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Integral curves of derivations*

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We integrate, by a constructive method, derivations of even degree on the sections of an exterior bundle by families of $\mathbf{Z}_2$-graded algebra automorphisms, dependent on a real parameter, and which satisfy a flow condition. We also study the case of local endomorphisms when their components of degree zero are derivations and with no component of negative degree, but then we have integral families of $\mathbf{R}$-linear automorphisms. This integration method can be applied to the Frölicher—Nijenhuis derivations on the Cartan algebra of differential forms, and to the integration of superfields on graded manifolds.

Introduction

The aim of this paper is to integrate constructively derivations on the algebra of smooth sections of an exterior bundle.

The problem was motivated by our wish of defining a type of variations of differential forms which should be the integrals of the Frölicher—Nijenhuis derivations [3]. In the category of supermanifolds, the corresponding problem is that of the existence of local flows of vector superfields. In [2], Bruzzo and Cianci proved the existence of such flows, but failed in giving a constructive method.

Let $E \to M$ be a real vector bundle with fibre dimension c, and let $\Lambda E \to M$ be the exterior bundle of E. Given an even derivation on $\Gamma(\Lambda E)$, D, we look for a local family of algebra automorphisms of $\Gamma(\Lambda E)$, T_t, such that $T_t' = T_t \circ D$, $T_0 = \mathrm{Id}$. Basically, the integration is made by the composition of an automorphism of $\Gamma(\Lambda E)$ defined by a bundle isomorphism of E and exponentials of homogeneous even derivations on $\Gamma(\Lambda E)$ of degree greater than zero.

With the odd derivations the situation is rather ugly. A family of $\mathbf{Z}_2$-algebra automorphisms cannot integrate an odd derivation. Nevertheless, the previous integration method can be generalized, in a natural way, to localizable endomorphisms of $\Gamma(\Lambda E)$ such as

$$D = D_{(0)} + D_{(1)} + \dots + D_{(q)},$$

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where $D_{(i)}$ is an endomorphism of degree i, and $D_{(0)}$ is a derivation. Note that we exclude the homogeneous component of degree -1. In this case we must relax the conditions for the solution; instead of algebra automorphisms, we must look for **R**-linear automorphisms. Odd derivations with no component of degree -1 are included in this case.

A fundamental property of the integration method, when applied to the algebra of differential forms, is the following: if the endomorphism (or the derivation) commutes with the exterior differential, then the integral family does so.

1. Definitions

Let $\pi: E \to M^n$ be a real vector bundle with $n \geqq 1$ and fibre dimension c, and let $\pi: \Lambda E \to M$ be the exterior bundle of E, where the fibre over $m \in M$ is the vector space $\Lambda E_m = \oplus \Lambda^p E_m = \oplus \underbrace{E_m \wedge \ldots \wedge E_m}_{p}$.

Let $\Gamma(\Lambda E)$ be the exterior **R**-algebra of smooth sections of ΛE (all objects are C^∞). It will be considered as **Z** or $\mathbf{Z}_2$-graded in the usual manner. If $a \in \Gamma(\Lambda E)$, we denote by $a_{(p)}$ the homogeneous component of degree p of a. If a is homogeneous, its degree will be denoted sometimes by $|a|$.

Unless otherwise stated, *linear* will mean **R**-linear.

A linear endomorphism $D: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is said to be *homogeneous of degree* $|D|$ if $|D(a_{(p)})| = p + |D|$. Every linear endomorphism D of $\Gamma(\Lambda E)$ can be uniquely decomposed as $D = D_{(-c)} + \ldots + D_{(0)} + \ldots + D_{(c)}$, where $D_{(i)}$ is a homogeneous endomorphism of degree $i \in \mathbf{Z}$. Adding the terms of the same parity, it can also be uniquely decomposed as $D = D_{(0)} + D_{(1)}$, where $D_{(i)}$ is a homogeneous endomorphism of degree $i \in \mathbf{Z}_2$.

A linear endomorphism D of $\Gamma(\Lambda E)$ is said to be *localizable* if there is a map $\varphi: M \to M$ such that, for each $a \in \Gamma(\Lambda E)$ and $m \in M$, $(Da)(m)$ is determined by the ∞-jet of a at $\varphi(m)$.

A homogeneous linear endomorphism $D: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is called a *homogeneous derivation of degree* $|D|$ if

$$D(ab) = (Da)\,b + (-1)^{|D|\,|a|}a(Db)\,; \quad a, b \in \Gamma(\Lambda E)\,.$$

Note that we usually drop the use of the wedge $\wedge$ to denote the product in the exterior algebra. We fix the following terminology: when we refer to the $\mathbf{Z}_2$-grading, we shall talk about *even* or *odd* objects if they are of degree 0 or 1, respectively. We shall reserve the expression "object of degree . . ." for the **Z**-grading.

Every homogeneous derivation is determined by its action on the elements of degree 0 and degree 1. Thus all derivations of degree less than -1 are zero.

A linear endomorphism $D = D_{(0)} + D_{(1)}$ of $\Gamma(\Lambda E)$ is called a *derivation* if $D_{(0)}(D_{(1)})$ is an even (odd) derivation. Every even derivation $D_{(0)}$ can be uniquely decomposed as $D_{(0)} = H_{(0)} + H_{(2)} + \ldots + H_{(2p)} + \ldots$, where $H_{(2p)}$ is a homogeneous derivation of degree $2p$. Every odd derivation $D_{(1)}$ can be uniquely decomposed as $D_{(1)} = H_{(-1)} + H_{(1)} + \ldots + H_{(2p-1)} + \ldots$, where $H_{(2p-1)}$ is a homogeneous derivation of degree $2p-1$.

Let $\varphi: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ be a graded algebra homomorphism and $D: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ be a homogeneous linear homomorphism such that $D(ab) = (Da)\,\varphi(b) + (-1)^{|D|\,|a|}\,\varphi(a)\,(Db)$. Then we say that D is a *homogeneous φ-derivation*.

