - 1.$$

Example 43 .
$P_{c,\mu}(\varepsilon) = P_{c_0,\mu}(\varepsilon) = P_{\ell_1,\mu}(\varepsilon) = P_{\ell^\infty,\mu}(\varepsilon) = 0$. for all $\varepsilon \in [0, 2]$.

Main features of these moduli

1. $$\delta_X\left(\frac{\varepsilon}{2}\right) \leq R'_X(\varepsilon) \leq R_X(\varepsilon) \leq R''_X(\varepsilon) \leq \Delta_{X,S}(\varepsilon).$$

2. $$\varepsilon_S(X) \leq R''_0(X) \leq R_0(X) \leq R'_0(X) \leq 2\varepsilon_0(X).$$

3. $0 \leq P_{X,\mu}(\varepsilon) \leq \frac{\varepsilon}{\mu(B_X)-\varepsilon}$. It follows that $P_{X,\mu}$ is continuous at 0.

18 Domínguez modulus of (NUS), [Domínguez 95].

It is the function given by

$$\Gamma_X(t) := \sup \left\{ \inf \left\{ \frac{\|x_1 + tx_n\| + \|x_1 - tx_n\|}{2} : n > 1 \right\} : (x_n) \text{ basic sequence in } B_X \right\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. If X is (US) then $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = 0$.
2. $(X, \|\cdot\|)$ is (NUS) if and only if X is reflexive and $\lim_{t\to 0} \frac{\Gamma_X(t)}{t} = 0$.

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1. For $1 < p < \infty$, $\Gamma_{\ell_p}(t) = (1 + t^p)^{\frac{1}{p}}$.

Main features of these modulus

1. $\rho_X(t) \geq \Gamma_X(t)$ for every $t \in [0, 2]$.

Other properties of this modulus are listed below, among the ones of the constant $WCS(X)$.

19 The Modulus of squareness, a universal one

It was defined in the joint paper by Przeslawski and Yost [Przeslawski-Yost 95]. For more information about it see the joint paper by Benítez, Przeslawski, Yost [Benítez et al. 98] which is the source of all this information. Given a normed space $(X, \|\cdot\|)$, one observes that for any $x, y \in X$ with $\|y\| < 1 < \|x\|$, there is a unique $z = z(x, y) \in S_X \cap [x, y]$, (where $[x, y]$ is the line segment joining x and y. We put

$$\omega(x, y) := \frac{\|x - z(x, y)\|}{\|x\| - 1}$$

and define $\chi_X : [0, 1) \longrightarrow [1, \infty)$ by

$$\chi_X(\beta) := \sup\{\omega(x, y) : \|y\| \leq \beta < 1 < \|x\|\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. $(X, \|\cdot\|)$ is uniformly convex if and only if

$$\lim_{\beta \to 1^-} (1 - \beta)\chi_X(\beta) = 0.$$

2. $(X, \|\cdot\|)$ is uniformly smooth if and only if $\chi'_X(0) = 0$.

3. $$\chi_{X^*}(\beta) = \frac{1}{\chi_X^{-1}\left(\frac{1}{\beta}\right)}.$$

4. If $\chi_X(\beta) < \frac{1}{1-\beta}$ for some β, then $(X, \|\cdot\|)$ has uniformly normal structure.

5. If a normed space $(X, \|\cdot\|)$ is uniformly non-square then for each $\beta \in (0, 1)$,

$$\chi_X(\beta) < \chi_1(\beta) := \frac{1 + \beta}{1 - \beta}.$$

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Example 44 $\chi_X(\beta) = \chi_2(\beta) := \frac{1}{\sqrt{1-\beta^2}}$ whenever $(X, \|\cdot\|)$ is an inner product space.

Example 45 . For any normed space $(X, \|\cdot\|)$ containing $\ell_1(2)$

$$\chi_X(\beta) = \chi_1(\beta).$$

Other facts concerning this modulus and/or constant

1. $\chi_X(\beta) = \sup\{\chi_M(\beta) : M \subset X, \dim(M) = 2\}$.

2. $\chi_X(\cdot)$ is a strictly increasing and convex function.

3. For all normed space $(X, \|\cdot\|)$

$$\chi_X(\beta) \leq \chi_1(\beta).$$

4. $$\chi_X(\beta) < \chi_1(\beta)$$

   everywhere on $(0,1)$ unless $(X, \|\cdot\|)$ contains arbitrarily close copies of $\ell_1(2)$.

5. $$\chi'_X(\beta) < \chi'_1(\beta)$$

   almost everywhere on $(0,1)$.

6. $$\chi_X(\beta) > \chi_2(\beta)$$

   everywhere on $(0,1)$ unless $(X, \|\cdot\|)$ is an inner product space.

7. $$\chi_X(\gamma) - \chi_X(\beta) \leq \chi_1(\gamma) - \chi_1(\beta)$$

   whenever $0 \leq \beta < \gamma < 1$.

8. Fix $\delta, \beta \in (0,1)$. If $\chi_X(\beta) > (1-\delta\beta)\chi_1(\beta)$, then $(X, \|\cdot\|)$ contains a two-dimensional subspace whose Banach -Mazur distance from $\ell_1(2)$ is less than $\frac{1}{1-(1+\beta)\delta}$.

9. If X and Y are two isomorphic Banach spaces whose Banach -Mazur distance is less than $1+2\delta^2$ for some $\delta \in [0,1]$, then for all $\beta \in (0,1)$

   $$|\chi_X(\beta) - \chi_Y(\beta)| \leq \frac{2(\delta+\delta^2)}{(1-\beta)^2}.$$

10. Let ρ_X the modulus of smoothness of a Banach space $(X, \|\cdot\|)$. Then, for all $\beta \in (0,1)$

    $$1 \leq \frac{\chi_X(\beta) - 1}{\rho_X(\beta)} \leq \frac{2}{1-\beta}.$$

    and $\chi'_X(0) = \rho'_X(0)$.

11. See other property of this modulus in the list of the properties of the coefficient $N(X)$. [Prus-Szczepanik 01]

20 Three further modulus for the Nearly Uniform Convexity

They were defined by J.M. Ayerbe and S. Francisco ([Ayerbe-Francisco 97]) as follows.

$D_{X,\mu} : [0, \mu(B_X)) \longrightarrow [0, \infty)$ is the function given by

$$D_{X,\mu}(\varepsilon) := \inf\{\|x\| - 1 : x \in X \backslash B_X, \tilde{\mu}(R_x) \geq \varepsilon\}.$$

where, for a bounded subset A of X, $\tilde{\mu}(A) := \sup\{\mu(C) : C = \text{co}(C), C \subset A\}$ and $\mu \in \{\alpha, \chi, \beta\}$.

Related coefficient with this modulus

The coefficients of noncompact convexity of $(X, \|\cdot\|)$ corresponding to this modulus are the numbers

$$D_{0,\mu}(X) := \sup\{\varepsilon \geq 0 : D_{X,\mu}(\varepsilon) = 0\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. The Banach space $(X, \|\cdot\|)$ is (NUC) if and only if $D_{0,\mu}(X) = 0$.
2. If $D_{0,\beta}(X) < \frac{1}{2}$, then the space $(X, \|\cdot\|)$ is reflexive and has normal structure.

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Example 46 .Let H be a separable and infinite dimensional Hilbert space. Then

$$\begin{array}{ll} D_{H,\alpha}(\varepsilon) = \frac{2\varepsilon^2}{4-\varepsilon^2} & \varepsilon \in [0, 2) \\ D_{H,\chi}(\varepsilon) = \frac{2\varepsilon^2}{1-\varepsilon^2} & \varepsilon \in [0, 1) \\ D_{H,\beta}(\varepsilon) = \frac{2\varepsilon^2}{2-\varepsilon^2} & \varepsilon \in [0, \sqrt{2}). \end{array}$$

$\boxed{\text{Other facts about these moduli}}$ (See [Ayerbe-Francisco 97].)

1. $D_{X,\mu}$ is nondecreasing in $[0, \mu(B_X))$, and $D_{X,\mu}(0) = 0$.
2. For every $\varepsilon \in \left[0, \frac{1}{2}\mu(B_X)\right)$ we have
$$0 \leq D_{X,\mu}(\varepsilon) \leq \frac{\varepsilon}{\mu(B_X) - \varepsilon}.$$
3. $D_{X,\mu}$ is continuous at $\varepsilon = 0$.
4. For every $\varepsilon \in [0, \mu(B_X))$ we have
$$D_{X,\mu}(\varepsilon) < \frac{2\varepsilon}{\mu(B_X)}.$$
5. $$\varepsilon_\beta(X) \leq 2D_{0,\beta}(X) \leq 4\varepsilon_\beta(X).$$

21 A further modulus for the Uniform Convexity

It was defined by S. Francisco ([Francisco 97]) as follows.

$D_X : [0, \infty) \longrightarrow [0, \infty)$ is the function given by

$$D_X(\varepsilon) := \inf\{\|x\| - 1 : x \in X \backslash B_X, \operatorname{diam}(R_x) \geq \varepsilon\}.$$

Related coefficient with this modulus

The coefficient of convexity of $(X, \|\cdot\|)$ corresponding to this modulus is the number

$$D_0(X) := \sup\{\varepsilon \geq 0 : D_X(\varepsilon) = 0\}.$$

Geometrical properties in terms of this modulus and/or this coefficient

1. The Banach space $(X, \|\cdot\|)$ is (UC) if and only if $D_0(X) = 0$.
2. If $D_0(X) < \frac{1}{2}$, then the space $(X, \|\cdot\|)$ is reflexive and has normal structure.

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Example 47 .

Let H be a separable Hilbert space. Then

$$D_H(\varepsilon) = \begin{cases} \frac{2}{\sqrt{4-\varepsilon^2}} - 1 & \varepsilon \in [0, \sqrt{3}] \\ \sqrt{1+\varepsilon^2} - 1 & \varepsilon \in [\sqrt{3}, \infty) \end{cases}$$

Other facts about this modulus (See [Francisco 97] and [Ayerbe-Francisco 97(b)]).

1. D_X is nondecreasing in $[0, \infty)$, and $D_X(0) = 0$.
2. For every $\varepsilon \in [0, 1]$ we have $0 \leq D_X(\varepsilon) \leq \frac{\varepsilon}{2-\varepsilon}$. Hence, if $\varepsilon \in (0, 1)$, then $D_X(\varepsilon) < \varepsilon$.
3. D_X is continuous at $\varepsilon = 0$.
4. $$\varepsilon_0(X) \leq 2D_0(X) \leq 8\varepsilon_0(X).$$
5. If $(X, \|\cdot\|)$ is a uniformly convex Banach space, then D_X is strictly increasing on $[0, \infty)$.

22 Modulus of uniform "nonoctahedralness"

It was defined in [Jiménez 99]. Denote by $\tilde{\delta}(X)$ the supremum of the set of numbers $\varepsilon \in [0,2]$ for which there exist points x_1, x_2, x_3 in B_X with $\min\{\|x_i - x_j\| \colon i \neq j\} \geq \varepsilon$. Define the function $\tilde{\delta}_X : [0, \tilde{\delta}(X)) \to [0,1]$ by

$$\tilde{\delta}_X(\varepsilon) = \inf\left\{1 - \frac{1}{3}\|x_1 + x_2 + x_3\| : \; x_i \in B_X, i = 1,2,3, \text{ and } \min\{\|x_i - x_j\| \colon i \neq j\} \geq \varepsilon\right\}.$$

Related coefficient with this modulus

$\tilde{\varepsilon}_0(X) := \sup\{\varepsilon \in [0, \tilde{\delta}(X)) \colon \tilde{\delta}_X(\varepsilon) = 0\}$.

EXAMPLES

Example 48 .

If $(H, \|.\|)$ is a Hilbert space with dimension greater than 2, for ε small enough

$$\tilde{\delta}_H(\varepsilon) \geq 1 - \sqrt{1 - \frac{\varepsilon^2}{3}}.$$

Example 49 .Consider the classical real sequence space ℓ_2 endowed with its usual Euclidean norm $\|\cdot\|$. Let $|\cdot|$ be a norm on ℓ_2 such that

$$\|x\| \leq |x| \leq b\|x\| \qquad (x \in \ell_2)$$

for some $b \geq 1$ and let $X = (\ell_2, |\cdot|)$. Then $\tilde{\varepsilon}_0(X) < 2$ for $b < \sqrt{\frac{7}{3}}$.

Example 50 .Now consider the space $E_\beta := (\ell_2, |\cdot|_\beta)$, where $|x|_\beta = \max\{\|x\|, \beta\|x\|_\infty\}$. Since $\|x\| \leq |x|_\beta \leq \beta\|x\|_2$ for all $x \in \ell_2$ then $\tilde{\varepsilon}_0(E_\beta) < 2$ for $\beta < \sqrt{\frac{7}{3}}$. On the other hand, for $\beta \geq \sqrt{2}$ we have $\varepsilon_0(E_\beta) = 2$.

Geometrical properties in terms of this modulus and/or this coefficient

1. ([García et al. 01]). $\tilde{\varepsilon}_0(X) < 1 \Rightarrow X$ has (UNS).

Main features of this modulus (See [Jiménez 99].)

1. For any $\varepsilon \in [0, \tilde{\delta}(X))$ we have that $\delta_X(\varepsilon) \leq \tilde{\delta}_X(\varepsilon)$.
2. $\tilde{\varepsilon}_0(X) \leq \varepsilon_0(X)$

   In some cases, this inequality is strict.
3. If $(X, \|\cdot\|)$ is a Banach space with the (WORTH) property such that $\tilde{\varepsilon}_0(X) < 2$ then $(X, \|\cdot\|)$ has the (WFPP).

   Recall that $(X, \|\cdot\|)$ has the (WORTH) property if

   $$\lim_n |\|x_n - x\| - \|x_n + x\|| = 0$$

   for all $x \in X$ and for all weakly null sequences (x_n) in X.
4. Suppose that there exist $\delta > 0$ and $\varepsilon \in (0,2)$ such that for all points $x, y, z \in S_X$ with $\|(1/3)(x + y + z)\| > 1 - \delta$ we have

   $$\|y - z\| \operatorname{dist}(x, [y, z]) < \varepsilon.$$

   Then $\tilde{\varepsilon}_0(X) < 2$. (Here $[y, z]$ is the affine span of $\{y, z\}$.)

23 Two Moduli of Gao for Normal Structure [Gao 01]

They are the functions $W : [0,2] \longrightarrow [0,1]$ and $W_1 : [0,2] \longrightarrow [0,1]$ respectively given for $0 \leq \varepsilon \leq 2$ by

$$W(\varepsilon) := \inf \{\sup\{< f_x, (x-y)/2 >: f_X \in \nabla_x\}, x, y \in S_X, \|x-y\| \geq \varepsilon\}$$

(which is called modulus of W-convexity) and

$$W_1(\varepsilon) := \sup \{\inf\{< f_x, (x-y)/2 >: f_X \in \nabla_x\}, x, y \in S_X, \|x-y\| \leq \varepsilon\}$$

(which is called modulus of W_1-convexity). Here ∇_x is the set of norm 1 support functionals of S_X at x.

