Programa

Lección 8
Los potenciales electromagnéticos

- 8.1 Introducción
- 8.2 Los potenciales electromagnéticos. Transformaciones de contraste
- 8.3 Ecuación de ondas para los potenciales. Soluciones retardadas
- 8.4 Campos de radiación
- 8.5 Radiación de sistemas sencillos: el dipolo eléctrico y el dipolo magnético

Programa

Lección 8
Los potenciales electromagnéticos

BIBLIOGRAFÍA

Griffiths	Lección	9
Jackson	Lección	6 y 9
Pomer	Lección	13
Reitz - Milford -Christy	Lección	16

Los potenciales electromagnéticos. Transformaciones de contraste

$$\nabla \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \qquad \nabla x\, \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad \nabla \vec{B} = 0 \qquad \nabla x\, \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

$$\vec{B} = \nabla x\, \vec{A} \qquad \nabla x(\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0 \qquad \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\boxed{\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \phi}$$

Los potenciales electromagnéticos

Los potenciales electromagnéticos. Transformaciones de contraste

$$\nabla \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \quad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \quad \nabla(-\nabla\phi - \frac{\partial \vec{A}}{\partial t}) = \frac{\rho}{\varepsilon_0} \quad \boxed{\nabla^2\phi + \frac{\partial(\nabla\vec{A})}{\partial t} = -\frac{\rho}{\varepsilon_0}}$$

$$\nabla x\vec{B} = \mu_o \vec{J} - \mu_o\varepsilon_o \frac{\partial \vec{E}}{\partial t} \qquad \vec{B} = \nabla x\, \vec{A} \qquad \nabla x\nabla x\vec{A} = \nabla(\nabla\vec{A}) - \nabla^2\vec{A})$$

$$\nabla x\nabla x\vec{A} = \nabla(\nabla\vec{A}) - \nabla^2\vec{A} = \mu_o\vec{J} - \mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t} = \mu_o\vec{J} - \mu_o\varepsilon_o(-\nabla\phi - \frac{\partial\vec{A}}{\partial t})$$

$$\boxed{(\nabla^2\vec{A} - \mu_o\varepsilon_o\frac{\partial^2\vec{A}}{\partial t^2}) - \nabla(\nabla\vec{A} + \mu_o\varepsilon_o\frac{\partial\phi}{\partial t}) = -\mu_o\vec{J}}$$

Los potenciales electromagnéticos

Los potenciales electromagnéticos. Transformaciones de contraste

$$(\phi, \vec{A}); (\phi', \vec{A}') \qquad \vec{A}' = \vec{A} + \vec{\alpha} \qquad \phi' = \phi + \beta$$

$$\nabla x \vec{\alpha} = 0 \qquad \vec{\alpha} = \nabla\lambda \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\nabla\beta + \frac{\partial \vec{\alpha}}{\partial t} = 0 \qquad \nabla(\beta + \frac{\partial \lambda}{\partial t}) = 0 \qquad \beta = -\frac{\partial \lambda}{\partial t} + K(t)$$

$$\vec{A}' = \vec{A} + \nabla\lambda \qquad \phi' = \phi - \frac{\partial \lambda}{\partial t}$$

Los potenciales electromagnéticos

Transformación (gauge) de Coulomb

$$\boxed{\nabla\vec{A} = 0}$$

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

$$\phi = \frac{1}{4\pi\,\varepsilon_0}\int\frac{\rho}{R}\,dv \qquad\qquad \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$$

$$(\nabla^2\vec{A} - \mu_o\varepsilon_o\frac{\partial^2\vec{A}}{\partial t^2}) = -\mu_o\vec{J} + \mu_o\varepsilon_o\nabla\frac{\partial\phi}{\partial t}$$

Los potenciales electromagnéticos

Transformación (gauge) de Lorentz

$$\nabla \vec{A} = -\mu_o \varepsilon_o \frac{\partial \phi}{\partial t}$$

Los potenciales electromagnéticos

Ecuación de ondas para los potenciales. Soluciones retardadas

$$\nabla^2\phi - \mu_o\varepsilon_o \frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A} - \mu_o\varepsilon_o \frac{\partial^2\vec{A}}{\partial t^2} = -\mu_o\vec{J}$$