Let $TM \to M$ be the tangent bundle over M and $\Gamma(\Lambda E \otimes TM)$ the space of smooth sections of $\Lambda E \otimes TM$. We can define in $\Gamma(\Lambda E \otimes TM)$ a **Z** and a $\mathbf{Z}_2$-grading. Every $K \in \Gamma(\Lambda E \otimes TM)$ can be expressed as a finite sum of decomposable homogeneous sections $k_{(p)} \otimes X$ where $k_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $X \in \Gamma(TM)$.

Let ∇ be a linear connection in E. If $K = k_{(p)} \otimes X$, we define the endomorphism $\nabla_K: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $\nabla_K a = k_{(p)}\,\nabla_X a$, and if $K \in \Gamma(\Lambda E \otimes TM)$, we define ∇_K by linear extension.

∇_K is a derivation and we call it the *proper derivation* associated to K through ∇. If K is a homogeneous element, in whatever grading, then ∇_K is a derivation of degree $|K|$.

Now, we shall define another type of derivations, the algebraic ones.

Let $\pi: E^* \to M$ be the dual bundle of E, and let $\Gamma(\Lambda E \otimes E^*)$ be the space of smooth sections of $\Lambda E \otimes E^*$. We can define in $\Gamma(\Lambda E \otimes E^*)$ a **Z**-grading and a $\mathbf{Z}_2$-grading.

Every $\Phi \in \Gamma(\Lambda E \otimes E^*)$ can be expressed as a sum of decomposable homogeneous sections $b_{(p)} \otimes \alpha$ where $b_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $\alpha \in \Gamma(E^*)$.

If $\Phi = b_{(p)} \otimes \alpha$, we define the endomorphism $i_\Phi: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $i_\Phi a = b_{(p)} i_\alpha a$, where $a \in \Gamma(\Lambda E)$, and where i_α is the interior multiplication; and if $\Phi \in \Gamma(\Lambda E \otimes E^*)$, we define i_Φ by linear extension.

i_Φ is a derivation that acts trivially on the smooth functions on M, and we call it the *algebraic derivation associated to* Φ. If Φ is a homogeneous element, in either grading, then i_Φ is a derivation of degree $|\Phi| - 1$ (modulo 2 if we are dealing with the $\mathbf{Z}_2$-grading).

We have then two special kinds of derivations, the proper and the algebraic ones.

2. Characterization of the derivations on $\Gamma(\Lambda E)$

We will present an analysis of the derivations analogous to that of Frölicher—Nijenhuis (cf. [3]).

Proposition 1. *Let D be a derivation on $\Gamma(\Lambda E)$, and let ∇ be a connection in E. Then, there are unique fields $K \in \Gamma(\Lambda E \otimes TM)$, and $\Phi \in \Gamma(\Lambda E \otimes E^*)$ such that $D = i_\Phi + \nabla_K$. So, D is localizable.*

Proof: Suppose that D is a homogeneous derivation of degree $k \geqq 0$, and let $\alpha^1, \ldots, \alpha^k \in \Gamma(E^*)$ be smooth sections. The map $f \to (Df)(\alpha^1, \ldots, \alpha^k)$, where $f \in C^\infty(M) \subset \Gamma(\Lambda E)$, is a derivation and then it defines a vector field on M, which we denote by $K(\alpha^1, \ldots, \alpha^k)$. The map from $\Gamma(E^*) \times \ldots \times \Gamma(E^*)$ to $\mathfrak{X}(M)$ defined by $(\alpha^1, \ldots, \alpha^k) \to K(\alpha^1, \ldots, \alpha^k)$ is $C^\infty(M)$-linear and skewsymmetric, whence it defines a section $K \in \Gamma(\Lambda E \otimes TM)$ that satisfies $\nabla_K f = Df$ for every $f \in C^\infty(M)$.

Then, the operator $D - \nabla_K$ is a derivation of degree k that acts trivially on $C^\infty(M)$; therefore it is a $C^\infty(M)$-linear endomorphism of $\Gamma(\Lambda E)$ which is determined by its action on the sections of degree 1. Then, if $s \in \Gamma(E)$, the map $s \to (D - \nabla_K) s$ defines a homomorphism from $\Gamma(E)$ into $\Gamma(\Lambda^{k+1}E)$ and therefore there is a section $\Phi \in \Gamma(\Lambda^{k+1}E \otimes E^*)$ such that $(D - \nabla_K) s = i_\Phi s$. The operator i_Φ is a derivation of degree k that acts trivially on $C^\infty(M)$ and as $D - \nabla_K$ on the sections of E. Then $D = \nabla_K + i_\Phi$.

The case of $k = -1$ is trivial. □

3. Characterization of the automorphisms of $\Gamma(\Lambda E)$

This characterization of the automorphisms of $\mathbf{Z}_2$-graded $\mathbf{R}$-algebras of $\Gamma(\Lambda E)$ is based on a similar characterization proposed by A. Uhlmann in [5] for the case of the algebra of differential forms on a manifold.

A bijection $T: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is called a $\mathbf{Z}_2$*-graded* $\mathbf{R}$*-algebra automorphism* (in short, *automorphism of* $\Gamma(\Lambda E)$) if the following properties hold:

1) $T(\alpha a + \beta b) = \alpha T(a) + \beta T(b)$, for each $a, b \in \Gamma(\Lambda E)$ and for each $\alpha, \beta \in \mathbf{R}$;
2) $T(ab) = T(a)\, T(b)$, for each $a, b \in \Gamma(\Lambda E)$;
3) T is compatible with the $\mathbf{Z}_2$-grading.

Consequences:

— $T(0) = 0$ and $T(1) = 1$.

— T sends nilpotent elements into nilpotent elements. As a consequence, $(Ta_{(p)})_{(q)} = 0$ if $q < p$ in the $\mathbf{Z}$-grading.

— The condition 3 is a consequence of the previous two when the dimension of the fibre is even.

— T is determined by its restriction to the sections of degree 0 and 1.