Geometrical properties in terms of this modulus and/or this coefficient

1. [Gao 01] The Banach space $(X, \|\cdot\|)$ is **uniformly nonsquare** whenever $W(2) > 0$.
2. [Gao 01] If the Banach space $(X, \|\cdot\|)$ is not uniformly nonsquare then $W_1(\varepsilon) = \frac{\varepsilon}{2}$ for all $\varepsilon \in [0,2]$.
3. [Gao 01] If the Banach space $(X, \|\cdot\|)$ fails to have weak normal structure then $W(1) = 0$ and $W(2) \leq \frac{1}{2}$. Hence a Banach with either $W(1) > 0$ or $W(2) > \frac{1}{2}$ has (weak) normal structure. In fact, under any of these condition $(X, \|\cdot\|)$ has *uniformly* normal structure.
4. [Gao 01] If the Banach space $(X, \|\cdot\|)$ fails to have weak normal structure then $W_1(\varepsilon) = \frac{\varepsilon}{2}$ for all $\varepsilon \in [0,2]$. Hence a Banach with either $W_1(\varepsilon) < \frac{\varepsilon}{2}$ for any $\varepsilon \in [0,2]$ has (weak) normal structure. In fact, under any of these condition $(X, \|\cdot\|)$ has *uniformly* normal structure.
5. [Prus-Szczepanik 01] A Banach space is uniformly convex if and only if $W(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$.
6. [Prus-Szczepanik 01] If
$$W(t) > \max\{0, \frac{1}{2}(t-1)\}$$
for some $t \in (0,2)$ then $(X, \|\cdot\|)$ has uniformly normal structure.

EXAMPLES

Example 51 .[Prus-Szczepanik 01]
If $(X, \|\cdot\|)$ is a Hilbert space

$$W(\varepsilon) = W_1(\varepsilon) = \left(\frac{\varepsilon}{2}\right)^2.$$

Example 52 .[Prus-Szczepanik 01]
Let X be the space $\mathbb{R}^2$ endowed with the norm given by the formula

$$\|(\alpha, \beta)\| := \max\{\frac{7}{6}|\alpha|, |\alpha+\beta|, \frac{7}{6}|\beta|\}.$$

Then

$$W(\varepsilon) = \begin{cases} 0 & 0 \leq \varepsilon \leq \frac{10}{7} \\ \frac{1}{2}\left(\varepsilon - \frac{10}{7}\right) & \frac{10}{7} < \varepsilon < \frac{12}{7} \\ \frac{7}{12}(\varepsilon - 1) & \frac{12}{7} < \varepsilon < 2 \\ 1 & \varepsilon = 2 \end{cases}$$

So W need not be convex.

Other facts concerning this modulus and/or constant

1. [Gao 01] In both of the above definitions the conditions $\|x-y\| \geq \varepsilon$ $\|x-y\| \leq \varepsilon$ may be replaced by the conditions $\|x-y\| = \varepsilon$.

2. [Gao 01] W is a continuous increasing function on $[0,2]$ with $W(0)=0$ and $W(2) \leq 1$.

3. [Gao 01] $W(\varepsilon) \leq \delta(\varepsilon)$ for $\varepsilon \in [0,2]$.

4. [Gao 01] W_1 is a continuous increasing function on $[0,2]$ with $W_1(0)=0$ and $W_1(2) \leq 1$.

5. [Gao 01] $W_1(\varepsilon) \leq \frac{\varepsilon}{2}$ for $\varepsilon \in [0,2]$.

6. [Prus-Szczepanik 01]

$$W(\varepsilon) = \inf\{\frac{1}{2}(1-n^-(x,y)) : x,y \in S_X, \|x-y\| \geq \varepsilon\}.$$

$$W_1(\varepsilon) = \sup\{\frac{1}{2}(1-n^+(x,y)) : x,y \in S_X, \|x-y\| \leq \varepsilon\}.$$

   Here

$$n^-(x,y) := \lim_{t\to 0^-} \frac{\|x+ty\|-1}{t}, \quad n^+(x,y) := \lim_{t\to 0^+} \frac{\|x+ty\|-1}{t},$$

7. [Prus-Szczepanik 01] For every Banach space, $W(2) = W_1(2) = 1$.

8. [Prus-Szczepanik 01] For every Banach space,

$$\frac{2}{3\varepsilon}(\tau W(\varepsilon-\tau) + (\varepsilon-\tau)W(\tau)) \leq \delta_X(\varepsilon) \leq W(\varepsilon)$$

   for all ε, τ with $0 \leq \tau < \varepsilon < 2$.

9. More accurate study of the continuity of these moduli can be found in [Prus-Szczepanik 01]

24 Comments on another modulus of Ji Gao (and Papini) for uniformly normal structure

For $\varepsilon \in [0,1]$ define

$$Q_\varepsilon(X) := \sup\{\|x+\varepsilon y\| + \|x-\varepsilon y\| : x, y \in S_X\}.$$

In the work [Gao t.a.1] in fact it is defined as

$$Q_\varepsilon(X) := \sup\{Q_\varepsilon(x) : x \in S_X\},$$

where $Q_\varepsilon(x) := \sup\{\|x+\varepsilon y\| + \|x-\varepsilon y\| : y \in S_X\}$.

Notice that $Q_1(X) = 2A_2(X)$ where $A_2(X)$ is the constant studied in [Baronti et al. 00]. (See section 36 below). This fact yields many (trivial) consequences about $Q_1(X)$.

Even more, in [Papini 01] are defined two moduli, which generalize the constants $A_1(X)$ and $A_2(X)$ given in [Baronti et al. 00]. More precisely

$$A_1^\varepsilon(X) := \inf_{y\in S_X} \sup_{\|x\|\le\varepsilon} \frac{\|y+x\| + \|y-x\|}{2} = \inf_{y\in S_X} \sup_{x\in\mathcal{E}} \frac{\|y+\varepsilon x\| + \|y-\varepsilon x\|}{2},$$

$$A_2^\varepsilon(X) := \sup_{y\in S_X} \sup_{\|x\|\le\varepsilon} \frac{\|y+x\| + \|y-x\|}{2} = \sup_{y\in S_X} \sup_{x\in\mathcal{E}} \frac{\|y+\varepsilon x\| + \|y-\varepsilon x\|}{2},$$

where $\mathcal{E}$ denotes the set of extreme points of the unit sphere of $(X, \|\cdot\|)$. One can see that Gao's modulus $Q_{(.)}(X)$ is in fact $2A_2(.)$.

Moreover, (see [Gao t.a.1], Proposition 1), $Q_\varepsilon(X) = 2\rho_X(\varepsilon) + 2$, where $\rho_X(.)$ is the well known Lindenstrauss modulus of smoothness. Hence, in some sense, the moduli $Q_{(.)}(X)$ and $A_2(.)$ are superfluous.

EXAMPLES

Example 53 For a Hilbert space H, $Q_\varepsilon(H) = 2\sqrt{1+\varepsilon^2}$.

Example 54 If X is either the Banach space ℓ_p or $L^p([0,1])$ then

$$Q_\varepsilon(X) = \begin{cases} 2(1+\varepsilon^p)^{\frac{1}{p}} & 1 < p \le 2 \\ 2^{\frac{1}{q}}\left((1+\varepsilon)^p + (1-\varepsilon)^p\right)^{\frac{1}{p}} & p > 2 \end{cases}$$

where $p+q = pq$.

Geometrical properties in terms of this modulus

(a) A Banach $(X, \|\cdot\|)$ space with $Q_\varepsilon(X) < 2(1+\varepsilon)$ for some $\varepsilon \in [0,1]$ is uniformly nonsquare.

(b) A Banach $(X, \|\cdot\|)$ space with $Q_\varepsilon(X) < 2+\varepsilon$ for some $\varepsilon \in [0,1]$ has uniformly normal structure.

Other facts concerning this modulus and/or constant

(a) $Q_\varepsilon(X) = Q_\varepsilon(X_\mathcal{U})$.

In [Papini 01] is presented a quite detailed study of the functions A_i^ε, $(i = 1, 2)$. Among other properties are listed the following.

(a) The functions $\varepsilon \mapsto A_i^\varepsilon(X)$ are nondecreasing and $1-$lipschitz.

(b) The function $\varepsilon \mapsto A_2^\varepsilon(X)$ is convex.

(c) For any Banach space $(X, \|\cdot\|)$ and $\varepsilon \in [0, 1]$,

$$1 \leq A_1^\varepsilon(X) \leq A_2^\varepsilon(X) \leq 1 + \varepsilon.$$

(d) If Y is a dense subspace of $(X, \|\cdot\|)$, then $A_i^\varepsilon(Y) = A_i^\varepsilon(X)$.

(e) If Y is a subspace of X, then $A_2^\varepsilon(Y) \leq A_2^\varepsilon(X)$.

(f) $A_2^\varepsilon(X) = \sup\{A_2^\varepsilon(Y) : \dim(Y) = 2,\ Y \text{subspace of} X\}$
$A_1^\varepsilon(X) \geq \inf\{A_1^\varepsilon(Y) : \dim(Y) = 2,\ Y \text{subspace of} X\}$.

(g) In any space $(X, \|\cdot\|)$, $A_2^\varepsilon(X) \geq \sqrt{1 + \varepsilon^2}$.

25 Near Uniform Noncreasyness and related moduli

I would like to express my gratitude to the authors of the paper [Prus-Szczepanik 05] for sending me the corresponding tex file. All the results listed in this section are taken from this file.

Assume that X lacks the Schur property. Then the family $\mathcal{N}_X$ of all weakly null sequences (x_n) in S_X is nonempty. Given $\epsilon \geq 0$ and $x \in X$, we put

$$d_X(\epsilon, x) := \inf_{(y_m)\in\mathcal{N}_X} \limsup_{m\to\infty} \|x + \epsilon y_m\| - \|x\|$$

and

$$b_X(\epsilon, x) = \sup_{(y_m)\in\mathcal{N}_X} \liminf_{m\to\infty} \|x + \epsilon y_m\| - \|x\|.$$

In case $x \in S_X$ the moduli d and b coincide with those which were studied in [Maluta et. al 01].

Let (x_n) be either a finite or infinite sequence in a Banach space X. We put $\mathrm{sep}(x_n) = \inf_{n\neq m} \|x_n - x_m\|$. Next, given a nonempty bounded subset A of X, by $\beta(A)$ we denote the separation measure of noncompactness of A, i.e. $\beta(A) = \sup\{\mathrm{sep}(x_n)\}$ where the supremum is taken over all infinite sequences (x_n) in A (see [Ayerbe et al. 97 (b)]). In the definition of UNC spaces the sets $S(x^*, y^*, \delta)$ are used. We shall define their counterparts for NUNC spaces. Let (x_n^*) be a bounded sequence in X^* and $\alpha \in \mathbb{R}$. We set

$$S((x_n^*), \alpha) := \left\{x \in B_X : \liminf_{n\to\infty} x_n^*(x) \geq \alpha\right\}.$$

Let X be a Banach space without the Schur property and $\epsilon \in [0, 1]$. We define

$$\Delta_X(\epsilon) := \inf\{1 - x^*(x)\}$$

where the infimum is taken over all elements $x \in X$ which are weak limits of sequences (x_n) in B_X with $\liminf_{n\to\infty} \|x_n - x\| \geq \epsilon$ and all elements $x^* \in X^*$ which are weak* limit points of sequences (x_n^*) in B_{X^*} with $\liminf_{n\to\infty} \|x_n^* - x^*\| \geq \epsilon$. Existence of such sequences follows form the assumption that X lacks the Schur property and the so called Josefson-Nissenzweig Theorem. In the definition of $\Delta_X(\epsilon)$ one can replace the unit balls B_X and B_{X^*} by the unit spheres S_X and S_{X^*}, respectively. It is clear that if a Banach space X is reflexive, then the moduli Δ_X and Δ_{X^*} are equal.

By $\mathcal{M}_X$ we denote the set of all weakly null sequences (y_n) in B_X such that

$$\limsup_{n\to\infty} \limsup_{m\to\infty} \|y_n - y_m\| \leq 1.$$

Let $x \in X$ and $\epsilon \geq 0$. We put

$$b_1(\epsilon, x) = \sup_{(y_m)\in\mathcal{M}_X} \liminf_{m\to\infty} \|x + \epsilon y_m\| - \|x\|.$$

Observe that "lim inf" can be replaced by "lim sup" in the definition of $b_1(\epsilon, x)$. It follows that $b_1(\epsilon, x)$ is a convex function of $\epsilon \in [0, +\infty)$. Moreover, $b_1(0, x) = 0$, so $b_1(\epsilon, x)/\epsilon$ is nondecreasing in the interval $(0, +\infty)$.

Geometrical properties in terms of these moduli and/or coefficients

1. It was shown that d is strongly related to nearly uniform convexity introduced in [Huff 80], which is an infinite dimensional counterpart of uniform convexity. Namely, a space X is nearly uniformly convex if and only if X is reflexive and $\inf_{x \in S_X} d(\epsilon, x) > 0$ for every $\epsilon > 0$. The dual property is called nearly uniform smoothness (see [Prus 89]). A space X is nearly uniformly smooth if and only if X is reflexive and

$$\lim_{\epsilon \to 0^+} \left(\frac{1}{\epsilon} \sup_{x \in S_X} b(\epsilon, x) \right) = 0.$$

2. **Definition 25.1** *Let X be a Banach space without the Schur property. We say that X is nearly uniformly noncreasy (NUNC for short) if for every $\epsilon > 0$ there is $t > 0$ such that for every $x \in S_X$ it is the case that $d(\epsilon, x) \geq t$ or $b(t, x) \leq \epsilon t$. Additionally, we treat spaces with the Schur property as being NUNC.*

   The class of NUNC Banach spaces contains all nearly uniformly convex spaces and all nearly uniformly smooth spaces. Moreover, it contains also all UNC spaces.

3. Let X be an infinite dimensional Banach space. If X does not contains an isomorphic copy of ℓ_1, then the following conditions are equivalent.

   (i) $\Delta_X(\epsilon) > 0$ for every $\epsilon \in (0, 1]$.

   (ii) The space X is NUNC.

4. A reflexive space X is NUNC if and only if X^* is NUNC.

EXAMPLES

Example 55 Consider the space $X = (\mathbb{R} \oplus c_0)_{l_1}$, i.e. the product $\mathbb{R} \times c_0$ endowed with the norm

$$\|(\alpha, u)\| = |\alpha| + \|u\|_{c_0}$$

where $\alpha \in \mathbb{R}$ and $u \in c_0$. This space is not reflexive, so it is not NUC, nor NUS, nor UNC. However, X is NUNC.

Example 56 Consider the space ℓ_p with $1 < p < \infty$. Then

$$\Delta_{\ell_p}(\epsilon) = 1 - (1 - \epsilon^p)^{1/p} (1 - \epsilon^q)^{1/q}$$

for every $\varepsilon 3silon \in [0, 1]$, where $1/p + 1/q = 1$. Notice that $\Delta_{\ell_2}(\epsilon) = \epsilon^2 \leq \Delta_{\ell_p}(\epsilon)$ for every $p \in (1, \infty)$ and every $\epsilon \in [0, 1]$.

Example 57 Let $Y = (\ell_1 \oplus \ell_2)_{\ell_2}$. Clearly, Y does not have the Schur property. Moreover $\Delta_Y(\varepsilon) \leq 1 - (1 - \varepsilon^2)^{1/2}$.

Main features of these moduli

1. Let X be a Banach space without the Schur property. If there exists $\epsilon \in (0, 1)$ such that for every $x \in S_X$ it is the case that $b_1(1, x) < 1 - \epsilon$ or $d(1, x) > \epsilon$, then X has the weak fixed point property.

2. Let X be a Banach space without the Schur property. Each of the following conditions is sufficient for the weak fixed point property.

(i) There exists $\gamma \in (0,1)$ such that if $(x_n^*) \subset S_{X^*}$ and $\operatorname{sep}(x_n^*) \geq \gamma$, then

$$\beta\left(S\left(\left(x_n^*\right), \gamma\right)\right) < \gamma.$$

(ii) $\lim_{\epsilon \to 1^-} \Delta_X(\epsilon) > 0$.