Los potenciales electromagnéticos

Satisfacen las soluciones retardadas
el gauge de Lorentz

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

$$\phi(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{R}\,dv' \qquad \vec{A}(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{\vec{J}(r')}{R}\,dv'$$

Los potenciales electromagnéticos

Satisfacen las soluciones retardadas el gauge de Lorentz

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{[\rho(r')]}{R}\, dv' \qquad \vec{A}(r,t) = \frac{\mu_0}{4\pi}\int \frac{[\vec{J}(r')]}{R}\, dv'$$

$$\phi_1(r,t) \approx \frac{1}{4\pi\varepsilon_0}\int_{V_1} \frac{\rho(r',t)}{R}\, dv' \qquad \nabla^2\phi_1 = -\frac{\rho}{\varepsilon_0}$$

$$\phi_2(r,t) = \frac{1}{4\pi\varepsilon_0}\int_{V_2} \frac{\rho\left(r', t - R/c\right)}{R}\, dv'$$

Los potenciales electromagnéticos

Satisfacen las soluciones retardadas el gauge de Lorentz

$$\nabla^2\phi_2 = \nabla^2\phi_{r'} = \nabla^2\phi_R \quad \nabla_R{}^2\left(\frac{\rho}{R}\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\frac{\rho}{R}\right) = \frac{1}{R_2}\frac{\partial}{\partial R}\left(R\frac{\partial\rho}{\partial R} - \rho\right) = \frac{1}{R}\frac{\partial^2\rho}{\partial R^2}$$

$$\nabla^2\phi_2 = \frac{1}{4\pi\,\varepsilon_0}\int_{v_2}\nabla^2\left(\frac{\rho}{R}\right)dv = \frac{1}{4\pi\,\varepsilon_0}\int_{v_2}\frac{1}{R}\left(\frac{\partial^2\rho}{\partial R^2}\right)dv$$

$\rho\left(r', t - R/c\right)$ *es solución de la ecuación de ondas*

$$\frac{\partial^2\rho}{\partial R^2} = \frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} \qquad \nabla^2\phi_2 = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{1}{4\pi\,\varepsilon_0}\int_{v_2}\frac{\rho\left(\vec{r}', t - R/c\right)}{R}dv\right] = \frac{1}{c^2}\frac{\partial^2\phi_2}{\partial t^2} \approx \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

Los potenciales electromagnéticos

Satisfacen las soluciones retardadas el gauge de Lorentz

$$\nabla^2\phi_2 = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$$

$$\nabla^2\phi = \nabla^2(\phi_1 + \phi_2) = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{1}{\varepsilon_0}\rho$$

Los potenciales electromagnéticos

Ecuación de ondas para los potenciales. Soluciones retardadas

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(r',t-\frac{R}{c}\right)}{R}\,dv' \qquad \phi(r,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\left[\rho\left(r'\right)\right]}{R}\,dv'$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int \frac{\left[\vec{J}\left(r'\right)\right]}{R}\,dv' \qquad \frac{1}{c^2}\frac{\partial\phi}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\left[\dot{\rho}\right]}{R}\,dv'$$

$$\partial_i t' = -\frac{R_i}{c\,R} \qquad \nabla\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int_V \frac{\nabla\left[\vec{J}\left(r'\right)\right]}{R}\,dv' - \frac{\mu_0}{4\pi}\int_V \left[\vec{J}\left(r'\right)\right]\nabla'\left(\frac{1}{R}\right)dv'$$

Los potenciales electromagnéticos

Ecuación de ondas para los potenciales. Soluciones retardadas

$$\nabla\left[\vec{J}(r')\right] = -\frac{\vec{R}\left[\dot{\vec{J}}\right]}{cR}$$

$$\nabla'\left[\vec{J}\right] = \nabla'\left[\vec{J}\right]_{t'} + \frac{\partial\left[\vec{J}\right]}{\partial t'}\nabla' t' = \nabla'\left[\vec{J}\right]_{t'} + \frac{\vec{R}\left[\dot{\vec{J}}\right]}{cR} = \nabla'\left[\vec{J}\right]_{t'} - \nabla\left[\vec{J}\right]$$

$$(\nabla'\vec{J})_{t'} + \left(\frac{\partial\rho}{\partial t'}\right)_{t'} = 0 \qquad\qquad \nabla\left[\vec{J}\right] = -\frac{\partial\left[\rho\right]}{\partial t'} - \nabla'\left[\vec{J}\right]$$