Now we shall see how, associated to every automorphism of $\Gamma(\Lambda E)$, there exists an isomorphism of E that induces a diffeomorphism on M.

Let $\pi_{(i)}$ be the projector of $\Gamma(\Lambda E)$ onto $\Gamma(\Lambda^i E)$.

$\pi_{(0)} \circ T: \Gamma(\Lambda^0 E) \to \Gamma(\Lambda^0 E)$ is an automorphism of $\Gamma(\Lambda^0 E)$. Every automorphism of $\Gamma(\Lambda^0 E) = C^\infty(M)$ maps maximal ideals into maximal ideals. All the maximal ideals of $C^\infty(M)$ are of the form $I_m = (f \in C^\infty(M) \mid f(m) = 0)$. Then $T(I_m) = I_{\varphi(m)}$. Thus, T defines a map $\varphi: M \to M$. In the same way, T^{-1} defines the inverse map of φ.

Moreover, if f is C^∞, then $(\pi_{(0)} \circ T)(f) = f \circ \varphi^{-1}$, and $f \circ \varphi^{-1}$ should be C^∞. Thus, φ is a diffeomorphism.

Lemma 1. *Let* $T: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ *be an automorphism, and let* $a, b \in \Gamma(\Lambda E)$ *be such that* $a|_U = b|_U$ *where* U *is an open subset of* M. *Then, for every* $m \in U$ *we have that* $(Ta)(\varphi(m)) = (Tb)(\varphi(m))$.

Proof. If $m \in U$, let h be a smooth function on M such that $h(m) = 1$ and such that its support is contained in U. The section $hc = h(a - b)$ vanishes identically; then

$$0 = T(hc)(\varphi(m)) = T(h)(\varphi(m))\, T(c)(\varphi(m)) .$$

Equating by increasing degrees step by step and having in mind that $\pi_{(0)}(T(h)(\varphi(m))) = h(m) = 1$, we get easily the result $T(c)(\varphi(m)) = 0$.

It is clear then that if $U \subset M$ is an open subset, we can define consistently an isomorphism, restriction of T to U, $T|_U: \Gamma(\Lambda\pi^{-1}(U)) \to \Gamma(\Lambda\pi^{-1}(\varphi U))$, where φ is the diffeomorphism of M associated to T.

Now, let us suppose that a and b are two homogeneous sections of degree 1 such that $a(m) = b(m)$. We shall prove that $(\pi_{(1)} \circ T)(a)(\varphi(m)) = (\pi_{(1)} \circ T)(b)(\varphi(m))$. The previous results justify the local technique of the following proof. Let $(e_1, \dots, e_c)$ be a local frame of E on $\pi^{-1}(U)$, where U is an open neigbourhood of m, such that $a(m) = b(m) = pe_1(m)$, with $p \in \mathbf{R}$. Let $a|_U = f_1 e_1 + \dots + f_c e_c$, where f_i are functions on U, such that $f_i(m) = p\delta_{1i}$. Then

$$(\pi_{(1)} \circ T)(a)(\varphi(m)) = \sum_{i=0}^{c} f_i \circ \varphi^{-1}(\varphi(m))\, (\pi_{(1)} \circ T)(e_i)(\varphi(m))$$
$$= p(\pi_{(1)} \circ T)(e_1)(\varphi(m)) = (\pi_{(1)} \circ T)(b)(\varphi(m)) .$$

If $e \in E_m$, we can define $\phi(e) \in E_{\varphi(m)}$ by $(\pi_{(1)} \circ T)(a)(\varphi(m))$, where a is any section of degree 1 such that $a(m) = e$. Then, we have a bundle isomorphism $\phi: E \to E$, π-related with the diffeomorphism φ.

Reciprocally, every bundle isomorphism $\phi: E \to E$, π-related with a diffeomorphism $\varphi: M \to M$, defines an automorphism $\phi^{\sim}: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by means of $\phi^{\sim}(a) = \phi \circ a \circ \varphi^{-1}$ for sections of degree 1, $\phi^{\sim}(f) = (f \circ \varphi^{-1})$ for functions, and the obvious extension to $\Gamma(\Lambda E)$.

We put $S = T \circ (\phi^{-1})^{\sim}$; then S is such that $(\pi_{(0)} \circ S)(f) = f$, for every function and $(\pi_{(1)} \circ S)(a_{(1)}) = a_{(1)}$, for every section of degree 1. The image of a homogeneous element of degree p can thus be written as

$$S(a_{(p)}) = a_{(p)} + \pi_{(p+2)}(S(a_{(p)})) + \pi_{(p+4)}(S(a_{(p)})) + \dots$$

Let us suppose that S is an automorphism of $\Gamma(\Lambda E)$ and k a natural number greater than 0 such that for every homogeneous element $a_{(p)}$ we have

$$S(a_{(p)}) = a_{(p)} + \pi_{(p+2k)}(S(a_{(p)})) + \pi_{(p+2k+2)}(S(a_{(p)})) + \dots$$

Let $D: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ be the operator defined by

$$Da = \sum_{p=0}^{c} (\pi_{(p+2k)} \circ S)(a_{(p)}) .$$

An easy computation shows that D is a homogeneous derivation of degree $2k$.

Proposition 1 assures that, given the connection ∇, there exist a unique $K = K_{(2k)} \in \Gamma(\Lambda^{2k}E \otimes TM)$ and a unique $\Phi = \Phi_{(2k+1)} \in \Gamma(\Lambda^{2k+1}E \otimes E^*)$ such that $D = i_\Phi + \nabla_K$.

Let $B = \exp(D) = \mathrm{Id} + D + (1/2)DD + \dots$ (this series is finite). Then, $\exp(D)$ is an automorphism of $\Gamma(\Lambda E)$ and we put $S_1 = S \circ B^{-1}$. S_1 is also an automorphism and $S_1(a_{(p)}) = a_{(p)} + \pi_{(p+2k+2)}(S_1(a_{(p)})) + \dots$ By iteration we reach at last the following result.