3. Assume that X is a Banach space without the Schur property. Let $\|\cdot\|$ be the initial norm in X and let a norm $|\cdot|$ in X satisfy the condition $\|x\| \leq |x| \leq \sigma \|x\|$ for every $x \in X$. We set $Y = (X, |\cdot|)$. If $\sigma < M_1(X)$, then Y has the weak fixed point property.

   Here we have used the constant $M_1(X)$, a generalization of the Domínguez Benavides coefficient. Let X be a Banach space without the Schur property. Given $t \geq 0$ and $x \in X$, we put

$$d_X^{-1}(t,x) = \max\left\{\epsilon \geq 0 : d_X(\epsilon, x) \leq t\right\}.$$

   Next, we set

$$M_1(X) := \sup_{0<\epsilon<1} \sup_{t>0} \inf_{x \in B_X} \max\left\{\frac{1}{d_X^{-1}(1-\|x\|+\epsilon, x)}, \sup_{s \geq t}\left(\frac{(1-\epsilon)s+1}{b_{1,X}(s,x)+\|x\|}\right)\right\}.$$

   It is easy to see that

$$M_1(X) = \lim_{\epsilon \to 0^+} \sup_{t>0} \inf_{x \in B_X} \max\left\{\frac{1}{d_X^{-1}(1-\|x\|+\epsilon, x)}, \sup_{s \geq t}\left(\frac{s+1}{b_{1,X}(s,x)+\|x\|}\right)\right\}.$$

   Observe also that

$$\begin{aligned} M_1(X) &\geq \sup_{t>0}\left(\frac{t+1}{\sup_{x \in B_X}\left(b_{1,X}(t,x)+\|x\|\right)}\right) \\ &= \sup_{t>0}\left(\frac{1+\frac{1}{t}}{\sup_{x \in B_X}\left(b_{1,X}\left(1, \frac{1}{t}x\right)+\left\|\frac{1}{t}x\right\|\right)}\right) \\ &= M(X). \end{aligned}$$

   Notice that this result improves the stability theorem given in [Jiménez-Llorens 00] (See fact 12) in the report of coefficient $M(X)$, 33).

4. Let X be a Banach space without the Schur property. If Y is a Banach space such that

$$d(X,Y) < M_1(X),$$

   then Y has the weak fixed point property.

Example 58 Let now $x \in \ell_p$ and $\epsilon \geq 0$. One has that

$$d_{\ell_p}(\epsilon, x) = (\|x\|^p + \epsilon^p)^{1/p} - \|x\|.$$

Consequently,

$$d_{\ell_p}^{-1}(t,x) = ((\|x\|+t)^p - \|x\|^p)^{1/p}$$

for every $t \geq 0$. Moreover,

$$b_{1,\ell_p}(\epsilon, x) = \left(\|x\|^p + \frac{\epsilon^p}{2}\right)^{1/p} - \|x\|.$$

Hence

$$M_1\left(\ell_p\right) = \lim_{\epsilon\to 0^+} \sup_{t>0} \inf_{x\in B_{\ell_p}} \max\left\{ \frac{1}{\left((1+\epsilon)^p - \|x\|^p\right)^{1/p}}, \sup_{s\geq t}\left(\frac{s+1}{\left(\|x\|^p + \frac{s^p}{2}\right)^{1/p}}\right)\right\}.$$

In particular $M_1(\ell_2) = \sqrt{2+\sqrt{2}}$.

Example 59 Consider the space $X = \left(\mathbb{R}\oplus c_0\right)_{l_1}$. One has

$$M_1(X) = \frac{1+\sqrt{5}}{2}. \tag{1}$$

If $x = (\alpha, u) \in X$ where $\alpha \in \mathbb{R}$, $u \in c_0$, then

$$d_X\left(t,x\right) = b_{1,X}\left(t,x\right) = \max\left\{\|u\|_{c_0}, t\right\} - \|u\|_{c_0} \tag{2}$$

for every $t \geq 0$. Hence $d_X^{-1}\left(t,x\right) = t + \|u\|_{c_0}$. It is also easy to see that $r_X(c) = \max\{0, c-1\}$ for every $c \geq 0$ and that $M\left(X\right) = 1$.

PART II. LIST OF COEFFICIENTS

26 Jung constant

It is the oldest of this list. It was defined by Jung in a work published in 1901 [Jung 1901]. We will follow the Amir's paper [Amir 85] to list some important facts about it.

For a bounded subset A of X, and a subset Y of X, we denote by $r_Y(A)$ the **relative Chebyshev radius of** A **with respect to** Y, that is,

$$r_Y(A) := \inf\left\{\sup\{\|x-y\| : x \in A\} : y \in Y\right\}.$$

The **Jung constant of** $(X, \|\cdot\|)$ is

$$J(X) := \sup\{2r_X(A) : A \subset (X), \operatorname{diam}(A) = 1\}.$$

Besides the Jung constant J one may study also the "self-Jung" constant

$$J_s(X) := \sup\{2r_A(A) : A \subset X,\ \operatorname{diam}(A) = 1\}.$$

EXAMPLES

Example 60 $J(\ell_2^n) = \sqrt{\frac{2n}{n+1}}$.

Example 61 If dim(X)=n, $J(X) = 1$ if and only if $X = \ell_\infty^n$.

Example 62 $J(\ell_2) = \sqrt{2}$. ([Ayerbe-Xu 93].)

Example 63

$$J(E_\beta) = \begin{cases} \beta\sqrt{2} & 1 \le \beta \le \sqrt{2} \\ 2 & \sqrt{2} < \beta < \infty \end{cases}$$

Example 64 [Papini 83]
$J(\ell_\infty) = J(L^\infty([0,1]) = 1.J_s(\ell_\infty) = J_s(L^\infty([0,1]) = 2.$

Example 65 [Papini 83]
$J(\ell_1) = J(L^1([0,1]) = J(c_0) = 2 = J_s(\ell_1) = J_s(L^1([0,1]) = J_s(c_0).$

Other facts concerning this modulus and/or constant

1. $1 \le J(X) \le 2$.
2. $J(X) = 1$ if and only if $X = \mathcal{C}(T)$ for a stonian T.
3. If the compact Hausdorff space T is not extremaly disconnected, then for every finite-codimensional subspace E of $\mathcal{C}(T)$ we have $J(E) = 2$.
4. $J(X) \le J_s(X)$.
5. Is $(X, \|\cdot\|)$ is a dual space,

$$J(X) = \sup\{2r_X(K) : K \text{ finite},\ K \subset X, \operatorname{diam}(K) = 1\}.$$

6. If $(X, \|\cdot\|)$ is a reflexive Banach space, then

$$J_s(X) = \sup\{2r_{\mathrm{conv}\,(K)}(K) : K \text{ finite}, \ K \subset X, \mathrm{diam}(K) = 1\}.$$

7. If $(X, \|\cdot\|)$ is a nonreflexive Banach space, then $J_s(X) = 2$.

8. For every infinite dimensional space $(X, \|\cdot\|)$, $J_s(X) \geq \sqrt{2}$.

9. For every n-dimensional space $(X, \|\cdot\|)$, $J_s(X) \leq \frac{2n}{n+1}$.

10. For every n and every $\varepsilon > 0$ we have

$$J_s(X) \leq 2 \max\left\{1 - \frac{1-\varepsilon}{n!\varepsilon}, 1 - \delta_X^{(n)}(\varepsilon)\right\}.$$

27 The Jordan-von Neumann constant

In connection with a famous work of Jordan and von Neumann concerning inner products, the von Neumann-Jordan constant $C_{NJ}(X)$ for a Banach space $(X, \|\cdot\|)$ was introduced by Clarkson ([Clarkson 37]) as the smallest constant C for which

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all x, y with $(x, y) \neq (0, 0)$. If C is the best possible in the right hand side of the above inequality then so it is $1/C$ in the left. Hence

$$C_{NJ}(X) := \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, \text{ no both zero} \right\}.$$

The statements without explicit reference has been taken from and [Kato-Takahashi 97] and [Kato et al. 01].

Geometrical properties in terms of this constant

1. $(X, \|\cdot\|)$ is a Hilbert space if and only if $C_{NJ}(X) = 1$.
2. $(X, \|\cdot\|)$ is uniformly non-square if and only if $C_{NJ}(X) < 2$.
3. [Dhompongsa et al. 03] (See also [?]). If $C_{NJ}(X) < \frac{3+\sqrt{5}}{4} \approx 1.309$ then $(X, \|\cdot\|)$ and its dual have uniformly normal structure.

   This result has been improved in [Dhompongsa, S.; Kaewkhao, A. 06c] as follows. (See also and [Saejung 06]).

   If $C_{NJ}(X) < \frac{1+\sqrt{3}}{2} \approx 1.366$ then $(X, \|\cdot\|)$ and its dual have uniformly normal structure.

EXAMPLES

Example 66 If $1 \leq p \leq \infty$, p' is the conjugate of p, and $\dim(L^p(\mu)) \geq 2$, then

$$C_{NJ}(L^p(\mu)) = 2^{\frac{2}{\min\{p,p'\}}-1}.$$

Example 67 Let $X_{\lambda,p}$ be the space ℓ_p with the norm

$$\|x\|_{\lambda,p} := \max\{\|x\|_p, \lambda\|x\|_\infty\}.$$

Then,

$$C_{NJ}(X_{\lambda,p}) = \min\{2, \lambda^2 2^{1-\frac{2}{p}}\}.$$

Example 68 For $1 \leq p \leq 2$, let $Y_{\lambda,p}$ be the space $L^p([0,1])$ with the norm

$$\|x\|_{\lambda,p} := \max\{\|x\|_p, \lambda\|x\|_1\}.$$

Then,

$$C_{NJ}(Y_{\lambda,p}) = \min\{2, \lambda^2 2^{\frac{2}{p-1}}\}.$$

Example 69 For $\lambda > 0$ let Z_λ be $\mathbb{R}^2$ with the norm

$$|x|_\lambda := (\|x\|_2^2 + \lambda \|x\|_\infty^2)^{\frac{1}{2}}.$$

Then,

$$C_{NJ}(Z_\lambda) = \frac{2(\lambda+1)}{\lambda+2}$$

Example 70 [Jiménez e. a. 06]

$$C_{\mathrm{NJ}}(\ell_{2,\infty}) = \frac{3}{2}.$$

Example 71 [Yang-Wang 06]

$$C_{NJ}(\ell_\infty - \ell_1) = \frac{3+\sqrt{5}}{4}.$$

Here $\ell_\infty - \ell_1$ is $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_\infty & x_1.x_2 \geq 0 \\ \|(x_1,x_2)\|_1 & x_1.x_2 < 0 \end{cases}$$

Example 72 [Yang-Wang 06]

$$C_{NJ}(\ell_2 - \ell_1) = \frac{3}{2}.$$

Here $\ell_2 - \ell_1$ is $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_2 & x_1.x_2 \geq 0 \\ \|(x_1,x_2)\|_1 & x_1.x_2 < 0 \end{cases}$$

Other facts concerning this modulus and/or constant

1. $1 \leq C_{NJ}(X) \leq 2$.
2. $$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(1+\|y\|^2)} : x \in S_X, y \in B_X \right\}.$$
3. $$C_{NJ}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2} : \|x\|^2 + \|y\|^2 = 1 \right\}.$$
4. $C_{NJ}(X) = C_{NJ}(X^*)$.
5. If X and Y are isomorphic Banach spaces then

$$\frac{C_{NJ}(Y)}{d(X,Y)^2} \leq C_{NJ}(X) \leq C_{NJ}(Y) d(X,Y)^2$$

where $d(X,Y)$ is the Banach-Mazur distance between X and Y.

6. If $(X, \|\cdot\|)$ is finitely representable in Y, then $C_{NJ}(X) \leq C_{NJ}(Y)$. Moreover $C_{NJ}(X) = C_{NJ}(X_{\mathcal{U}})$.

7. If $\dim(X) = 2$ then
$$\frac{1}{d(X, \ell_2^2)^2} \leq C_{NJ}(X) \leq d(X, \ell_2^2)^2.$$

8. (Unpublished)

   $\varepsilon_0(X) \leq 2\sqrt{C_{NJ}(X) - 1}$.

9. In [Kato et al. 01] was raised the following problem: Compute $C_{NJ}(X)$ for the Bynum spaces. See partial answers in [Jiménez e. a. 06 , Yang-Wang 06, Dhompongsa et al. 03b].

10. See also Section 29 for a fixed point for multivalued nonexpansive mappings in terms of $C_{NJ}(X)$.

27.1 A parametrization of the von Neumann Jordan constant

Among others in [Dhompongsa et al. 03] the authors have generalized this definition in the following sense.

$$C_{NJ}(a, X) := \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X,\ \text{not all zero and } \|y - z\| \leq a\|x\|\right\}.$$

where a is a nonnegative parameter.

Obviously $C_{NJ}(X) = C_{NJ}(0, X)$. Sometimes $C_{NJ}(a, X)$ at some point $a > 0$ is easier to compute than $C_{NJ}(0, X)$.

Example 73 *If H is a Hilbert space, and $a \in [0, 2]$ then*
$$C_{NJ}(a, H) = 1 + \frac{4a}{4 + a^2}.$$

Example 74 For $1 < p < 2$ and let the norm on $\mathbb{R}^2$ defined by
$$|(x_1, x_2)| := \begin{cases} |x_1| + |x_2| & x_1 x_2 \geq 0 \\ (|x_1|^p + |x_2|^p)^{\frac{1}{p}} & x_1 x_2 < 0 \end{cases}$$

Then, $\delta_X(1) = 0$, $C_{NJ}(X) = 1 + 2^{\frac{2}{p-2}}$, $J(X) \geq 2^{\frac{1}{p}}$ and $C_{NJ}(1, X) < 2$.

Geometrical properties in terms of this modulus

1. [Dhompongsa et al. 03] If $C_{NJ}(a, X) < 2$ for some $a \geq 0$ then $C_{NJ}(X) < 2$ and consequently $(X, \|\cdot\|)$ is uniformly nonsquare.

2. [Dhompongsa et al. 03] If
$$C_{NJ}(r, X) \leq \frac{(1+r)^2 + (3r-1)^2}{2(1+r^2)}$$
for some $r \in \left(\frac{1}{2}, 1\right]$ then $(X, \|\cdot\|)$ has normal structure.

3. [Dhompongsa et al. 03] If $C_{NJ}(.,X)$ is concave and

$$C_{NJ}(a,X) < \frac{3+\sqrt{5}+(5-\sqrt{5})a}{4}$$

for some $a \in [0,1]$ then $(X, \|\cdot\|)$ has uniformly normal structure.

4. [Dhompongsa et al. 03b]

If $C_{NJ}(a,X) < (1+a)^2/(1+a^2)$ for some $a \in (0,1]$, then $(X, \|\cdot\|)$ has uniform normal structure. This result has been improved in ([Yang-Wang 06] (Theorem 2.4) as follows.

If

$$C_{NJ}(X) < \frac{2+(1+a)^2+\sqrt{4+(1+a)^4}}{4}$$

for some $a \in [0, \sqrt{2}-1]$ or

$$C_{NJ}(a,X) < (1+a)^2/(1+a^2)$$

for some $a \in [\sqrt{2}-1, 1]$, then $(X, \|\cdot\|)$ has uniformly normal structure.

Other facts concerning this modulus and/or constant

1. [Dhompongsa et al. 03] For all $a \geq 0$

$$1 + \frac{4a}{4+a^2} \leq C_{NJ}(a,X) \leq 2$$

and $C_{NJ}(2,X) = 2$.