Los potenciales electromagnéticos

Ecuación de ondas para los potenciales. Soluciones retardadas

$$\frac{\mu_0}{4\pi}\int_V \frac{\nabla\left[\vec{J}\right]}{R}\,dv' = -\frac{\mu_0}{4\pi}\int_V \frac{\left[\dot{\rho}\right]}{R}\,dv' - \frac{\mu_0}{4\pi}\int_V \frac{\nabla'\left[\vec{J}\right]}{R}\,dv'$$

$$\nabla\,\vec{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} = -\frac{\mu_0}{4\pi}\int_V \frac{\nabla'\left[\vec{J}\right]}{R}\,dv' - \frac{\mu_0}{4\pi}\int_V \left[\vec{J}\right]\nabla'\left(\frac{1}{R}\right)dv'$$

$$-\frac{\mu_0}{4\pi}\int_V \frac{\nabla'\left[\vec{J}\right]}{R}\,dv' - \frac{\mu_0}{4\pi}\int_V \left[\vec{J}\right]\nabla'\left(\frac{1}{R}\right)dv' = -\frac{\mu_0}{4\pi}\int_V \nabla'\left(\frac{\left[\vec{J}\right]}{R}\right)dv' = -\frac{\mu_0}{4\pi}\int_{S_\infty}\left(\frac{\left[\vec{J}\right]}{R}\right)d\vec{S}' = 0$$

Los potenciales electromagnéticos

Campos de radiación

$$\nabla \, x \, \vec{A} = \frac{\mu_0}{4\pi} \int_V \left[\vec{J}\right] x \, \frac{\vec{R}}{R^3} \, dv' + \frac{\mu_0}{4\pi} \int_V \frac{\nabla x \left[\vec{J}\right]}{R} \, dv'$$

$$\nabla x \left[\vec{J}\right] = \nabla \, t' x \left[\dot{\vec{J}}\right] = \left[\dot{\vec{J}}\right] x \, \frac{\vec{R}}{cR}$$

$$\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi} \int_V \left[\vec{J}\right] x \frac{\vec{R}}{R^3} \, dv' + \frac{\mu_0}{4\pi \, c} \int_V \frac{\left[\dot{\vec{J}}\right] x \vec{R}}{R^2} \, dv' \qquad -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_0}{4\pi} \int_V \frac{\left[\dot{\vec{J}}\right]}{R} \, dv'$$

Los potenciales electromagnéticos

Campos de radiación

$$-\nabla\phi = \frac{1}{4\pi\,\varepsilon_0}\int_V \frac{[\rho]\vec{R}}{R^3}\,dv' + \frac{1}{4\pi\varepsilon_0\,c}\int_V \frac{\left[\dot{\rho}\right]\vec{R}}{R^2}\,dv'$$

$$\frac{\partial\rho}{\partial t'} = -\left(\nabla'\left[\vec{J}\right]\right)_{t'} = \frac{\vec{R}\left[\dot{\vec{J}}\right]}{cR} - \nabla'\left[\vec{J}\right] \qquad \int_V \left[\dot{\rho}\right]\frac{\vec{R}}{R^2}\,dv' = \int_V \left[\dot{\vec{J}}\right]\left(\frac{\vec{R}}{cR^3}\right)dv' - \int_V \left(\nabla'\left[\vec{J}\right]\right)\left(\frac{\vec{R}}{R^2}\right)dv'$$

$$\int_V \left(\nabla'\left[\vec{J}\right]\frac{\vec{R}}{R^2}\right)dv' = \int_V \nabla'\left(\left[\vec{J}\right]\frac{\vec{R}}{R^2}\right)dv' - \int_V \left(\left[\vec{J}\right]\nabla'\right)\frac{\vec{R}}{R^2}\,dv' = \int_V \frac{\left[\vec{J}\right]}{R^2}\,dv' - 2\int_V \frac{\left[\vec{J}\right]\vec{R}}{R^4}\,dv'$$