Theorem 1. *Let* $T: \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ *be an automorphism, and let* ∇ *be a connection in the bundle* $\pi: E \to M$, *then there exist:*

a) *a bundle isomorphism* $\phi: E \to E$,

b) *a section* $K_{(2k)} \in \Gamma(\Lambda^{2k}E \otimes TM)$, *for each* $k > 0$, *and*

c) *a section* $\Phi_{(2k+1)} \in \Gamma(\Lambda^{2k+1}E \otimes E^*)$, *for each* $k > 0$,

such that $T = B_{(r)} \circ \ldots \circ B_{(2)} \circ B_{(1)} \circ \phi^{\sim}$ *where* $\phi^{\sim}$ *is the automorphism induced by* ϕ,

$$B_{(k)} = \exp\,(i_{\Phi_{(2k+1)}} + \nabla_{K_{(2k)}}),$$

and r is the integral part of $c/2$.

In the decomposition $T = B_{(r)} \circ \ldots \circ B_{(2)} \circ B_{(1)} \circ \phi^{\sim}$ the factors are uniquely determined.

4. Characterization of the automorphisms of $A(M)$

We shall study the particular case when E is the cotangent bundle, i.e. $\Gamma(\Lambda E) = A(M)$, the algebra of differential forms on M. We will apply the above results, but also we will deduce new ones that are peculiar to $A(M)$. For instance, the existence of the exterior differential, d, allows us to define another class of automorphisms, those commuting with d.

Lemma 2. (*of extension*). *Let* $T_0: A_0(M) \to A(M)$ *be a* $\mathbf{Z}_2$*-graded* $\mathbf{R}$*-algebra monomorphism such that* $\pi_{(0)} \circ T_0: A_0(M) \to A_0(M)$ *is an isomorphism. Then there exists a unique automorphism* $T: A(M) \to A(M)$ *that agrees with* T_0 *on* $A_0(M)$ *and commutes with* d.

Proof. $\pi_{(0)} \circ T_0$ determines a diffeomorphism $\varphi: M \to M$ such that $(\pi_{(0)} \circ T_0)(f) = f \circ \varphi^{-1}$. Let $\varphi^*: A(M) \to A(M)$ be the pull-back of φ and let us consider the composition $\varphi^* \circ T_0: A_0(M) \to A(M)$. We have that $(\pi_{(0)} \circ \varphi^* \circ T_0)(f) = (\varphi^* \circ \pi_{(0)} \circ T_0)(f) = \varphi^*(f \circ \varphi^{-1}) = f \circ \varphi^{-1} \circ \varphi = f$, whenever $f \in C^\infty(M)$. Then $(\varphi^* \circ T_0)(f) = f + D_{(2)}f + \ldots$ and thus $D_{(2)}: A_0(M) \to A(M)$ is a derivation. There is a unique $K \in \Gamma(\Lambda^2 T^*M \otimes TM)$ such that we can write $D_{(2)} = i_K \circ \mathrm{d}$. We extend $D_{(2)}$ to $A(M)$ by putting $D_{(2)} = i_K \circ \mathrm{d} + \mathrm{d} \circ i_K$ and we build the automorphism $B_{(1)} = \exp\,(D_{(2)})$ which commutes with d. Now we consider $B_{(1)}^{-1} \circ \varphi^* \circ T_0$ and iterate the process until we obtain, necessarily, the identity.

Then $T_0 = B_{(r)} \circ \ldots \circ B_{(1)} \circ (\varphi^{-1})^*|_{A_0(M)}$ and thus $T = B_{(r)} \circ \ldots \circ B_{(1)} \circ (\varphi^{-1})^*$ is the desired automorphism. The uniqueness is obvious.

We have done almost all the necessary work to state and prove a theorem which is the "integral" version of Frölicher–Nijenhuis' characterization of the derivations on $A(M)$.

Theorem 2. *Let* $T: A(M) \to A(M)$ *be an automorphism. Then T can be uniquely decomposed as* $T = T_d \circ T_c$, *where* T_d *is an automorphism commuting with* d *and where* T_c *is an automorphism that acts as the identity on* $C^\infty(M)$.

Proof. Let T_0 be the monomorphism obtained by restriction of T to the subalgebra $A_0(M)$. Applying the extension lemma, Lemma 2, we get an automorphism $T_d: A(M) \to A(M)$ such that $T_d \circ \mathrm{d} = \mathrm{d} \circ T_d$, and T_d acts as T_0 on $C^\infty(M)$.

Now, we put, $T_c = T_d^{-1} \circ T$. T_c is an automorphism that acts as the identity on $C^\infty(M)$. Thus, we have $T = T_d \circ T_c$.

Thus, every automorphism of $A(M)$ that commutes with d can be uniquely written as the composition of the pull-back of a diffeomorphism of M with exponentials of homogeneous even derivations that commute with d.

Now, let us apply, with a slight modification, the previous results to these automorphisms.

Corollary 1. *Let* $T: A(M) \to A(M)$ *be an automorphism; then there exist:*

a) *a diffeomorphism* $\varphi: M \to M$,

b) *a bundle isomorphism* T^*M *which induces the identity on M and which can be represented by an element* $\Phi_{(1)} \in \Gamma(T^*M \otimes TM)$ *such that* $\det \Phi_{(1)} \neq 0$ *everywhere*,

b) *a unique section* $K_{(2k)} \in \Gamma(\Lambda^{2k}T^*M \otimes TM)$, *for each* $k > 0$, *and*

c) *a unique section* $\Phi_{(2k+1)} \in \Gamma(\Lambda^{2k+1}T^*M \otimes TM)$, *for each* $k > 0$,

such that $T = B_{(r)} \circ \ldots \circ B_{(2)} \circ B_{(1)} \circ \Phi_{(1)} \circ \varphi^*$, *where* φ^* *is the pull-back of* φ, $\Phi_{(1)}$ *is the extension of* $\Phi_{(1)}$ *to* ΛT^*M,

$$B_{(k)} = \exp\,(i_{\Phi_{(2k+1)}} + \mathscr{L}_{K_{(2k)}}),$$

and r is the integral part of $n/2$.