2. [Dhompongsa et al. 03] $C_{NJ}(.,X)$ is a continuous nondecreasing function.

3. [Dhompongsa et al. 03] $C_{NJ}(a,X) = C_{NJ}(a,\tilde{X})$, $(a \geq 0)$.

4. [Dhompongsa et al. 05] Let $X_1, ..., X_N$ be Banach spaces with $C_{NJ}(a, X_i) < 2$ for all $i = 1, ..., N$. If Z is uniformly convex, then $C_{NJ}(a, (X_1 \bigoplus \ldots \bigoplus X_N)_Z) < 2$.

Here Z stands for a Banach space with a normalized 1-unconditional basis $(e_i)_{i\in I}$. For Banach spaces X_i $(i \in I)$, the Z-direct sum is defined by

$$\left(\bigoplus_{i\in I} X_i\right)_Z := \left\{(x_i) \in \prod_i X_i : \sum_i \|x_i\| e_i \text{ converges in } Z\right\}$$

endowed with $\|(x_i)\| := \|\sum_i \|x_i\|e_i\|_Z$.

See section 30 below for further properties.

27.2 A related modulus, or another parametrization of the von Neumann Jordan constant

Given a Banach space $(X, \|\cdot\|)$, in [Yang-Wang 06] is defined the function $\gamma : [0,1] \to [1,4]$ by

$$\begin{aligned}\gamma_X(t) \quad &:= \sup\left\{\tfrac{\|x+y\|^2+\|x-y\|^2}{2} : x \in S_X, y \in tS_X\right\} \\ &= \sup\left\{\tfrac{\|x+ty\|^2+\|x-ty\|^2}{2} : x \in S_X, y \in S_X\right\}\end{aligned}$$

This function, (which is called 'constant' by their authors (?)), has the following properties. (See again [Yang-Wang 06] for all the information summarized in this subsection).

Example 75 $(X, \|\cdot\|)$ is a Hilbert space if and only if $\gamma_X(t) \equiv 1+t^2$

Example 76 $\gamma_{\ell_\infty}(t) = (1+t)^2$.

Example 77 For $2 \le p < \infty$, $\gamma_{\ell_p}(t) = \left(\frac{(1+t)^p+(1-t)^p)}{2}\right)^{\frac{2}{p}}$.

Example 78 For $2 \le p < \infty$ and $n \ge 2$, $\gamma_{\ell_p^n}(t) = \left(\frac{(1+t)^p+(1-t)^p)}{2}\right)^{\frac{2}{p}}$.

Example 79 $\gamma_{\ell_\infty - \ell_1}(t) = \frac{1}{2}(1+(1+t)^2)$. Here $\ell_\infty - \ell_1$ is $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_\infty & x_1.x_2 \ge 0 \\ \|(x_1,x_2)\|_1 & x_1.x_2 < 0. \end{cases}$$

Example 80 $\gamma_{\ell_2 - \ell_1}(t) = \frac{(}{1} + t + t^2)$. Here $\ell_2 - \ell_1$ is $\mathbb{R}^2$ endowed with the norm

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_2 & x_1.x_2 \ge 0 \\ \|(x_1,x_2)\|_1 & x_1.x_2 < 0. \end{cases}$$

Geometrical properties in terms of this modulus

1. A Banach space is uniformly smooth whenever $\lim_{t\to 0} \dfrac{\gamma_X(t)-1}{t} = 0$.
2. $(X, \|\cdot\|)$ is not uniformly non-square if and only if $\gamma_X(t) = (1+t)^2$.
3. If there exists $t \in (0,1]$ such that $2\gamma_X(t) < 1+(1+t)^2$ then $(X, \|\cdot\|)$ has super-normal structure, and therefore uniformly normal structure.

Other facts concerning this modulus and/or constant

1. For all $t \in [0,1]$, $1 \le 1+t^2 \le \gamma_X(t) \le (1+t)^2 \le$.
2. $C_{NJ}(X) = \sup\left\{\frac{\gamma_X(t)}{1+t^2} : 0 \le t \le 1\right\}$.
3. $\gamma_X(t) := \sup\left\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2} : x \in S_X, y \in B_X\right\}$
4. $\gamma_X(t) := \sup\left\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2} : x, y \in B_X\right\}$.
5. $\gamma_X(.)$ is a non-decreasing function.

6. $\gamma_X(.)$ is a convex function.

7. $\gamma_X(.)$ is continuous on $[0,1]$.

8. $t \mapsto \frac{\gamma_X(t)-1}{t}$ is non decreasing on $(0,1]$.

9. $\gamma_X(t) := \sup\left\{\frac{\|x+ty\|^2+\|x-ty\|^2}{2} : x, y \in ex(B_X)\right\}$, where $ex(B_X)$ stands for the set of the extremal points of B_X.

10. If Y is a Banach space which is finitely representable in $(X, \|\cdot\|)$ then $\gamma_X(t) \geq \gamma_Y(t)$ for every $t \in [0,1]$.

28 Bynum's coefficient of (uniformly) normal structure

Notation: Let A be a nonempty bounded subset of X.

$$r(A) := \inf_{a \in A} \{\sup\{\|a - y\| : y \in A\}\}$$

is called the Chebyshev self-radius of the set A.

Bynum in 1980 defined [Bynum 80] the following coefficient of (uniformly) normal structure:

$$N(X) := \inf \left\{ \frac{\operatorname{diam}(C)}{r(C)} : C \text{ bounded, convex}, C \subset X, \operatorname{diam}(C) > 0 \right\}$$

Geometrical properties in terms of this constant

Definition 28.1 *A Banach space $(X, \|\cdot\|)$ with $N(X) > 1$ is said to have* **uniformly normal structure***.*

Separation of this property

The Banach $(X, \|\cdot\|)$ obtained by the ℓ_2 direct sum of the sequence of Banach spaces (ℓ_n) is (UCED), and hence it has normal structure. Nevertheless $N(X) = 1$.

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1. [Bynum 80]. For a Hilbert space H, $N(H) = \sqrt{2}$.
2. [Casini-Maluta 85]. For $1 \leq \beta < \sqrt{2}$, $N(E_\beta) = \frac{\sqrt{2}}{\beta}$. (For $\beta \geq \sqrt{2}$, $N(E_\beta) = 1$).
3. ([Prus 90], [Domínguez 91]). For $1 < p < \infty$,

$$N(\ell_p) = N(L^p([0,1]) = \min\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}.$$

Other facts concerning this modulus and/or constant

1. ([Bae 84], [Maluta 84]). If $N(X) > 1$ then X is reflexive.
2. ([Casini-Maluta 85]). If C is a nonempty weakly compact convex subset of X and $T : C \to C$ is uniformly k-lipschitzian with $k < \sqrt{N(X)}$ then T has a fixed point in C.
3. Bynum's lower bound of $N(X)$ ([Bynum 80]). If X is a reflexive Banach space,

$$N(X) \geq \frac{1}{1 - \delta_X(1)}.$$

4. Amir's lower bound of $N(X)$ ([Amir 85])

$$N(X) \geq \sup \left\{ \left(\max \left\{ 1 - \frac{1}{n!n}(1 - \varepsilon), 1 - \delta^{(k)}(\varepsilon) \right\} \right)^{-1} : 0 \leq \varepsilon \leq 1 \right\}.$$

5. (See [Kato et al. 01]).

$$N(X) \geq \frac{1}{\sqrt{C_{NJ}(X) - \frac{1}{4}}}.$$

This implies that if $C_{NJ}(X) < 5/4$, then X and X^* have uniform normal structure.

6. [Khamsi 87]. If $\liminf_{t\to 0}\frac{\rho_X(t)}{t} < \frac{1}{2}$ then $N(X) > 1$.

7. [Prus 91] $N(X) \geq \alpha - (\alpha^2 - 4)^{1/2}$, where $\alpha := \inf\{\frac{\varepsilon}{2} + 2 - \delta(\varepsilon) : 1 \leq \varepsilon \leq 3/2\}$.

8. [Prus 96].
$$N(X) \geq \left(\inf\left\{1 + \rho_X(t) - \frac{t}{2} : t \in \left[0, \frac{1}{2}\right]\right\}\right)^{-1}.$$

9. [Bynum 80]. For X, Y isomorphic Banach spaces, $N(X) \leq d(X, Y)N(Y)$, where $d(X, Y)$ is the Banach -Mazur distance between the Banach spaces X, Y.

10. [Maluta 84]. If $\dim(X) = \infty$ then $N(X) \leq \sqrt{2}$.

11. W.L. Bynum defined in [Bynum 80] the coefficient $BS(X)$ *as the supremum of the set of all numbers M with the property that for each bounded sequence (x_n) with asymptotic diameter A, there is some y in the closed convex hull of the (range of the) sequence such that $M \limsup_n \|x_n - y\| \leq A$.*

    Here
    $$\operatorname{diam}_a((x_n)) := \lim_{n\to\infty} (\sup \|x_i - x_j\| : i, j \geq n)$$
    is called the asymptotic diameter of the sequence (x_n), and
    $$r_a((x_n)) := \inf\{\limsup \|x_n - x\| : x \in \overline{\text{co}}\{x_n : n = 1, \ldots\}\}$$
    is called the asymptotic radius of the sequence (x_n).

    An equivalent definition is
    $$BS(X) := \inf\left\{\frac{\operatorname{diam}_a((x_n))}{r_a((x_n))} : (x_n) \text{ is a nonconvergent bounded sequence in } X\right\}.$$

    It was shown in [Lim 83] that for all Banach space X, $N(X) = BS(X)$.

12. [Casini-Maluta 85]. Suppose that, C is a weakly compact convex subset of X and $T : C \to C$ is k-uniformly Lipschitzian on C with $k < \sqrt{N(X)}$, then T has a fixed point.

13. [Prus-Szczepanik 01] For every Banach space,
    $$N(X) \geq \frac{1}{1 - s}$$
    where
    $$s := \sup\{\frac{2}{3}(\tau W(1 - \tau) + (1 - \tau)W(\tau)) : 0 < \tau < 1\}.$$

14. [Prus-Szczepanik 01] For every Banach space,
    $$N(X) \geq 1 + \frac{s_1^2}{24 + 2s_1^2}$$
    where
    $$s_1 := \sup\{\min\{2 - t, W(t) - \frac{t - 1}{2} : 1 \leq t \leq 2\}.$$

15. [Prus-Szczepanik 01] For every Banach space,

$$N(X) \geq 1 + \frac{4s_2^2}{75 + 8s_2^2}$$

where

$$s_2 := \sup\{\frac{t}{2} - W_1(t) : 0 \leq t \leq 2\}.$$

16. [Prus-Szczepanik 01] For every Banach space,

$$N(X) \geq 1 + \frac{s_3^2}{6 + 3\sqrt{3} + 2s_3^2}$$

where

$$s_3 := \sup\{(1-\beta)(1-(1-\beta)\chi_X(\beta)) : 0 \leq \beta < 1\}.$$

17. [Yang-Wang 06] For each $a \in [0, 1]$,

$$N(X) \geq \sqrt{\frac{\max_{r\in[a,1]} f(r)}{C_{NJ}(a, X)}}$$

where for $r \in [0, 1]$,

$$f(r) := \frac{(1+r)^2 + (1+a)^2}{2(1+r^2)}.$$

29 Bynum's weakly convergent sequence coefficient

It was defined by L.B. Bynum in [Bynum 80] as follows

$WCS(X)$ is the supremum of the set of all numbers M with the property that for each weakly convergent sequence (x_n) with asymptotic diameter A, there is some y in the closed convex hull of the (range of the) sequence such that $M \limsup_n \|x_n - y\| \leq A$.

This is probably one of the Banach space constants most widely studied, although with considerable confusion because there are many equivalent definitions. We will follow in this summary the work by Sims and Smith [Sims-Smith 99]. See also the Ph. D. dissertation of M.A. Smyth.

An equivalent definition is (see [Lim 83])

$$WCS(X) := \inf\left\{\frac{\operatorname{diam}_a((x_n))}{r_a((x_n))} : (x_n) \text{ is a weakly (non strongly) convergent sequence in } X\right\}.$$

Geometrical properties in terms of this constant

1. **Definition 29.1** *We quote [Sims-Smith 99], p. 500.*

   > *Some authors have said that a space X has weak uniform normal structure if $WCS(X) > 1$. We shall say that X satisfies Bynum's condition if this inequality holds.*

2. Recall that a Banach space has the generalized Gossez-Lami Dozo property (GGLD) for short if for every weakly null sequence (x_n) such that $\lim \|x_n\| = 1$ we have that $D[(x_n)] > 1$, where

   $$D[(x_n)] := \limsup_m \left(\limsup_n \|x_m - x_n\|\right).$$

   This property was defined by A. Jiménez-Melado in [Jiménez 92]. In the same work was defined the following coefficient in order to get stability results for (GGLD) in terms of the Banach-Mazur distance:

   $$\beta(X) := \inf\{D[(x_n)] : x_n \overset{w}{\rightharpoonup} 0,\ \|x_n\| \to 1\}.$$

   Obviously $(X, \|\cdot\|)$ has property (GGLD) if $\beta(X) > 1$. Moreover (GGLD) implies (WNS).

Separation of this property

Example 81 The space c_0 equivalently renormed by

$$\|(x_n)\| := \|(x_n)\|_\infty + \sum_n \frac{|x_n|}{2^n},$$

It enjoys Opial condition, and hence (WNS), but lacks (GGLD). (See [Jiménez 92]).

Example 82 *[Giles et al. 85]* In order to produce an example of a discontinuous metric projection Brown devised a geometrically interesting equivalent renorming of the Hilbert sequence space ℓ^2. Given the natural basis (e_n) and writing

$$M := \{(x_n) \in \ell^2 : x_1 = 0\}$$

and

$$M_k := \text{span } \{e_1, e_k\}$$

for $k \geq 3$, in ℓ^2 can be given an equivalent rotund norm $\|\cdot\|$ such that its restriction to M remains the original ℓ^2 norm $\|.\|_2$ and its restriction to M_k is an $\ell^{p(k)}$ norm where $p(k) \to \infty$ as $k \to \infty$. This space is not uniformly rotund in every direction, but it is reflexive and it has normal structure. Moreover this space is not locally uniformly rotund.

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Example 83 [Bynum 80]. For $p \geq 1$, $WCS(\ell_p) = 2^{\frac{1}{p}}$.

Example 84 For a Hilbert space H, $WCS(H) = \sqrt{2}$.

Example 85 [Ayerbe-Xu 93].

$$WCS(E_\beta) = \begin{cases} \frac{\sqrt{2}}{\beta} & 1 \leq \beta\sqrt{2} \\ 1 & \sqrt{2} < \beta < \infty \end{cases}$$

Example 86 [Ayerbe et al. 97 (b)]. $WCS(c_0) = 1$.

Example 87 [Domínguez et al. 96]. $WCS(\ell_{p,q}) = \min\{2^{\frac{1}{p}}, 2^{\frac{1}{q}}\}$.

Example 88 ([Prus 90], [Domínguez 91]). Let (Ω, Σ, μ) be a σ-finite measure space. Let $1 \leq p < +\infty$ and assume that $L^p(\Omega)$ is infinite dimensional. Then

$$WCS(L^p(\Omega)) = N(L^p(\Omega)) = \min\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}.$$

if either $p \geq 2$ or μ is not purely atomic.