Los potenciales electromagnéticos

Campos de radiación

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\,\varepsilon_0}\int_V \frac{[\rho]\vec{R}}{R^3}\,dv' + \frac{1}{4\pi\varepsilon_0 c^2}\int_V \frac{\left(\left[\dot{\vec{J}}\right]\vec{R}\right)\vec{R}}{R^3}\,dv' -$$

$$-\frac{\mu_0}{4\pi}\int_V \frac{\left[\dot{\vec{J}}\right]}{R}\,dv' + \frac{1}{4\pi\varepsilon_0 c}\int_V \frac{\left[\vec{J}\right]}{R^2}\,dv' - \frac{2}{4\pi\varepsilon_0 c}\int_V \frac{\left(\vec{J}\,\vec{R}\right)}{R^4}\,dv'$$

$$\left(\vec{R}\left[\vec{J}\right]\right)\vec{R} - \vec{R}x\left(\left[\vec{J}\right]x\vec{R}\right) = 2\left(\left[\vec{J}\right]\vec{R}\right)\vec{R} - R^2\left[\vec{J}\right]$$

Los potenciales electromagnéticos

Campos de radiación

$$\vec{E}(\vec{r},t) = \frac{1}{4\pi\,\varepsilon_0}\int_V \frac{[\rho]\vec{R}}{R^3}\,dv' + \frac{1}{4\pi\varepsilon_0\,c}\int_V \frac{\left([\vec{J}]\vec{R}\right)\vec{R} - \vec{R}\,x\left([\vec{J}]\,x\,\vec{R}\right)}{R^4}\,dv' + \frac{\mu_0}{4\pi}\int_V \frac{\left(\left[\dot{\vec{J}}\right]x\,\vec{R}\right)x\vec{R}}{R^3}\,dv'$$

$$\vec{B}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi\,c}\int_V \frac{\left[\dot{\vec{J}}\right]x\vec{R}}{R^2}\,dv' \qquad \vec{E}_{rad}(\vec{r},t) = \frac{\mu_0}{4\pi}\int_V \frac{\left(\left[\dot{\vec{J}}\right]x\,\vec{R}\right)x\vec{R}}{R^3}\,dv'$$

Los potenciales electromagnéticos

Campos de radiación

$$\rho(\vec{r},t) = \rho_\omega(\vec{r})e^{-i\omega t}$$

$$\phi(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_\omega(\vec{r}')e^{-i\omega\left(t-R/c\right)}}{R}dv' = \frac{e^{-i\omega t}}{4\pi\varepsilon_0}\int_V \frac{\rho_\omega(\vec{r}')e^{i\omega R/c}}{R}dv'$$

$$\phi_\omega(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_\omega(\vec{r}')e^{ikR}}{R}dv' \qquad \vec{A}_\omega(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_\omega(\vec{r}')e^{ikR}}{R}dv'$$

$$\vec{E}_{\omega\ rad}(\vec{r}) = -\frac{ikc\mu_0}{4\pi}\int_V \frac{\left(\vec{J}_\omega x\,\vec{R}\right)x\vec{R}\,e^{ikR}}{R^3}dv' = -\frac{i\mu_0 c}{4\pi\,k}\int_V \frac{\left(\vec{J}_\omega x\,\vec{k}\right)x\vec{k}\,e^{ikR}}{R}dv'$$

$$\vec{B}_{\omega\ rad}(\vec{r}) = -\frac{ik\mu_0}{4\pi}\int_V \frac{\vec{J}_\omega x\,\vec{R}\,e^{ikR}}{R^2}dv' = -\frac{i\mu_0}{4\pi}\int_V \frac{\vec{J}_\omega x\,\vec{k}\,e^{ikR}}{R}dv'$$

Los potenciales electromagnéticos

Radiación de sistemas sencillos: el dipolo eléctrico y el dipolo magnético

$$i d\vec{l} = \frac{dq}{dt} d\vec{l}' \rightarrow \frac{d\vec{p}}{dt}$$

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi} \int \frac{[i]\, d\vec{l}'}{R}\, dv' \rightarrow \frac{\mu_0 \left[\dot{\vec{p}}\right]}{4\pi R} = \frac{\mu_0 \left[\dot{\vec{p}}\right]\left(t - R/c\right)}{4\pi R}$$

$$\frac{1}{c^2}\frac{\partial \phi}{\partial t} = -\nabla \vec{A} = -\frac{\mu_0 \nabla \left[\dot{\vec{p}}\right]}{4\pi\ R} + \frac{\mu_0 \left[\dot{\vec{p}}\right]\vec{R}}{4\pi\ R^3} = \frac{\mu_0 \left[\ddot{\vec{p}}\right]\vec{R}}{4\pi\ cR^2} + \frac{\mu_0 \left[\dot{\vec{p}}\right]\vec{R}}{4\pi\ R^3}$$