Applying Theorem 2, we can also factorize both T_d and T_c and get another factorization of T.

5. Integration of derivations

A well-known example can help us to understand the technical reasons that give rise to the complicated definition of curve of automorphisms of $\Gamma(\Lambda E)$ which we shall use.

Let $X \in \mathfrak{X}(M)$ be a vector field not necessarily complete. It is well-known that $\mathscr{L}_X$ is a derivation on $A(M)$ of degree zero. The problem of finding a family $\phi_t^{\sim}$ of automorphisms of $A(M)$ such that

$$\left.\frac{\partial \varphi_t^{\sim}(s)}{\partial t}\right|_{t_0} = \varphi_{t_0}^{\sim}(\mathscr{L}_X s),$$

i.e., the problem of integrating the derivation $\mathscr{L}_X$, has the obvious solution $\varphi_t^{\sim} = \varphi_t^*$, the pull-back of the flow of X, φ_t. Nevertheless, this affirmation is technically inaccurate, because if X is not complete then there is no $t \neq 0$ such that φ_t is defined in all points of M.

All that we know is the existence, for $t \neq 0$, of a maximal open set $\mathscr{D}_t$ such that $\varphi_t: \mathscr{D}_t \to \mathscr{D}_{-t}$ is a diffeomorphism, and we also know the remaining properties of $\mathscr{D}_t$ and their relations with φ_t and X (cf. [6]).

Definition. Let $\pi: E \to M$ be a vector bundle. A *curve of (local) linear automorphisms of* $\Gamma(\Lambda E)$ is a one-parameter family of $\mathbf{R}$-*linear* isomorphisms $T_t: \Gamma(\Lambda\pi^{-1}(\mathscr{D}_{-t})) \to \Gamma(\Lambda\pi^{-1}(\mathscr{D}_t))$, $t \in \mathbf{R}$, where $\mathscr{D}_t$ is an open subset of M, with the following conditions:

a) $\mathscr{D}_0 = M$, $T_0 = \mathrm{Id}$, $\bigcup_{t>0} \mathscr{D}_t = \bigcup_{t<0} \mathscr{D}_t = M$, $\mathscr{D}_{t_1} \subset \mathscr{D}_{t_2}$ if $t_1 \geqq t_2 \geqq 0$, or if $t_1 \leqq t_2 \leqq 0$, and $\{t \in \mathbf{R} \mid m \in \mathscr{D}_t\}$ is an open interval for every $m \in M$;

b) if $a \in \Gamma(\Lambda E)$, let us write $T_t(a) = T_t(a|_{\mathscr{D}_{-t}})$; then, if $m \in M$, the curve $t \to (T_t a)(m) \in \Lambda E_m$, for t such that $m \in \mathscr{D}_t$, is smooth; and the map $m \in \mathscr{D}_{t_0} \to (\partial/\partial t)|_{t_0} (T_t a)(m)$ belongs to $\Gamma(\Lambda\pi^{-1}(\mathscr{D}_{t_0}))$.

A curve of linear automorphisms of $\Gamma(\Lambda E)$ is called a *curve of* $\mathbf{Z}_2$*-graded algebra automorphisms*, briefly, *curve of automorphisms*, if, for every $t \in \mathbf{R}$, T_t is a $\mathbf{Z}_2$-graded $\mathbf{R}$-algebra isomorphism.

Let us denote by $T'_{t_0}: \Gamma(\Lambda\pi^{-1}(\mathscr{D}_{-t_0})) \to \Gamma(\Lambda\pi^{-1}(\mathscr{D}_{t_0}))$ the operator given by $(T'_{t_0} a)(m) = (\partial/\partial t)|_{t_0} (T_t a)(m)$.

It is easy to check that when T_t is a curve of $\mathbf{Z}_2$-graded algebra automorphisms, then T'_{t_0} is an even T_{t_0}-derivation, i.e. $T'_{t_0}(ab) = (T'_{t_0} a)\, T_{t_0} b + (T_{t_0} a)\, T'_{t_0} b$. Therefore, we cannot obtain the odd derivations as derivatives of curves of algebra automorphisms. Since our wish is to include in the same scheme even and odd derivations, we have chosen curves of $\mathbf{R}$-linear automorphisms instead.

Now the question is this: given a derivation D on $\Gamma(\Lambda E)$, does a curve of linear automorphisms T_t exist such that $T'_t = T_t \circ D$?

When we try to answer this question it will be made clear, owing to the method, the following facts:

1°) The integration process that we shall describe demands that the component of D of degree -1 must vanish. This makes no matter in the case of even derivations because they lack of that component.

2°) We can generalize the question to localizable linear endomorphisms of $\Gamma(\Lambda E)$ instead of derivations (see Section 2), provided that they can be expressed as a sum of homogeneous endomorphisms of degree greater than or equal to zero, and that the part of degree zero is a derivation.

Theorem 3. *Let D be a localizable linear endomorphism of $\Gamma(\Lambda E)$, such that*

$$D = D_{(0)} + D_{(1)} + \dots + D_{(p)} + \dots,$$

with $D_{(0)}$ a derivation. Then there exists a unique curve of linear automorphisms of $\Gamma(\Lambda E)$, T_t, such that $T_0 = \mathrm{Id}$, $T'_t = T_t \circ D$.

Proof. 1°) *Integration of the part of degree zero.*
From Proposition 1, given the connection ∇ in E, there exist $K \in \Gamma(\Lambda^0 E \otimes TM)$ and $\Phi \in \Gamma(\Lambda^1 E \otimes E^*)$ such that $D_{(0)} = i_\Phi \pm \nabla_K$. Note that K is a vector field on M; so let $\varphi_t: \mathscr{D}_t \to \mathscr{D}_{-t}$ be the flow of K.