Example 89 [Domínguez 98]. Let X_p be ℓ_2 endowed with the norm

$$\|x\| := \left(|x_1|^p + \left(\sum_{n=2}^{\infty} |x_n|^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

where $p > 2$. Then

$$WCS(X_p) = \sqrt{2}.$$

Example 90 [Cui-Hudzik 99]. For $1 < p < \infty$ the *Cesàro sequence space* ces_p was defined by J. S.Shue in 1970. It is useful in the theory of matrix operators an others.

$$\text{ces}_p := \left\{ x \in \ell^0 : \|x\| := \left(\sum_{n=1}^{\infty} \left[\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right]^p \right)^{\frac{1}{p}} < \infty \right\}.$$

One has, for $1 < p < \infty$,

$$WCS(\text{ces}_p) = 2^{\frac{1}{p}}.$$

Other facts concerning this modulus and/or constant

1. ([Sims-Smith 99]).

$$WCS(X) = \inf\left\{\frac{\mathrm{diam}_a((x_n))}{r_a((x_n))} : x_n \xrightarrow{w} 0,\ x_n \not\to 0\right\}.$$

'diam_a' can be replaced with 'diam' in the above equality.

2. (See [Sims-Smith 99] and the references therein).

   The following constant are equal.

   (1) $WCS(X)$.

   (2) $\inf\{\mathrm{diam}_a((x_n)) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$.

   (3) $\beta(X)$.

   (4) $\inf\{\gamma((x_n)) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$.

   (5) $\inf\{\alpha((x_n)) : x_n \xrightarrow{w} 0,\ \|x_n\| \to 1\}$.

3. [Bynum 80]. For a reflexive Banach space $(X, \|\cdot\|)$,

$$1 \le N(X) \le BS(X) \le WCS(X) \le 2.$$

4. [Ayerbe-Domínguez 93]. If X is a reflexive infinite dimensional space such that the duality mapping is continuous, then

$$1 \le N(X) \le \frac{2}{J(X)} \le WCS(X) \le 2.$$

5. [Dhompongsa et al. 06b]

$$\left(WCS(X)\right)^2 \ge \frac{2C_{NJ}(X)+1}{2C_{NJ}(X)^2}.$$

6. [Domínguez-López 92].

$$WCS(X) \ge \frac{1}{\prod_{n=0}^{\infty}(1-\delta_X(2^{-n})}.$$

7. [Bynum 80]. Let X, Y be isomorphic Banach spaces. Then,

$$WCS(X) \le d(X,Y)WCS(Y).$$

   (See also [Ayerbe et al. 97 (b)], p. 119.)

8. [Jiménez 92]. If $WCS(X) = \beta(X) > 1$ and $d(X,Y) < \beta(X)$ then Y has property GGLD.

9. For any Banach space we have

$$WCS(X) \ge 1 + r_X(1).$$

   (See [Lin et al. 95]). This inequality may be strict, as in the following example.

Example 91 *([Kuczumow 99]).* Let $X = \ell_2 \oplus \mathbb{R}$ equipped with the norm

$$\|(x,r)\| := \|x\|_2 + \max\left\{|r| - \frac{1}{2}\|x\|_2, 0\right\}$$

where $\|.\|_2$ denotes the Euclidean ℓ_2 norm. The space X has the nonstrict Opial property, $r_X(1) = \frac{1}{4}$ and $WCS(X) = \sqrt{2}$.

10. ([Domínguez-López 92]). If $(X, \|\cdot\|)$ is a non-Schur Banach space

$$WCS(X) \geq \lim_{\varepsilon \to 1^-} \frac{1}{1 - \Delta_{X,\beta}(\varepsilon)}.$$

11. ([Domínguez 95].) Let X be a reflexive Banach space. Then

$$\sup_{0 \leq \varepsilon \leq a} \frac{t\varepsilon WCS(X^*)}{4} - \Delta_{X,\sigma}(\varepsilon) \leq \Gamma_{X^*}(t) \leq \sup_{0 \leq \varepsilon \leq a} \frac{t\varepsilon}{WCS(X)} - \Delta_{X,\sigma}(\varepsilon)$$

for every $t > 0$, where $a = \sigma(B_X)$. If, in addition, X^* satisfies the Opial condition then

$$\sup_{0 \leq \varepsilon \leq a} \frac{t\varepsilon}{2} - \Delta_{X,\sigma}(\varepsilon) \leq \Gamma_{X^*}(t).$$

12. ([Domínguez 95]). Let X be a reflexive Banach space. Then

$$WCS(X) \lim_{t \to 0} \frac{\Gamma_{X^*}(t)}{t} \leq \varepsilon_{0,\sigma}(X) \leq \frac{4}{WCS(X^*)} \lim_{t \to 0} \frac{\Gamma_{X^*}(t)}{t}.$$

13. ([Domínguez 95]). Let X be a reflexive Banach space satisfying the Opial condition. Then

$$\sup_{0 \leq \varepsilon \leq a} \frac{t\varepsilon WCS(X) WCS(X^*)}{4} - \Delta_{X,\beta}(\varepsilon) \leq \Gamma_{X^*}(t) \leq \sup_{0 \leq \varepsilon \leq a} t\varepsilon - \Delta_{X,\beta}(\varepsilon)$$

for every $t > 0$, where $a = \sigma(B_X)$. If, in addition, X^* satisfies the Opial condition then

$$\sup_{0 \leq \varepsilon \leq a} \frac{t\varepsilon WCS(X)}{2} - \Delta_{X,\beta}(\varepsilon) \leq \Gamma_{X^*}(t).$$

14. ([Domínguez 92]). Let $(X_n, |.|_n)$ be a sequence of reflexive Banach spaces. Let $1 < p < \infty$. Let X be the ℓ_p-direct sum of $(X_n, |.|_n)$. Then,

$$WCS(X) = \inf\{WCS(X_i), 2^{\frac{1}{p}} : i \in \mathbb{N}\}.$$

15. ([Domínguez 92]). Let $(X_1, |.|_1), \ldots, (X_n, |.|_n)$ be a finite sequence of reflexive Banach spaces. Let Z be $\mathbb{R}^n$ with a monotonous norm. Then,

$$WCS\left((X_1 \oplus \ldots \oplus X_n)_Z\right) = \min\{WCS(X_i) : i = 1, \ldots, n\}.$$

16. ([Domínguez-Xu 95], [Ayerbe et al. 97 (b)]). Suppose that $WCS(X) > 1$, C is a weakly compact convex subset of X and $T : C \to C$ is asymptotically regular on C. If

$$\liminf |T^n| < \sqrt{WCS(X)},$$

then T has a fixed point.

17. [Dhompongsa et al. 06b] Let E be a nonempty weakly compact convex subset of a Banach space $(X, \|\cdot\|)$ with

$$C_{NJ}(X) < 1 + \frac{WCS(X)^2}{4}.$$

Assume that $T : E \to KC(E)$ is a nonexpansive mapping. Then T has a fixed point. (Here $KC(E)$ is the family of nonempty compact convex subsets of E.

30 James and Gao-Lau coefficients

Recall that a Banach space $(X, \|\cdot\|)$ is called *uniformly non-square* ([James 64]) if there exists a $\delta \in (0,1)$ such that for any $x, y \in S_X$ either $\frac{1}{2}\|x+y\| \le 1-\delta$ or $\frac{1}{2}\|x-y\| \le 1-\delta$. The constant

$$J(X) := \sup\{\|x+y\| \wedge \|x-y\| : x, y \in S_X\}.$$

where $\|x+y\| \wedge \|x-y\| := \min\{\|x+y\|, \|x-y\|\}$ is called the *non-square* or *James constant* of $(X, \|\cdot\|)$.

Moreover Schäfer (1970) introduced another definition of uniformly non-squareness. $(X, \|\cdot\|)$ *is called* uniformly non-square *(in the sense of Schäfer) if there exists* $\lambda > 1$ *such that for every* $x, y \in S_X$ *we have either* $\|x+y\| > \lambda$ *or* $\|x-y\| > \lambda$. *The constant*

$$S(X) = \inf\{\max\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}$$

is called the Schäfer constant *of the space* $(X, \|\cdot\|)$.

Ji Gao (1982, 1983), and also joint with K.S. Lau ([Gao-Lau 90], [Gao-Lau 91]) studied $G(X)$ and other similar constants in order to simplify the descriptions of the unit ball of $(X, \|\cdot\|)$ due to Schäfer [Schäfer 76], in terms of "girths" and perimeters. (See also [Casini 86]).

The four constants introduced in [Gao-Lau 90] where

$$g(X) := \inf\{\inf\{\max\{\|x+y\|, \|x-y\|\} : y \in S_X\}, x \in S_X\}$$

$$G(X) := \sup\{\inf\{\max\{\|x+y\|, \|x-y\|\} : y \in S_X\}, x \in S_X\}$$

$$j(X) := \inf\{\sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S_X\}, x \in S_X\}$$

$$J(X) := \sup\{\sup\{\min\{\|x+y\|, \|x-y\|\} : y \in S_X\}, x \in S_X\}$$

The notations which use these authors are not standard. In fact, we have

Coefficient	[Casini 86]	[Gao-Lau 90]	[Kato et al. 01]
$\inf\limits_{x\in S_X}\{\inf\limits_{y\in S_X}\{\|x+y\| \vee \|x-y\|\}\}$	$g(X)$	$g(X)$	$S(X)$
$\sup\limits_{x\in S_X}\{\inf\limits_{y\in S_X}\{\|x+y\| \vee \|x-y\|\}\}$	$G(X)$	$G(X)$	–
$\inf\limits_{x\in S_X}\{\sup\limits_{y\in S_X}\{\|x+y\| \wedge \|x-y\|\}\}$	$g'(X)$	$j(X)$	–
$\sup\limits_{x\in S_X}\{\sup\limits_{y\in S_X}\{\|x+y\| \wedge \|x-y\|\}\}$	$G'(X)$	$J(X)$	$J(X)$

The symbol $J(X)$ often has been used for denoting the Jung constants also. The words 'Jung', 'James' and 'Ji' begin with the same letter 'J'. We have here a source of confusion.

Geometrical properties in terms of these constants

1. ([Casini 86]). If $g(X) > 1$ then $(X, \|\cdot\|)$ is uniformly non square.
2. ([Gao-Lau 90]). $(X, \|\cdot\|)$ is uniformly non-square if and only if $J(X) < 2$.
3. In ([Gao-Lau 91]) was showed that if $J(X) < \frac{3}{2}$ then $(X, \|\cdot\|)$ has uniform normal structure. This result has been improved in [Dhompongsa et al. 03b]: If

$$J(X) < \frac{1+\sqrt{5}}{2}$$

then $(X, \|\cdot\|)$ has uniformly normal structure.

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Example 92 ([Casini 86], [Gao-Lau 90]). $g(L^1) = G(L^1) = 1$. $g'(L^1) = J(L^1) = 2$.

Example 93 ([Casini 86], [Gao-Lau 90]). $g(\ell_1) = G(\ell_1) = g'(\ell_1) = J(\ell_1) = 2$.

Example 94 ([Casini 86], [Gao-Lau 90]). $g(\ell_\infty) = g'(\ell_\infty) = 1$, $G(\ell_\infty) = J(\ell_\infty) = 2$.

Example 95 ([Casini 86], [Gao-Lau 90]). For $p \in (1,\infty)$, $g(\ell_p) = \min\{2^{1/p}, 2^{1-1/p}\}$.

Example 96 ([Casini 86], [Gao-Lau 90]). For $p \in (1,\infty)$, $J(\ell_p) = \max\{2^{1/p}, 2^{1-1/p}\}$.

Example 97 ([Casini 86], [Gao-Lau 90]). For $p \in (1,\infty)$, $g'(\ell_p) = G(\ell_p) = 2^{1/p}$.

Example 98 [Dhompongsa et al. 03b]. For $p \in (1,\infty)$, $J(\ell_{p\infty}) \geq 2^{1/p} \geq (1+\sqrt{5})/2$ if $p \leq p_0$. $J(\ell_{p\infty}) \geq 1 + \left(\frac{1}{2}\right)^{1/p} \geq (1+\sqrt{5})/2$ if $p \geq p_0$. Here p_0 is the solution of the equation $2^{1/p} = 1 + \left(\frac{1}{2}\right)^{1/p}$.

Example 99 ([Gao-Lau 90]). $g((\mathbb{R}^2, \|.\|_\infty)) = g'((\mathbb{R}^2, \|.\|_\infty)) = 1$, and $G((\mathbb{R}^2, \|.\|_\infty)) = J((\mathbb{R}^2, \|.\|_\infty)) = 2$.

Example 100 [Dhompongsa et al. 03b] $J(\ell_\infty - \ell_p) = 1 + (1/2)^{\frac{1}{p}}$. Here $\ell_\infty - \ell_p$ is the 2 dimensional Day-James space whose norm is defined by

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_p & x_1x_2 \geq 0 \\ \|(x_1,x_2)\|_\infty & x_1x_2 \leq 0. \end{cases}$$

Example 101 [Dhompongsa et al. 03b] $J(\ell_p - \ell_q) \leq 2\left(\frac{2^{p/q}}{2+2^{p/q}}\right)^{\frac{1}{p}}$ for $1 \leq q \leq p < \infty$. Here the norm is defined by

$$\|(x_1,x_2)\| := \begin{cases} \|(x_1,x_2)\|_p & x_1x_2 \geq 0 \\ \|(x_1,x_2)\|_q & x_1x_2 \leq 0. \end{cases}$$

Example 102 ([Gao-Lau 90]). Let X be $\mathbb{R}^2$ endowed with any norm whose unit sphere is affinely homeomorphic to a hexagon with one of its vertices at $(1,0)$. Then, $g(X) = g'(X) = 4/3$ and $G(X) = J(X) = 3/2$.

Example 103 ([Gao-Lau 90]). Let X be $\mathbb{R}^2$ endowed with any norm whose unit sphere is affinely homeomorphic to a convex symmetric body in the two dimensional Euclidean space, which is invariant under a rotation of $\pi/4$. (For instance, a circle or an octagon will satisfy this condition). Then,

$$g(X) = g'(X) = G(X) = J(X) = \sqrt{2}.$$

Example 104 ([Gao-Lau 90]). If $1 \leq p < \infty$, $G(L^p) = \max\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$.
([Gao-Lau 90]). $g(L^\infty) = 1$, and $g'(L^\infty) = 2$, $G(L^\infty) = J(L^\infty) = 2$.

Example 105 [Kato et al. 01] Let $\mathbb{R}^2$ with the norm defined by

$$\|x\| := \begin{cases} \|x\|_2 & x_1x_2 \geq 0 \\ \|x\|_1 & x_1x_2 \leq 0 \end{cases}$$

Then $J((\mathbb{R}^2, \|.\|)) = \sqrt{\frac{8}{3}}$.

Example 106 [Kato et al. 01] For $2 \le p < \infty$ let $X_{\lambda,p}$ be the space ℓ_p with the norm

$$\|x\|_{\lambda,p} := \max\{\|x\|_p, \lambda\|x\|_\infty\}.$$

Then, $J(X_{\lambda,p}) = \min\{2, \lambda\, 2^{1-\frac{1}{p}}\}$.

Example 107 [Kato et al. 01] For $1 \le p \le 2$ let $Y_{\lambda,p}$ be the space $L^p([0,1])$ with the norm

$$\|x\|_{\lambda,p} := \max\{\|x\|_p, \lambda\|x\|_1\}.$$

Then, $J(Y_{\lambda,p}) = \min\{2, \lambda\, 2^{\frac{1}{p}}\}$.