Los potenciales electromagnéticos

Radiación de sistemas sencillos: el dipolo eléctrico y el dipolo magnético

$$\phi = \frac{[\vec{p}]\vec{R}}{4\varepsilon_0\pi\ R^3} + \frac{\left[\dot{\vec{p}}\right]\vec{R}}{4\varepsilon_0\pi\ cR^2} \qquad \vec{B} = \frac{\mu_0\left[\dot{\vec{p}}\right]x\vec{R}}{4\pi\ R^3} + \frac{\mu_0\left[\ddot{\vec{p}}\right]x\vec{R}}{4\pi\ cR^2}$$

$$\partial_i\left(\frac{[p_j]R_j}{R^n}\right) = \partial_i[p_j]\left(\frac{R_j}{R^n}\right) + [p_j]\frac{\delta_{i,j}}{R^n} - n\ [p_j]\frac{R_i R_j}{R^{n+2}} =$$

$$= \frac{[p_j]}{R^n} - \frac{n([\vec{p}]\vec{R})R_i}{R^{n+2}} - \frac{\left(\left[\dot{\vec{p}}\right]\vec{R}\right)R_i}{cR^{n+1}}$$

Los potenciales electromagnéticos

Radiación de sistemas sencillos: el dipolo eléctrico y el dipolo magnético

$$\nabla\left(\frac{[\vec{p}]\vec{R}}{R^{n}}\right)=[\vec{p}]-\frac{n\left([\vec{p}]\vec{R}\right)\vec{R}}{R^{n+2}}-\frac{\left(\left[\dot{\vec{p}}\right]\vec{R}\right)\vec{R}}{cR^{n+1}}$$

$$-\nabla\phi=\frac{3\left(\left([\vec{p}]\vec{R}\right)\vec{R}-R^{2}[\vec{p}]\right)}{4\pi\,\varepsilon_{0}R^{5}}+\frac{3\left(\left[\dot{\vec{p}}\right]\vec{R}\right)\vec{R}-R^{2}\left[\dot{\vec{p}}\right]}{4\pi\,\varepsilon_{0}cR^{4}}+\frac{\left(\left[\ddot{\vec{p}}\right]\vec{R}\right)\vec{R}}{4\pi\,\varepsilon_{0}c^{2}R^{3}}$$

$$\vec{E}=\frac{3\left([\vec{p}]\vec{R}\right)\vec{R}-R^{2}[\vec{p}]}{4\pi\,\varepsilon_{0}R^{5}}+\frac{3\left(\left[\dot{\vec{p}}\right]\vec{R}\right)\vec{R}-R^{2}\left[\dot{\vec{p}}\right]}{4\pi\,\varepsilon_{0}cR^{4}}+\frac{\left(\left[\ddot{\vec{p}}\right]x\,\vec{R}\right)x\vec{R}}{4\pi\,\varepsilon_{0}c^{2}R^{3}}$$

Los potenciales electromagnéticos

Radiación de sistemas sencillos: el dipolo eléctrico y el dipolo magnético

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\left[\dot{\vec{p}}\right] + \frac{R^2\left[\ddot{\vec{p}}\right]}{c}\right)\frac{\vec{u}x\,\vec{R}}{R^3}$$

$$\vec{E} = \left(\left[\vec{p}\right] + \frac{R\left[\dot{\vec{p}}\right]}{c}\right)\frac{3\left(\vec{u}\,\vec{R}\right)\vec{R} - R^2\vec{u}}{4\pi\ \varepsilon_0 R^5} + \left[\ddot{\vec{p}}\right]\frac{\left(\vec{u}x\,\vec{R}\right)x\vec{R}}{4\pi\ \varepsilon_0 c^2 R^3}$$