We can define a π-projectable vector field on E by $Y(e) = K^H(e) - \iota_e(\Phi(e))$, $e \in E$, where K^H is the horizontal lift of K defined by ∇, and where ι_e is the vertical injection, i.e. $\iota_e(\Phi(e))$ is the tangent vector at $t = 0$ to the curve $t \to e + t\Phi(e) \in E_{\pi(e)}$.

This vector field defines a flow on E, ϕ_t. Its maximal domain is $\pi^{-1}(\mathscr{D}_t)$ and we have $\pi \circ \phi_t = \varphi_t \circ \pi$. Moreover, if $m \in \mathscr{D}_t$, ϕ_t defines a vector space isomorphism from E_m to $E_{\varphi_t(m)}$. Using local expressions we can see that the proof of this affirmation is identical to the usual one for the existence of the parallel displacement. Then, we can define the associated isomorphisms, $\phi_t^{\sim}$, which satisfy

$$\phi_t^{\sim} a = \phi_{-t} \circ a \circ \varphi_t, \tag{1}$$

where ϕ_t is built by extension of ϕ_t from E_m to ΛE_m. Thus, if $a \in \Gamma(\Lambda\pi^{-1}(\mathscr{D}_{-t}))$, $\phi_t^{\sim} a \in \Gamma(\Lambda\pi^{-1}(\mathscr{D}_t))$. Moreover, if a is a section of degree 1, the tangent vector at $t = 0$ to the curve

$$t \to \phi_t^{\sim}(a)(m) = \phi_{-t}(a(\varphi_t(m))),$$

is $-Y(a(m)) + a_*(K_m)$, and by definition of Y, this is equal to

$$a_*(K_m) - K^H(a(m)) + \iota_{a(m)}(\Phi(a(m))).$$

Now, applying the Proposition IX, page 336 of [4], that tangent vector is equal to

$$\iota_{a(m)}(\nabla_K a + i_\Phi a)(m) = \iota_{a(m)}(D_{(0)} a)(m).$$

Thus, it is clear that $\phi_t^{\sim}$ is a curve of automorphisms with associated domains $\mathscr{D}_t$, and that for sections on $\mathscr{D}_{-t}$ we have $(\mathrm{d}/\mathrm{d}t)(\phi_t^{\sim}) = \phi_t^{\sim} \circ D_{(0)} = D_{(0)} \circ \phi_t^{\sim}$. Moreover, $\phi_t^{\sim}$ is a curve of $\mathbf{Z}_2$-graded $\mathbf{R}$-algebra automorphisms, and not just $\mathbf{R}$-linear automorphisms.

2°) *Reduction of the equation*
From now on, every time we take the value at t of an operator, it will be understood that its domain is restricted to the sections on $\mathscr{D}_t$ or $\mathscr{D}_{-t}$, according to the case.

Let $S_t = T_t \circ \phi_{-t}^{\sim}$. This new curve of linear automorphisms should satisfy the equation

$$S'_t = T_t \circ D \circ \phi_{-t}^{\sim} - T_t \circ D_{(0)} \circ \phi_{-t}^{\sim} = S_t \circ (\phi_t^{\sim} \circ (D - D_{(0)}) \circ \phi_{-t}^{\sim}).$$

Let us write $H_t = \phi_t^{\sim} \circ (D - D_{(0)}) \circ \phi_{-t}^{\sim}$. H_t is an endomorphism, dependent on the real parameter t, which can be written as a sum of homogeneous endomorphisms of degree greater than or equal to 1. Using (1) and having in mind that $D - D_{(0)}$ is localizable, we see that H_t is localizable. From now on, we will not bother about the endomorphisms being localizable, because, in general, the proof is immediate; we shall briefly call them endomorphisms.

3°) *Induction*
Let us suppose that we have the equation $S'_t = S_t \circ D_t$, where D_t is an endomorphism, dependent on t, which can be written as a sum of homogeneous endomorphisms of degree greater than or equal to $k > 0$:

$$D_t = (D_t)_{(k)} + (D_t)_{(k+1)} + \dots + (D_t)_{(k+p)} + \dots$$

Let L_t be the unique homogeneous endomorphism of degree k such that

$$(\mathrm{d}/\mathrm{d}t)|_{t=t_0} (L_t a)_m = ((D_t)_{(k)} a)_m, \quad (L_0 a)_m = 0; \tag{2}$$

for all $a \in \Gamma(\Lambda E)$ and all $m \in M$. This equation can be solved by integration between zero and t.

Let $B_t = \exp(L_t)$, which is well defined because L_t is a nilpotent operator. Define $U_t = S_t \circ B_t^{-1}$. U_t should satisfy the equation

$$U_t' = S_t \circ D_t \circ B_t^{-1} + S_t \circ (B_t^{-1})' = U_t \circ B_t \circ (D_t + (B_t^{-1})' \circ B_t) \circ B_t^{-1}.$$

Let $H_t = B_t \circ (D_t + (B_t^{-1})' \circ B_t) \circ B_t^{-1}$. We will see that H_t is an endomorphism, dependent on t, which can be written as a sum of homogeneous endomorphisms of degree greater than k.

Since $B_t^{-1} = \exp(-L_t)$, then

$$(B_t^{-1})' = -(D_t)_{(k)} + (1/2)((D_t)_{(k)} L_t + L_t(D_t)_{(k)}) + \dots,$$

therefore

$$(B_t^{-1})' \circ B_t = (B_t^{-1})' \circ \exp(L_t) = -(D_t)_{(k)} + \dots,$$

where the dots represent endomorphisms of $\Gamma(\Lambda E)$ of degree strictly greater than k.

Then,

$$D_t + (B_t^{-1})' \circ B_t = \{(D_t)_{(k)} + (D_t)_{(k+1)} + \dots\} + \{-(D_t)_{(k)} + \dots\} = (D_t)_{(k+1)} + \dots$$

Neither $B_t = \exp(L_t)$ nor $B_t^{-1} = \exp(-L_t)$ have homogeneous terms of negative degree, therefore the conjugation by B_t of a homogeneous endomorphism of degree $k + i$ is an endomorphism which can be written as a sum of homogeneous endomorphisms of degree greater than or equal to $k + i$.