Other facts concerning this modulus and/or constant

1. ([Casini 86]).
$$\begin{array}{llll} 1 \le & g(X) & \le G(X) & \\ & \downarrow\, \le & \downarrow\, \le & \\ & g'(X) & \le J(X) & \le 2 \end{array}$$
2. ([Casini 86]). $g(X) \le \sqrt{2} \le J(X)$.
3. ([Papini 02]). $1 + \delta_X\left(\frac{1}{2}\right) \le g(X)$.
4. ([Casini 86]). If Y is a subspace of X, $g(X) \le g(Y)$ and $J(Y) \le J(X)$. Similar inequalities do not hold in general for g' and G.
5. ([Casini 86]). If $(X, \|\cdot\|)$ is uniformly non-square and $\widetilde{\varepsilon} \in (0,2)$ is the unique solution of $\varepsilon = 2(1 - \delta_X(\varepsilon))$, then $g(X) = \frac{2}{\widetilde{\varepsilon}} \in (1, \sqrt{2}]$.
6. (See [Casini 86]. Also in [Gao-Lau 90] and [Kato et al. 01], including proofs).
$$J(X)g(x) = 2.$$
7. ([Casini 86]). $J(X) \le \rho_X(1) + 1$.
8. ([Gao-Lau 90]). Let $T : X \to Y$ be an isomorphism. Then,
$$\frac{1}{\|T\|\|T^{-1}\|} \le \frac{g(Y)+2}{g(X)+2} \le \|T\|\|T^{-1}\|.$$
The same inequalities also hold if g is replaced by G, g' or J.
9. [Prus 91]
$$N(X) \ge J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}.$$
10. [Kato et al. 01]
$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\}, x, y \in B_X\}.$$
11. (See [Saejung 06])
$$R(X)w(X) \le J(X).$$
Here $w(X)$ is the Sims' measure of "worthwileness" (see section (31)) ([Sims 94]) and $R(X)$ the García-Falset coefficient of near uniform of smoothness ([García 94, García 97]).

12. ([Gao-Lau 90]). For any Banach space $(X, \|\cdot\|)$, $\sqrt{2} \leq J(X) \leq 2$.

13. ([Gao-Lau 90]). For any Banach space $(X, \|\cdot\|)$,

$$J(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) < 1 - \frac{\varepsilon}{2}\}.$$

14. ([Gao-Lau 91]). $J(X) < \varepsilon$ if and only if $\delta_X(\varepsilon) > 1 - \frac{\varepsilon}{2}$.

15. ([Prus 91]). $N(X) \leq J(X) + 1 - ((G(X)+1)^2 - 4)^{\frac{1}{2}}$.

16. ([Gao-Lau 91]). For any isomorphism T from X to ℓ_p or L^p, $1 < p < \infty$,

$$J(X) \leq \|T\|\|T^{-1}\| \max\left\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\right\}.$$

17. ([Gao-Lau 91]). For any Banach space $(X, \|\cdot\|)$ and for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$,

$$J(X_{\mathcal{U}}) = J(X).$$

18. ([Gao-Lau 90]). If X and Y are Banach spaces and $T : X \to Y$ is a (bicontinuous) isomorphism, then

$$(\|T\|\|T^{-1}\|)^{-1} \leq \frac{J(X)+2}{J(Y)+2} \leq \|T\|\|T^{-1}\|.$$

19. ([Gao-Lau 91]). If $J(Y) < \frac{3}{2}$ and

$$\Delta(X,Y) < \ln\left(\frac{7}{2J(Y)+2}\right),$$

then $(X, \|\cdot\|)$ has uniformly normal structure.

Here $\Delta(X,Y) := \inf\left\{\ln(\|T\|\|T^{-1}\|) : T \in \mathcal{I}(X,Y)\right\}$

20. ([Kato et al. 01]). For any Banach space we have

$$2J(X) - 2 \leq J(X^*) \leq 1 + \frac{J(X)}{2}$$

21. ([Kato et al. 01]). For any non-trivial Banach space $(X, \|\cdot\|)$,

$$\frac{1}{2}J(X)^2 \leq C_{NJ}(X) \leq \frac{J(X)^2}{(J(X)-1)^2+1},$$

where $C_{NJ}(X)$ is the von Neumann-Jordan constant of $(X, \|\cdot\|)$.

22. ([Saejung 06]). For any Banach space $(X, \|.\|)$,

$$C_{NJ(X)} \leq 1 + \frac{(J(X))^2}{4}.$$

where $C_{NJ}(X)$ is the von Neumann-Jordan constant of $(X, \|\cdot\|)$.

23. ([Kato et al. 01]). If X and Y are isomorphic Banach spaces, then

$$\frac{J(X)}{d(X,Y)} \leq J(Y) \leq d(X,Y)J(X),$$

where $d(X,Y)$ is the Banach-Mazur distance between X and Y.

30.1 A parametrization of the James constant

In [Dhompongsa et al. 03b] the authors introduce a generalized James constant which in fact is a function depending on a positive parameter a.

For $a \geq 0$ let

$$J(a, X) := \sup\{\|x + y\| \wedge \|x - z\| : x, y, z \in B_X,\ \|y - z\| \leq a\|x\|\}.$$

We will follow the work [Dhompongsa et al. 03b] to summarize the properties of this modulus. Note that,

(1) $J(0, X) = J(X)$.
(2) $J(., X)$ is a nondecreasing function.
(3) If $J(a, X) < 2$ for some $a \geq 0$ then $J(X) < 2$.

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Example 108 For a Hilbert space H, $J(a, H) = \sqrt{2 + a}$.

Example 109 For $1 < p < 2$ let X_p be $\mathbb{R}^2$ equipped with the norm $\|x\| = \|(x_1, x_2)\| := \|x\|_1$ or $\|x\|_p$ according to $x_1x_2 \geq 0$ or $x_1x_2 < 0$. Let $X_2(p) = \ell_2(X_p)$.

Then we have that $J(X_2(p)) \geq 2^{\frac{1}{p}}$ and $J(1, X_2(p)) < 2$. Hence $J(X_2(p))$ can be arbitrarily close to 2, (letting $p \to 1^+$) whereas $J(1, X_2(p)) < 2$.

Other facts concerning this modulus and/or constant

1. For all $a \geq 0$,
$$J(a, X)^2 \leq 2C_{NJ}(a, X).$$
2. For $0 \leq a \leq b$,
$$J(b, X) + \frac{a}{2} \leq J(a, X) + \frac{b}{2}.$$
In particular the function $J(., X)$ is continuous on $[0, \infty)$.
3. For $a \in (0, 1]$, $C_{NJ}(a, X) \geq \frac{(1+a)^2}{1+a^2}$ if and only if $J(1, X) = 2$.
4. $J(2 - \delta, X) < 2$ for all $\delta > J(X)$.
5. $J(2 - \delta, X) < 2$ for all $\delta > \varepsilon_0(X)$.
6. Let X, Y be isomorphic Banach spaces and $d(X, Y)$ their Banach Mazur distance. If for some $a \in [0, 1]$,
$$d(X, Y) < \frac{3 + a}{2J(a\, d(X, Y), Y)},$$
or
$$d(X, Y) < \frac{1 + \sqrt{5}}{2J(Y)},$$
then $J(a, X) < \frac{3+a}{2}$ or $J(X) < \frac{1+\sqrt{5}}{2}$ respectively. In particular if Y is a Hilbert space and
$$d(X, Y) < \frac{1 + \sqrt{5}}{2\sqrt{2}},$$
the X has uniformly normal structure.

31 A measure of the degree of WORTHwileness, [Sims 94]

Recall that a Banach space $(X, \|\cdot\|)$ has the (WORTH) property if

$$\lim_{n\to\infty} |\ \|x_n - x\| - \|x_n + x\|\ | = 0$$

for all x in X and for all weakly null sequence (x_n).

The (WORTH) property was introduced by B. Sims in [Sims 88], and the following coefficient (see [Sims 94]) measures the degree of "WORTHwileness" of the Banach space $(X, \|\cdot\|)$.

$$w(X) := \sup\{\lambda : \lambda \liminf \|x_n + x\| \leq \liminf \|x_n - x\| : x_n \stackrel{w}{\rightharpoonup} 0 \;\; x \in X\}$$

Independently, in [Jiménez-Llorens 96] the authors considered the constant

$$\mu(X) := \inf\{r > 0 : \limsup \|x_n + x\| \leq r \limsup \|x_n - x\| : x_n \stackrel{w}{\rightharpoonup} 0 \;\; x \in X\}.$$

It is easy to see that $\mu(X) = \frac{1}{w(X)}$.

Geometrical properties in terms of this constant

1. [Sims 94] $(X, \|\cdot\|)$ has (WORTH) if and only if $w(X) = 1$.
2. [Sims 94] A Banach space $(X, \|\cdot\|)$ has property (k) if there exists $k \in [0,1)$ such that whenever $x_n \stackrel{w}{\rightharpoonup} 0$, $\|x_n\| \to 1$ and $\liminf_n \|x_n - x\| \leq 1$ we have $\|x\| \leq k$. One has that $(k) \Rightarrow (WNS)$.

   The Banach space $(X, \|\cdot\|)$ has property (k) if

   $$w(X) > \max\left\{\frac{1}{2}\varepsilon, 1 - \delta_X(\varepsilon)\right\}$$

   for some positive ε.

Separation of this property [Sims 94]

For $p \neq 2$, the spaces $\mathcal{L}^p[0,1]$ do not enjoy (WORTH). They do however enjoy property (k).

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Example 110 For any finite dimensional X, $w(X) = 1$.

Example 111 $\mu(\ell_{2\infty}) = \sqrt{2}$

Other facts concerning this modulus and/or constant

1. For all $x \in X$ and for all weakly null sequence (x_n) in X. As

   $$\|x_n + x\| \leq \|x_n - x\| + 2\|x\| \leq \|x_n - x\| + 2 \liminf \|x_n - x\|$$

   we have that $1 \leq \mu(X) \leq 3$. (In other words, $1 \geq w(X) \geq \frac{1}{3}$).
2. One can replace $\liminf$ with $\limsup$ in the definition of $w(X)$.
3. (See [Jiménez-Llorens 96]). If $(X, \|\cdot\|)$ admits a λ-unconditional basis, then $\mu(X) \leq \lambda$. Consequently, $w(X) \geq \frac{1}{\lambda}$.

4. (See [Jiménez-Llorens 96].) A Banach space $(X, \|\cdot\|)$ has the (FPP) whenever

$$\frac{\varepsilon_0(X)}{4} + \frac{1}{2w(X)} < 1.$$

5. (See

6. JLS06.) A Banach space $(X, \|\cdot\|)$ has normal structure provided that $J(X) < 1 + \frac{1}{\mu(X)}$ where $J(X)$ stands for the James coefficient of $(X, \|\cdot\|)$.

7. (See

8. JLS06.) A Banach space $(X, \|\cdot\|)$ has normal structure provided that $C_{NJ}(X) < 1 + \frac{1}{\mu(X)^2}$ where $C_{NJ}(X)$ stands for the von Neumann-Jordan constant of $(X, \|\cdot\|)$.

32 A coefficient related with (NUS) property

It was defined by J. García-Falset in [García 94]

$$R(X) := \sup\{\liminf_{n\to\infty} \|x_n + x\| : x, x_n \in B_X (n = 1, \ldots), x_n \xrightarrow{w} 0, \}.$$

Geometrical properties in terms of this constant

1. Let $(X, \|\cdot\|)$ a Banach space. The following conditions are equivalent:
   (a) X is (WNUS).
   (b) X is reflexive and $R(X) < 2$.

   Recall that $(X, \|\cdot\|)$ is (WNUS) if for some $\varepsilon > 0$ there exists $\mu > 0$ such that if $0 < t < \mu$ and (x_n) is a basic sequence in B_X then there exists $k > 1$ so that $\|x_1 + x_n\| \leq 1 + \varepsilon t$.

2. (See [Dhompongsa, S.; Kaewkhao, A. 06c])
   A Banach space $(X, \|\cdot\|)$ has property m_∞ if and only if $R(X) = 1$.

   Recall that a Banach space $(X, \|\cdot\|)$ has property m_p (resp., m_∞) if for all $x \in X$, whenever (x_n) is a weakly null sequence in X,

$$\limsup_n \|x + x_n\|^p = \|x\|^p + \limsup_n \|x_n\|^p$$

   (resp.,

$$\limsup_n \|x + x_n\| = \max\left(\|x\|, \limsup_n \|x_n\|\right).$$

EXAMPLES

Example 112 If X is finite dimensional, $R(X) = 1$.

Example 113 $R(c_0) = R(\ell_1) = 1$.

Example 114 For $1 < p < \infty$, $R(\ell_p) = 2^{\frac{1}{p}}$.

Example 115 $R(c) = 2$.

Example 116 $R(\ell_{p,\infty}) \leq 2^{\frac{1}{p}}$.

Example 117 *([Domínguez 96])*$R(\ell_{2,1}) = 2$.

Other facts concerning this modulus and/or constant

(See [García 94] and [García 97]).

1. $1 \leq R(X) \leq 2$.

2. (See [Mazcuñán 02]) If $(X, \|\cdot\|)$ is a non Schur Banach space then

$$R(X) = \sup\{\liminf_n \|x_n + x\|\}$$

   where the supremum is taken over all $x \in S_X$ and weak null sequences (x_n) with

$$\liminf_n \|x_n\| = 1.$$

3. Let $(X, \|\cdot\|)$ be a weakly orthogonal Banach Lattice. Then $R(X) \leq \alpha(X)$ where $\alpha(X)$ is the Riesz angle of X.

4. Let $(X, \|\cdot\|)$ be a Banach space with the property (WORTH). Then $R(X) \leq G(X)$.

5. $(X, \|\cdot\|)$ is a Schur space if and only if $(X, \|\cdot\|)$ has the (KK) property and $R(X) = 1$.

   Recall that a Banach space is said to have the (KK) property whenever $\|x_n - x\| \to 0$ whenever $x_n \overset{w}{\rightharpoonup} x$ and $\|x_n\| \to \|x\|$.

6. If $R(X) < 2$ then X has the weak Banach-Saks property.

7. $\varepsilon_0(X) < 1 \Rightarrow R(X) < 2$.

8. $R(X) < 2 \Rightarrow (X, \|.\|)$ has (WFPP).

9. A Banach space $(X, \|\cdot\|)$ has the (FPP) if there exists a isomorphic Banach space Y such that $d(X, Y)R(Y) < 2$.

10. [Dhompongsa et al. 05] Let $X_1, ..., X_N$ be Banach spaces with $R(X_i) < 2$ for all $i = 1, ..., N$. If Z is uniformly convex, then $R(X_1 \bigoplus \ldots \bigoplus X_N)_Z < 2$, and hence $(X_1 \bigoplus \ldots \bigoplus X_N)_Z$ has the fixed point property for nonexpansive mappings.

    Here Z stands for a Banach space with a normalized 1-unconditional basis $(e_i)_{i \in I}$. For Banach spaces X_i $(i \in I)$, the Z-direct sum is defined by

    $$\left(\bigoplus_{i \in I} X_i\right)_Z := \left\{(x_i) \in \prod_i X_i : \sum_i \|x_i\| e_i \text{ converges in } Z\right\}$$

    endowed with $\|(x_i)\| := \|\sum_i \|x_i\| e_i\|_Z$.

11. If $\frac{\varepsilon_0(X)}{w(X)} < 2$ then $R(X) < 2$.

12. (See [Saejung 06])
    $$(R(X))^2(1 + (w(X))^2) \leq 4C_{NJ}(X).$$

    Here $w(X)$ is the Sims's coefficient of "worthwileness" (See [Sims 94]) and $C_{NJ}(X)$ is the von Neumann-Jordan constant of $(X, \|\cdot\|)$.