4°) *End of the process*

We are limited by the fibre dimension. Thus, by induction we shall arrive to $H_t = 0$. Undoing the changes, we have that

$$T_t = (B_{(r)})_t \circ \dots \circ (B_{(2)})_t \circ (B_{(1)})_t \circ \phi_t^{\sim},$$

where ϕ_t is the flow of the vector field associated to $D_{(0)}$, and $(B_{(i)})_t$ are the exponentials of the homogeneous endomorphisms of degree i, that have been obtained along the process.

By construction, T_t is the solution of the equation $T_t' = T_t \circ D$, and we have $T_0 = \mathrm{Id}$.

5°) *Uniqueness*

Let T_t and H_t be two curves of linear automorphisms of $\Gamma(\Lambda E)$ such that

$$T_t' = T_t \circ D, \quad T_0 = \mathrm{Id}, \quad \text{and} \quad H_t' = H_t \circ D, \quad H_0 = \mathrm{Id}.$$

From the equality $H_t^{-1} \circ H_t = \mathrm{Id}$ we deduce that $(H_t^{-1})' = -D \circ H_t^{-1}$, and thus $(T_t \circ H_t^{-1})' \equiv 0$. Therefore, $T_t \circ H_t^{-1}$ does not depend on t and since $T_0 \circ H_0^{-1} = \mathrm{Id}$, we have $T_t = H_t$.

Fix $s \in \mathbf{R}$ and put $G_t = T_s \circ T_t$ and $F_t = T_{s+t}$. G_t and F_t are solutions of the same equation, $T_t' = T_t \circ D$, with the same initial condition, T_s, for $t = 0$. By uniqueness, $T_s \circ T_t = T_{s+t}$.

If we are interested in arbitrary initial conditions like $T_0 = R$, where R is a linear automorphism of $\Gamma(\Lambda E)$, then, making the change $S_t = R^{-1} \circ T_t$ in the equation $T_t' = T_t \circ D$, we have that S_t is the solution of $S_t' = S_t \circ D$, with the initial condition $S_0 = \mathrm{Id}$.

The localizable endomorphism D commutes with its integral curve. In fact, let us suppose that $T_s \circ T_t = T_{s+t} = T_t \circ T_s$ and $T_0' = D$. Then, $T_s' = (d/dt)|_{t=0}\, T_{s+t} = T_s \circ D = D \circ T_s$.

With even derivations the result can be refined.

Theorem 4. *Let D be an even derivation on $\Gamma(\Lambda E)$. Then there exists a unique curve of automorphisms of the $\mathbf{Z}_2$-graded $\mathbf{R}$-algebra $\Gamma(\Lambda E)$, T_t, such that $T_0 = \mathrm{Id}$, $T_t' = T_t \circ D$.*

Proof. The difference lies in the third step. We will do it again. Let us suppose that the equation is $S_t' = S_t \circ D_t$, where D_t is an even derivation that can be written as a sum of homogeneous derivations of even degree greater than or equal to $2k$:

$$D_t = (D_t)_{(2k)} + (D_t)_{(2k+2)} + \dots + (D_t)_{(2k+2p)} + \dots$$

As before, by Proposition 1, there exist $K_t \in \Gamma(\Lambda^{2k} E \otimes TM)$ and $\Phi_t \in \Gamma(\Lambda^{2k+1} E \otimes E^*)$ such that $(D_t)_{(2k)} = \nabla_{K_t} + i_{\Phi_t}$.

Let $L_t \in \Gamma(\Lambda^{2k} E \otimes TM)$ and $\Psi_t \in \Gamma(\Lambda^{2k+1} E \otimes E^*)$ be such that

$$(d/dt)|_{t=t_0} L_t = K_t, \quad L_0 = 0; \qquad (2a)$$

$$(d/dt)|_{t=t_0} \Psi_t = \Phi_t, \quad \Psi_0 = 0. \qquad (2b)$$

Let B_t be the curve of automorphism $\exp(\nabla_{L_t} + i_{\Psi_t})$, and define $U_t = S_t \circ B_t^{-1}$. U_t must satisfy the equation:

$$U_t' = U_t \circ B_t \circ (D_t + (B_t^{-1})' \circ B_t) \circ B_t^{-1}.$$

Let $H_t = B_t \circ (D_t + (B_t^{-1})' \circ B_t) \circ B_t^{-1}$. The derivative of the automorphism B_t^{-1} is an even B_t^{-1}-derivation; then $(B_t^{-1})' \circ B_t$ is an even derivation. So is H_t, because it is the conjugation by the automorphism B_t of the even derivation $D_t + (B_t^{-1})' \circ B_t$. As in Theorem 3, the homogeneous components of H_t are of even degree greater than $2k$, and the process is repeated. Thus, the curve T_t factories as $(B_{(2s)})_t \circ \dots \circ (B_{(2)})_t \circ \phi_t^{\sim}$, where each factor is a $\mathbf{Z}_2$-graded algebra automorphism. Therefore T_t is a curve of automorphisms, as claimed.

In the particular case of $A(M)$, if D is an even derivation that commutes with the exterior differential, then, along the process of integration we always obtain automorphisms of the type T_d. Therefore, we can state the following

Corollary 2. *Let D be an even derivation on $A(M)$ which commutes with the exterior differential; then there exists a unique curve of automorphisms of $A(M)$, T_t, such that $T_0 = \mathrm{Id}$, $T_t' = T_t \circ D$, $T_t \circ d = d \circ T_t$.*

The preceding results are applicable to the integration of even vector superfields in graded manifolds, via Batchelor's Theorem [1]. For instance, when we apply our technique to the example shown in [2], we reach the same integral supercurve as is easily checked.