13. (See [Dhompongsa, S.; Kaewkhao, A. 06c]). If $(X, \|\cdot\|)$ has property m_p, $(1 \leq p < \infty)$, then $R(X) \leq 2^{\frac{1}{p}}$. Moreover, if in addition $(X, \|\cdot\|)$ does not have Schur property, then $R(X) = 2^{\frac{1}{p}}$.

14. (See [Dhompongsa, S.; Kaewkhao, A. 06c]). If $(X, \|\cdot\|)$ has property WORTH and $C_{NJ}(X) < 2$, then $R(X) < 2$.

33 Domínguez generalization, [Domínguez 96].

Given a nonnegative real number a, one defines

$$R(a, X) := \sup\{\liminf \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences (x_n) in the unit ball of X such that

$$D(x_n) = \limsup_n \left(\limsup_m \|x_n - x_m\| \right) \leq 1\,.$$

Moreover we put

$$M(X) := \sup\left\{ \frac{1+a}{R(a, X)} : a \geq 0 \right\}.$$

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Example 118 .For $1 < p < \infty$,

$$R(a, \ell_p) = \left(a^p + \frac{1}{2}\right)^{\frac{1}{p}} \Rightarrow M(\ell_p) = \left(1 + 2^{\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.$$

Example 119 .

$$R(a, \ell_{p,\infty}) = (1 + a^p)^{\frac{1}{p}} \Rightarrow M(\ell_{p,\infty}) = 2^{\frac{p-1}{p}}.$$

Example 120 *([Domínguez-Japón 98]).*
For a Hilbert space X, $M(X) = \sqrt{3}$.

$$M(E_\beta) = \begin{cases} \sqrt{3} & \beta \leq \sqrt{\frac{3}{2}} \\ \frac{\sqrt{2}}{\beta}\left(1 + \sqrt{\frac{\beta^2-1}{2}}\right) & \sqrt{\frac{3}{2}} < \beta < \sqrt{2} \\ 1 + \frac{1}{\sqrt{2}} & \sqrt{2} \leq \beta. \end{cases}$$

Example 121 *([Domínguez-Japón 98]).*
For $1 \leq p \leq q < \infty$,

$$R(a, \ell_{p,q}) = \left(a^p + \left(\frac{1}{2}\right)^{q/p}\right)^{\frac{1}{p}},$$

and for $q < p$,

$$R(a, \ell_{p,q}) = \begin{cases} \left(\frac{a^p}{2^{1-\frac{p}{q}}} + \frac{1}{2}\right)^{1/p} & \text{if } a \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \\ \left(a^q + \frac{1}{2}\right)^{1/q} & \text{if } a > \left(\frac{1}{2}\right)^{\frac{1}{q}}. \end{cases}$$

Example 122 *([Domínguez-Japón 98]).*
For $1 \leq p < \infty$ and $1 \leq q < \infty$,

$$M(\ell_{p,q}) = \begin{cases} 2^{\frac{1}{q}}\left[1 + \left(\frac{1}{2}\right)^{\frac{p}{q(p-1)}}\right]^{\frac{p-1}{p}} & p \leq q \\ 2^{\frac{1}{p}}\left[1 + \left(\frac{1}{2}\right)^{\frac{p}{q(p-1)}}\right]^{\frac{p-1}{p}} & q < p. \end{cases}$$

Example 123 .

$R(a, c_0) = a \Rightarrow M(c_0) = 2.$

Example 124 *[Domínguez 96b].*

Let $\ell_{p,\infty}$ the Bynum space defined in Example 1 and let p_n be a sequence in $(1, \infty)$ converging to 1. Consider the reflexive Banach space

$$X = \{(x_n) \in \prod_{n=1}^{\infty} \ell_{p_n\infty} : \sum_{n=1}^{\infty} \|x_n\|^2_{p_n\infty} < \infty\}.$$

with the norm $\|(x_n)\| = \sqrt{\sum_{n=1}^{\infty} \|x_n\|^2_{p_n\infty}}$. Since $M(\ell_{p\infty}) = 2^(1 - 1/p)$, and $\ell_{p_n\infty} \subset X$ for every $n \in \mathbb{N}$ we have

$$M(X) \leq \inf\{M(\ell_{p_n\infty} : n \in \mathbb{N}\} = \inf\{2^{(1-1/p_n)} : n \in \mathbb{N}\} = 1.$$

Other facts concerning this modulus and/or constant

(See [Domínguez 96].)

1. If $R(a, X) < 1 + a$ then has the (WFPP).
2. Let X, Y be isomorphic Banach spaces. Then
$$R(a, Y) \leq d(X, Y)R(a, X)$$
for every nonnegative number a.
3. If $M(X) > 1$, then has the (WFPP).
4. If $M(X) > 1$ and Y is a Banach space which is isomorphic to X and $d(X, Y) < M(X)$, then Y has the (WFPP).
5. $R(0, X) = \frac{1}{WCS(X)}$
6. $M(X) \geq WCS(X)$.
7. $M(\ell_{2,\infty}) = \sqrt{2} > WCS(\ell_{2,\infty}) = 1$.
8. Let X be a reflexive Banach space and denote
$$\Gamma(X) := \inf \left\{1 + \Gamma(s) - \frac{s}{2} : s \in [0, 1]\right\}.$$
Then
   (a) For every $a \geq 0$, $R(a, X) \leq 1 + a\Gamma(X)$.
   (b) $M(X) \geq \frac{1}{\Gamma(X)}$.
   (c) $M(X) > 1$ if $\Gamma'(0) < \frac{1}{2}$.
9. Let X be a reflexive Banach space and denote
$$\Gamma'(X) := \inf \left\{1 + \Gamma(s) - \frac{sWCS(X)}{2} : s \in [0, 1]\right\}.$$
Then

(a) $M(X) \geq \frac{1}{\Gamma'(X)}$.

(b) $M(X) > 1$ if $\Gamma'(0) < \frac{WCS(X)}{2}$.

10. ([Ayerbe et al. 97 (b)]). Let $(X, \|\cdot\|)$ be a Banach space and denote

$$\beta_X := \inf \left\{ 1 + \beta_X^k(s) - \frac{s}{2k} : s \in [0,1] \right\}.$$

Then

$$M(X) \geq \frac{1+2k}{1+2k\beta_X}.$$

11. Let $(X, \|\cdot\|)$ be a reflexive Banach space. If $c \in (0,1)$ satisfies $r_{X^*}(c) > 0$ then

$$R(a, X) \leq \max \left\{ 1 + ac, a + \frac{1}{1 + r_{X^*}(c)} \right\}.$$

In particular, if $r_{X^*}(1) > 0$ then $R(a, X) < 1 + a$ and $M(X) > 1$.

12. (See [Jiménez-Llorens 00].) If we define, for $B \geq 1$ the number $C_X(B) := \{c \geq 0 : r_X(c) \leq B - 1\}$ and $|\cdot|$ is an equivalent renorming of the Banach space $(X, \|\cdot\|)$ such that for all $x \in X$

$$\|x\| \leq |x| \leq B\|x\|,$$

and

$$B < \sup \left\{ \frac{1+a}{R\left(\frac{a}{B} C_X(B), X\right)} : a \geq 0 \right\},$$

then $(X, |\cdot|)$ has the (WFPP).

13. [Mazcuñán 03] If $\varepsilon_0(X) < 2$ then $M(X) > 1$, and hence $(X, \|\cdot\|)$ has the fixed point property for nonexpansive mappings.

14. [Dhompongsa et al. 05] Let $X_1, ..., X_N$ be Banach spaces with, $R(a, X_i) < 1 + a$ for some $a \in (0, 1]$, $(i = 1, ..., N)$. If Z is uniformly convex, then the Z-direct sum $(X_1 \bigoplus \ldots \bigoplus X_N)_Z$ has the fixed point property for nonexpansive mappings.

Here Z stands for a Banach space with a normalized 1-unconditional basis $(e_i)_{i \in I}$. For Banach spaces X_i $(i \in I)$, the Z-direct sum is defined by

$$\left(\bigoplus_{i \in I} X_i \right)_Z := \left\{ (x_i) \in \prod_i X_i : \sum_i \|x_i\| e_i \text{ converges in } Z \right\}$$

endowed with $\|(x_i)\| := \|\sum_i \|x_i\| e_i\|_Z$.

34 Lifschitz and Domínguez-Xu coefficients for uniformly lipschitzcian mappings

In 1975, Lifschitz ([Lifschitz 75]) introduced the coefficient $\kappa(M)$ for a metric space (M,d) as follows

$$\begin{aligned}\kappa(M) := \sup\{\beta > 0 : \exists \alpha > 1 \; s.t. \; \forall x, y \in M, \forall r > 0, d(x,y) > r \\ B[x,\beta r] \cap B[y,\alpha r] \subset B[z,r] \text{ for some } z \in M\}\end{aligned}$$

He proved that if (M,d) is a complete bounded metric space and T is a k-uniformly Lipschitzian selfmapping of M with $k < \kappa(M)$, then T has a fixed point. For a Banach space $(X, \|\cdot\|)$, it is often denoted by $\kappa_0(X) := \inf\{\kappa(M)\}$ where the infimun is taken over all the closed convex and bounded subsets of X.

The following definitions are closely inspired on this Lifschitz's coefficient, and were given in [Domínguez-Xu 95]

Let M be a bounded convex subset of X. A number $b \geq 0$ is said to have property (P) with respect to M if there exists some $a > 1$ such that for all $x, y \in M$ and $r > 0$ with $\|x - y\| \geq r$ and each weakly convergent sequence (x_n) in M for which

$$\begin{aligned}\limsup \|x_n - x\| \leq ar \\ \limsup \|x_n - y\| \leq br\end{aligned}$$

there exists some $z \in M$ such that $\liminf \|x_n - z\| \leq r$.

Then one defines

$\tilde{\kappa}(M) := \sup\{b > 0 : b \text{ has property } (P) \text{ w.r.t. } M\}$.

$\tilde{\kappa}(X) := \inf\{\tilde{\kappa}(M) : M \text{ as above}\}$.

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Example 125 ([Ayerbe-Xu 93])

$$\kappa_0(L^p) \geq \begin{cases} \left[1 + \frac{1+\tau_p^{p-1}}{(1+\tau_p)^{p-1}}\right]^{\frac{1}{p}} & p > 2 \\ \sqrt{p} & p \leq 2 \end{cases}$$

where $\tau_p \in (0,1)$ is the unique solution of the equation

$$(p-2)\tau^{p-1} + (p-1)\tau^{p-2} = 1.$$

Example 126 ([Lifschitz 75]). $\kappa_0(H) = \sqrt{2}$ whenever H is an infinite dimensional Hilbert space.

Example 127 ([Domínguez-Xu 95]). For $1 < p < \infty$, $\tilde{\kappa}(\ell_p) = 2^{\frac{1}{p}}$.

Main features of these coefficients

1. ([Ayerbe et al. 97 (b)], p. 145.) Let $(X, \|\cdot\|)$ a Banach space and h a solution of the equation

$$h\left(1 - \delta_X\left(\frac{1}{h}\right)\right) = 1.$$

Then $h \leq \kappa_0(X)$. Furthermore $\varepsilon_0(X) < 1$ if and only if $\kappa_0(X) > 1$.

2. ([Ayerbe et al. 97 (b)], p. 145, [Ayerbe-Xu 93].)

$$\begin{gathered} 1 \leq \kappa_0(X) \leq N(X) \leq WCS(X) \leq 2 \\ 1 \leq \kappa_0(X) \leq N(X) \leq \sqrt{2} \\ 1 \leq \kappa_0(X) \leq N(X) \leq \frac{2}{J(X)} \leq 2 \\ \kappa(X) \leq \frac{2}{J(X)} \end{gathered}$$

3. ([Lifschitz 75]). $\kappa_0(X) \geq \frac{1}{1-\delta_X(1)}$.

4. In [Domínguez-Xu 95] one can see easier equivalent definitions of $\tilde{\kappa}(X)$ when $(X, \|\cdot\|)$ is a Banach space satisfying the uniform Opial condition.

5. ([Domínguez-Xu 95]). If $(X, \|\cdot\|)$ is a Banach space satisfying the uniform Opial condition then $\tilde{\kappa}(X) = 1 + r_X(1)$.

6. ([Domínguez-Xu 95]). Let $(X, \|\cdot\|)$ a Banach space with the uniform Opial condition and M be a bounded convex subset of X. Then $\tilde{\kappa}(M) \geq h$ where h is the unique solution of the equation
$$t\left[1 - \Delta_X\left(\frac{1}{t}\right)\right] = 1.$$

7. ([Domínguez-Xu 95]). Let $(X, \|\cdot\|)$ a Banach space with the uniform Opial condition and M be a bounded convex subset of X. Then $\tilde{\kappa}(M) \geq h_0$ where
$$h_0 := \sup\left\{t > 1 : \Delta'_X\left(\frac{WCS(X)}{t}\right) + \frac{1}{t} > 1\right\}.$$

8. ([Domínguez-Xu 95], [Ayerbe et al. 97 (b)]) If $(X, \|\cdot\|)$ is a Banach space satisfying the Opial condition, or X is reflexive then
$$\tilde{\kappa}(X) \leq WCS(X).$$

   Example 128 *Let X be ℓ_2 renormed by*
$$\|x\| := \left[|x_1|^p + \left(\sum_{n=2}^{\infty} |x_n|^2\right)^{\frac{p}{2}}\right]^{\frac{1}{p}}.$$
   where $2 < p < \infty$.
$$\tilde{\kappa}(X) \leq 2^{\frac{1}{p}} < WCS(X) = \sqrt{2}.$$

9. ([Ayerbe et al. 97 (b)], Theorem IX.2.3). If $(X, \|\cdot\|)$ is a Banach space satisfying the uniform Opial condition, then
$$\tilde{\kappa}(X) \geq \frac{1}{1 - \Delta_{X,\beta}(1^-)}.$$

10. ([Domínguez-Xu 95]) Suppose that C is a weakly compact convex subset of the Banach space $(X, \|\cdot\|)$ and $T : C \to C$ is a uniformly Lipschitzian mapping. If
$$\liminf_n |T^n| < \tilde{\kappa}(C),$$
then T has a fixed point in C. (Here $|T^n|$ denotes the exact Lipschitz constant of T^n).

11. ([Domínguez 98].) Let $(X, \|\cdot\|)$ a Banach space, C a closed convex bounded subset of X and $T : C \to C$ a k-uniformly lipschitzcian mapping. If

$$k < \frac{1 + \sqrt{1 + 4N(X)(\kappa_0(X) - 1)}}{2}$$

then T has a fixed point.

12. ([Domínguez 98].)

$$\kappa_0(X) \le \frac{1 + \sqrt{1 + 4N(X)(\kappa_0(X) - 1)}}{2} \le N(X).$$

13. ([Zhao 92]) If $\beta \ge \frac{1}{2}\sqrt{5}$, then $\kappa_0(E_\beta) = 1$.

14. ([Domínguez 98].) If $1 \le \beta \le \frac{1}{2}\sqrt{5}$, then

$$\kappa_0(E_\beta) = \left(1 + \frac{1}{\lambda^2} - \frac{2}{\lambda^2}\sqrt{\lambda^2 - 1}\right)^{\frac{1}{2}}.$$

15. ([Domínguez 98].) Let $(X, \|\cdot\|)$ be a reflexive Banach space, C a bounded closed convex subset of X and $T : C \to C$ an asymptotically regular mapping. If

$$\liminf |T^n| < \frac{1 + \sqrt{1 + 4WCS(X)(\tilde{\kappa}(X) - 1)}}{2}$$

then T has a fixed point.

35 Coefficients for asymptotic normal structure, [Budzyńska et al. 98].

They were defined in [Budzyńska et al. 98] in order to get a quantitative description of the asymptotic normal structure, playing a similar role to that Bynums's coefficients for normal structure.

For a subsequence (x_{n_i}) of a bounded sequence (x_n) in X, we will denote by $r_a((x_{n_i}))$ the asymptotic radius of this subsequence with respect to the set $\overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$, i.e.

$$r_a((x_{n_i})) = \inf\left\{\limsup_i \|x - x_{n_i}\| : x \in \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})\right\}.$$

Moreover the bounded sequence (x_n) in X is asymptotically regular (a.r.) whenever $x_n - x_{n+1} \to 0_X$. Now we define

$$\begin{aligned} AN(X) := \sup\{k : k.\inf_{(x_{n_i})} r_a((x_{n_i})) \leq \mathrm{diam}_a((x_n)) \\ \text{for each a.r. bounded sequence } (x_n)\} \end{aligned}$$

If we add in this definition the condition that the sequence (x_n) is such that the set $\overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\})$ is weakly compact, then we get the **asymptotic normal structure coefficient with respect to the weak topology**

$$\begin{aligned} w - AN(X) := \sup\{k : k.\inf_{(x_{n_i})} r_a((x_{n_i})) \leq \mathrm{diam}_a((x_n)) \\ \text{for each a.r. bounded sequence } (x_n) \\ \text{with } \overline{\mathrm{co}}(\{x_n : n \in \mathbb{N}\}) \text{ weakly compact}\} \end{aligned}$$

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Example 129 $AN(\ell_p) = 2^{\frac{1}{p}}$.

Example 130 For $\beta, p > 1$ let us consider ℓ_p endowed with the norm

$$\|x\|_\beta := \max\left\{\|x\|_\infty, \frac{1}{\beta}\|x\|_p\right\}.$$

We denote this space by X^p_β. If we take the space $Z := X^2_{\sqrt{2}} \times \ell_1$ equipped with the ℓ_1-norm then this space is nonreflexive. Thus, $AN(Z) = 1$. But $w - AN(Z) = \sqrt{2}$.

Geometrical properties in terms of these constants

1. **Definition 35.1** *If $AN(X) > 1$ we say that $(X, \|\cdot\|)$ has* **uniform asymptotic normal structure***, (UAN) for short.*

2. **Definition 35.2** *If $w - AN(X) > 1$ we say that $(X, \|\cdot\|)$ has* **uniform asymptotic normal structure with respect to the weak topology***, (w-UAN) for short.*

Other facts concerning this modulus and/or constant

1. $1 \leq AN(X) \leq w - AN(X)$.

2. $1 \leq WCS(X) \leq w - AN(X)$.

3. In the definitions of $w - AN(X)$ we can replace $\text{diam}_a((x_n))$ by $\text{diam}((x_n))$.
4. If a Banach space $(X, \|\cdot\|)$ has $AN(X) > 1$ then it is reflexive.
5. $w - AN(X) > 1 \Rightarrow (WFPP)$.
6. Let $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ be isomorphic Banach spaces. Then we have
$$\begin{aligned} &AN(X) \leq d(X,Y)AN(Y), \\ &w - AN(X) \leq d(X,Y).w - AN(Y). \end{aligned}$$
7. $AN(X) = \infty$ if and only if $(X, \|\cdot\|)$ is finite dimensional.
8. $w - AN(X) = \infty$ if and only if $(X, \|\cdot\|)$ is a Schur space.

36 Coefficient for Semi-Opial property, [Budzyńska et al. 98].

It was defined in [Budzyńska et al. 98]. Recall that a Banach space is said to have the semi-Opial (weak semi-Opial) property, (SO) ((w-SO)) for short, if for each bounded nonconstant asymptotically regular sequence (x_n) in X (with weakly compact closed convex hull), there exists a subsequence (x_{n_i}) weakly convergent to x, such that

$$\liminf_i \|x - x_{n_i}\| < \operatorname{diam}(\{x_n\}).$$

Now we define the **semi-Opial coefficient with respect to the weak topology** as follows

$$\begin{aligned} w - SOC(X) := \sup\{k : k.\inf{}_{(x_{n_i}),x_{n_i} \overset{w}{\rightharpoonup} y}\, r_a\left(y,(x_{n_i})\right) \leq \operatorname{diam}_a((x_n)) \\ \text{for each a.r. bounded sequence } (x_n) \\ \text{with } \overline{\operatorname{co}}(\{x_n : n \in \mathbb{N}\}) \text{ weakly compact}\} \end{aligned}$$

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Example 131 $w - SOC(\ell_p) = 2^{\frac{1}{p}}$.

Example 132

$$w - SOC(X_\beta^p) = \max\left[1, \min\left(2^{\frac{1}{p}}, \frac{4^{\frac{1}{p}}}{\beta}\right)\right].$$

Example 133 If we take the space $Z := X^2_{\sqrt{2}} \times \ell_1$ equipped with the ℓ_1-norm then this space is nonreflexive. Thus, $AN(Z) = 1$. But $w - SOC(Z) = \sqrt{2}$.

Geometrical properties in terms of this constant

1. **Definition 36.1** *If $w - SOC(X) > 1$ we say that $(X, \|\cdot\|)$ has* uniform semi-Opial property, *with respect to the weak topology, (w-USO) for short.*

Other facts concerning this modulus and/or constant

1. $1 \leq WCS(X) \leq w - SOC(X) \leq w - AN(X)$.
2. In the definitions of $w - SOC(X)$ we can replace $\operatorname{diam}_a((x_n))$ by $\operatorname{diam}((x_n))$.
3. If a Banach space $(X, \|\cdot\|)$ has the nonstrict Opial property, then $w - SOC(X) = w - AN(X)$.
4. The space $L^p([0,1])$ $1 < p < \infty$, $p \neq 2$, has $w - USO$ property but does not satisfy the nonstrict Opial condition.
5. Let $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ be isomorphic Banach spaces. Then we have
$$w - SOC(X) \leq d(X,Y).w - SOC(Y).$$
6. $w - SOC(X) = \infty$ if and only if $(X, \|\cdot\|)$ is a Schur space.
7. Suppose that $X = W \oplus Z$ where W is a closed subspace of X, Z is a Schur space, and the projection onto W has norm 1. Then we have $w - SOC(X) = w - SOC(W)$
8. Let $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ be Banach spaces. If $(X, \|\cdot\|)$ is $w - USO$ and $WCS(Y) > 1$ then $(X \times Y)_{\ell_p}$ $(1 \leq p < \infty)$, is also $w - USO$.

37 Measuring the triangles inscribed in a semicircle, [Baronti et al. 00].

Two parameters for normed spaces were defined in [Baronti et al. 00]. These parameters measure how big the sum of the distances from a point of the unit sphere to two antipodal points can be. In other terms, their value depends on the perimeter of triangles with the diameter as one side and the third vertex on the sphere.

Now, following [Baronti et al. 00] (see also [Papini 01]), we define

$$\begin{aligned} A_1(X) &:= \tfrac{1}{2}\inf_{x\in S_X}\left(\sup_{y\in S_X}(\|x+y\|+\|x-y\|)\right) \\ A_2(X) &:= \tfrac{1}{2}\sup_{x\in S_X}\left(\sup_{y\in S_X}(\|x+y\|+\|x-y\|)\right) \end{aligned}$$

In [Baronti 81] were early defined the constants $A(X) := \sup\{\frac{\|x+y\|+\|x-y\|}{2} : x, y \in S_X, x \perp y\}$ and $A'(X) = A_2(X)$.

Geometrical properties in terms of this constant

1. $A_2(X) = 2$ characterizes the spaces which are not uniformly non-square.
2. $A_1(X) = A_2(X) = \sqrt{2}$ in inner product spaces.
3. If $A_1(X) = 2$, then $(X, \|\cdot\|)$ is not uniformly non-square; moreover $\dim(X) = \infty$.

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Example 134 [Baronti et al. 00]
Let $(X, \|\cdot\|)$ be the space $\mathbb{R}^2$ with the norm given by the hexagon, that is:

$$\|(x,y)\| := \max\{|x|, |y|, |x-y|\}.$$

Then we have $\delta_X(\varepsilon) = \max\{0, \frac{1}{2}(\varepsilon-1)\}$. Therefore, for all $\varepsilon \in [1,2]$:

$$1 + \frac{1}{2}\varepsilon - \delta_X(\varepsilon) = \frac{3}{2} = A_2(X).$$

Example 135 [Baronti et al. 00]
$A_1((L^1[0,1], \|.\|_1)) = 2$.

Example 136 [Baronti et al. 00]
If X is one of the spaces $L^1([0,1])$, $\mathcal{C}([0,1])$, $\mathcal{C}_0([0,1])$, c_0, c, ℓ_∞, then $A_2(X) = 2$.

Example 137 [Baronti et al. 00]
If X is one of the spaces c_0, c, ℓ_∞, then $A_1(X) = \frac{3}{2}$.
If X is one of the spaces ℓ_1, $L^1([0,1])$, $\mathcal{C}([0,1])$, $\mathcal{C}_0([0,1])$, then $A_1(X) = 2$.

Example 138 For $1 < p < \infty$
$A_2(\ell_p) = \max\{2^{\frac{1}{p}}, 2^{1-\frac{1}{p}}\}$.

Example 139 For $1 < p < 2$ we have

$$A_1(\ell_p) = 2^{\frac{1}{p}}.$$

For $2 < p < \infty$,

$$2^{\frac{1}{p}} \leq A_1(\ell_p) \leq \frac{1}{2} \sup_{0 \leq t \leq 1} \{(|1+t|^p + 1 - |t|^p)^{\frac{1}{p}} + (|1-t|^p + 1 - |t|^p)^{\frac{1}{p}}\}.$$

Example 140 For $(X, \|.\|) = (\mathbb{R}^2, \|.\|_\infty)$ and $(X, \|.\|) = (\mathbb{R}^2, \|.\|_1)$ one has that

$$A_2(X) = 2 > \frac{3}{2} = A_1(X).$$

Other facts concerning this modulus and/or constant

1. [Papini 01] $G(X) \leq A_2(X)$. No general relation exists between $A_1(X)$ and $G(X)$.
2. [Baronti et al. 00]

$$\begin{array}{l} A_1(X) := \frac{1}{2} \inf_{x \in S_X} \left(\sup_{y \in B_X} (\|x+y\| + \|x-y\|) \right). \\ A_2(X) := \frac{1}{2} \sup_{x \in B_X} \left(\sup_{y \in B_X} (\|x+y\| + \|x-y\|) \right). \end{array}$$

3. [Baronti et al. 00] $1 \leq A_1(X) \leq A_2(X) \leq 2$ always.
4. $A_2(X) = \rho_X(1) + 1$, where $\rho_X(.)$ is the Lindenstrauss modulus of smoothness.
5. $A_2(X) = 1 + \sup \left\{ \frac{\varepsilon}{2} - \delta_{X^*}(\varepsilon) : 0 \leq \varepsilon \leq 2 \right\}$.
6. For every Banach space $(X, \| \cdot \|)$, we have

$$A_2(X) = 1 + \sup \left\{ \frac{\varepsilon}{2} - \delta_X(\varepsilon) : \sqrt{2} \leq \varepsilon \leq 2 \right\}$$

7. $A_2(X) = A_2(X^*)$.
8. In any Banach space X, $A_1(X)A_2(X) \geq 2$.
9. In particular, $A_2(X) \geq \sqrt{2}$.
10. If Y is a subspace of X, $A_2(Y) \leq A_2(X)$.
11. $A_2(X) = \sup\{A_2(Y) : Y \text{ subspace of } X, \dim(Y) = 2\}$.
12. $A_1(X) \geq \frac{3+\sqrt{21}}{6} \simeq 1.264$.
13. If dim(X)=2, then $2A_2(X) \leq \frac{p(X)}{2}$, were $p(X) := 2\gamma(-x.x)$ is the "self length" of the unit sphere S_X. That is, $\gamma(-x, x)$ is the length of the curve joining $-x$ and x along S_X. Recall that $p(X) \in [6, 8]$.
14. If dim(X)=2, then $A_1(X) \leq \frac{1+\sqrt{1+4p(X)}}{4} \leq \frac{1+\sqrt{33}}{4} \simeq 1.686$.
15. $\delta_X(A_2(X)) \geq 1 - \frac{A_2(X)}{2}$

16. $\lim_{\varepsilon \to 2^-} \delta_X(\varepsilon) \geq 2 - A_2(X)$.

17. $\delta_X(\varepsilon) > 0$ whenever $\varepsilon > 2A_2(X) - 2$.

18. $A_2(X) \geq 1 + \frac{1}{2}\varepsilon_0(X)$. In particular $A_2(X) < \frac{3}{2}$ implies that $\delta_X(1) > 0$ and hence $\varepsilon_0(X) < 1$ (and hence $(X, \|\cdot\|)$ has normal structure).

19. $2A_2(X) \geq \varepsilon + 2\sqrt{1 - \frac{\varepsilon^2}{4}}$ for every $\varepsilon \in (0, 2)$.

20. (Unpublished)

    $A_2(X) \leq \sqrt{2C_{NJ}(X)}$

    where $C_{NJ}(X)$ stands for the Jordan- von Neumann constant of $(X, \|\cdot\|)$.

21. If Y is a dense subspace of X, then $A_i(Y) = A_i(X)$ $(i = 1, 2)$.

22. Let X, Y isomorphic Banach spaces. Then, for $i = 1, 2$ we have

$$|A_i(X) - A_i(Y)| \leq (4 - i)(d(X, Y) - 1).$$

23. If $A_1(X) = 2$ then X is not uniformly non square and $\dim(X) = \infty$.

38 Prus-Szczepanik coeffcients for uniformly normal structure, [Prus-Szczepanik 01].

Let X Y isomorphic Banach spaces. By $\mathcal{G}$ we denote the family of all normed spaces $(\mathbb{R}^2, \|.\|)$ whose norm coincides with the maximum norm in at least two quadrants of the standard coordinate system. Let $\mathcal{F}_2(X)$ the family of all two dimensional subspaces of X.

Put

$$G(X) := \inf\{d(Y, Z) - 1 : Y \in \mathcal{F}_2(X), Z \in \mathcal{G}\}.$$

In [Prus-Szczepanik 01] is defined

$$G_1(X) := \inf \left\{\max\{\|x - y\|, 3 - \|2x - y\|, 3 - \|x - 2y\| : x, y \in S_X\}\right\}.$$

Geometrical properties in terms of these constants

1. If X does not have normal structure then $G(X) = 0$.
2. If $(X, \|.\|)$ is not uniformly nonsquare, then $G_1(X) = 1$.

Other facts concerning this modulus and/or constant

1. $0 \le G(X) \le 1$.
2. $G(E) = \sqrt{2} - 1$ when E is a two dimensional Euclidean space.
3. $G(X) \le \sqrt{2} - 1$ if X is infinite dimensional.
4. $1 \le G_1(X) \le 2$.
5. If $(X, \|.\|)$ is not reflexive, then $G_1(X) = 1$.
6. $G_1(X) \le N(X)$. (Hence $G_1(X) > 1$ implies that $(X, \|.\|)$ has uniformly normal structure.
7. Let $(X, \|.\|)$ be a Banach space with $G_1(X) < 3/2$. Then
$$G(X) + 1 \le \frac{G_1(X)}{3 - 2G_1(X)}.$$
8. $$N(X) \ge 1 + \frac{G(X)}{3 + 2G(X)}.$$

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