6. Example

In order to make the integration process clearer, we will work out an example. Let M be $\mathbf{R}^2$ and let $\{x, y\}$ be the Cartesian coordinate functions. We will integrate the localizable linear endomorphism of $A(\mathbf{R}^2)$, $D = D_{(0)} + D_{(1)} = (i_{\Phi_{(1)}} + \mathscr{L}_{K_{(0)}}) + (\mathscr{L}_{K_{(1)}})$, where $K_{(0)} = \partial/\partial x$, $\Phi_{(1)} = \mathrm{d}x \otimes \partial/\partial y$ and $K_{(1)} = x\,\mathrm{d}y \otimes \partial/\partial x$.

Let ∇ be the linear connection which, in the canonical frame $\{\partial/\partial x, \partial/\partial y\}$, has coefficients $\Gamma^i_{jk} \equiv 0$. Then, let us consider the vector field Y on $T^*\mathbf{R}^2$ given by

$$Y_e = (\partial/\partial x)^H_e - \iota_e(\Phi_{(1)}(e)), \quad e \in T^*\mathbf{R}^2 .$$

Let $(x, y; u, v)$ be coordinates on $T^*\mathbf{R}^2$, defined as follows: if $e = p\,\mathrm{d}_m x + q\,\mathrm{d}_m y \in T^*_m\mathbf{R}^2$, then $x(e) = x(m)$, $y(e) = y(m)$, $u(e) = p$, $v(e) = q$. Thus, the vector field Y is equal to $\partial/\partial x - v(\partial/\partial u)$, and its flow is given by

$$\phi_t(x, y, p, q) = (x + t, y, -qt + p, q) .$$

The integral curve of automorphisms of $D_{(0)}$ is $\phi_t^\sim(s) = \phi_{-t} \circ s \circ \varphi_t$, where ϕ_t is the extension of ϕ_t to $A(\mathbf{R}^2)$. Here φ_t is the flow of $\partial/\partial x$, i.e. the diffeomorphism defined by $(x, y) \to (x + t, y)$. Thus we have $\phi_t^\sim(f) = f \circ \varphi_t$, $\phi_t^\sim(\mathrm{d}x) = \mathrm{d}x$, and $\phi_t^\sim(\mathrm{d}y) = \mathrm{d}y + t\,\mathrm{d}x$.

The second step of Theorem 3 consists of computing the conjugation of $D - D_{(0)} = D_{(1)}$ by $\phi_t^\sim$, $\phi_t^\sim \circ D_{(1)} \circ \phi_{-t}^\sim$. We put $P_t' = \phi_t^\sim \circ D_{(1)} \circ \phi_{-t}^\sim$. Immediately we check that

$$P_t'(f + a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}x \wedge \mathrm{d}y) = (x + t) f_x(\mathrm{d}y + t\,\mathrm{d}x) + ((x + t)(tb_x - a_x) + (tb - a))\,\mathrm{d}x \wedge \mathrm{d}y ,$$

where $f_x = \partial f/\partial x$.

Note that, in this particular case, P_t' can also be written as

$$(i_{\Psi_{(2)}} + \mathscr{L}_{L_{(1)}}) ,$$

where $L_{(1)} = [(x + t)(\mathrm{d}y + t\,\mathrm{d}x)] \otimes \partial/\partial x$, and $\Psi_{(2)} = t\,\mathrm{d}x \wedge \mathrm{d}y \otimes \partial/\partial x$. So, in spite of the fact that $D_{(1)}$ has no algebraic part, its conjugate by $\phi_t^\sim$ does.

Integrating P_t' with respect to t, according to equation (2), or (2a) and (2b), we obtain the endomorphism P_t defined by

$$\begin{aligned} &P_t(f + a\,\mathrm{d}x + b\,\mathrm{d}y + c\,\mathrm{d}x \wedge \mathrm{d}y) \\ &= (tx + (t^2/2)) f_x\,\mathrm{d}y + ((t^2/2)\,x + (t^3/3)) f_x\,\mathrm{d}x \\ &\quad + (((t^2/2)\,x + (t^3/3))\,b_x - (tx + (t^2/2))\,a_x + (t^2/2)\,b - ta)\,\mathrm{d}x \wedge \mathrm{d}y . \end{aligned}$$

Since $P_t \circ P_t = 0$, then $(B_{(1)})_t = \mathrm{Id} + P_t$.

The third step of the construction consists in computing the expression

$$(B_{(1)})_t \circ [P_t' + (B_{(1)}^{-1})_t' \circ (B_{(1)})_t] \circ (B_{(1)}^{-1})_t .$$

A simple calculation shows that it is equal to the endomorphism $-P_t' \circ P_t$, and we will denote it by H_t'. This endomorphism acts trivially on sections of degree greater than 0, and acts on functions by the formula:

$$H_t'(f) = -\left[(x + t)\left(\frac{t^2}{2}\,x + \frac{t^3}{6}\right) f_{xx} + \left(t^2 x + \frac{2}{3}\,t^3\right) f_x\right] \mathrm{d}x \wedge \mathrm{d}y .$$

The equation we must now solve is $(B_{(2)})_t' = (B_{(2)})_t \circ H'$, and we get the $\mathbf{R}$-linear automorphism $(B_{(2)})_t = \exp(H_t) = \mathrm{Id} + H_t$, where H_t is the endomorphism defined by the equations (2),

$$H_t(f) = -\left[\left(\frac{t^3}{6}\,x^2 + \frac{t^4}{6}\,x + \frac{t^5}{30}\right) f_{xx} + \left(\frac{t^3}{3}\,x + \frac{t^4}{6}\right) f_x\right] \mathrm{d}x \wedge \mathrm{d}y .$$

Note that, although initially D has no homogeneous component of degree 2, when we reduce the equation there appears an endomorphism, H_t, of degree 2. This is characteristic of the integration process.

Thus, the integral curve of $\mathbf{R}$-linear automorphisms of the endomorphism D is

$$T_t = (B_{(2)})_t \circ (B_{(1)})_t \circ \phi_t^\sim = (\mathrm{Id} + H_t) \circ (\mathrm{Id} + P_t) \circ \phi_t^\sim .$$

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