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Poisson-Nijenhuis Structures and the Vinogradov Bracket

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Abstract: We express the compatibility conditions that a Poisson bivector and a Nijenhuis tensor must fulfil in order to be a Poisson-Nijenhuis structure by means of a graded Lie bracket. This bracket is a generalization of Schouten and Frölicher-Nijenhuis graded Lie brackets defined on multivector fields and on vector valued differential forms respectively.

Key words: *Schouten-Nijenhuis bracket, Frölicher-Nijenhuis bracket, graded Lie algebras, bihamiltonian manifolds*
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1. Introduction

A Poisson-Nijenhuis structure on a differentiable manifold is a pair formed by a Poisson bivector and a Nijenhuis tensor that satisfy certain compatibility conditions. Such kind of structures has been studied in [5] and they have its origin in previous works by Magri in the theory of completely integrable Hamiltonian systems. The condition that a bivector must fulfil in order to be a Poisson bivector can be expressed by means of a suitable graded Lie bracket: the Schouten-Nijenhuis bracket. We have the same situation with the Nijenhuis tensor: there is a graded Lie bracket, the Frölicher-Nijenhuis bracket, that allows us to write the condition that a vector valued differential 1-form must fulfil in order to be a Nijenhuis tensor (see [9] and [10] for their definitions and properties).

In this paper, we express the compatibility conditions between both tensor fields, the Poisson bivector and the Nijenhuis tensor, by means of a graded Lie bracket defined by A.M. Vinogradov ([3], [11]). This graded Lie bracket is not defined on tensor fields but on graded differential operators on the algebra of differential forms, and in a certain sense, it is a generalization of the other two.

Let us denote by P and N the Poisson bivector and the Nijenhuis tensor, and let i_P and i_N denote their associated tensorial graded differential operators. We show that the compatibility conditions are precisely the vanishing conditions of the higher order part of the Vinogradov bracket of i_P and i_N. Moreover, we characterize the Poisson-Nijenhuis structures for which the bracket vanishes. In fact, we show that this bracket vanishes if and only if the trace of the recursion operator, i.e., the Nijenhuis tensor N, is a Casimir function for the Poisson bivector P.

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Both i_P and the bracket of i_P and i_N are graded differential operators of order 2. So, a first step is to obtain a decomposition of the graded differential operators of order 2 and degree -1 or -2 analogous to that of Frölicher and Nijenhuis for the derivations on the algebra of differential forms. This is done in Section 3. The application of this decomposition to the Poisson-Nijenhuis structures is presented in Section 4. In Section 5, we present a characterization of Poisson-Nijenhuis structures just by a single condition on the commutator of the sum of the differentials associated to the Poisson bivector and to the Nijenhuis tensor with itself. This allows to define a differential bicomplex for Poisson-Nijenhuis manifolds. Finally we apply this approach to bihamiltonian manifolds and give new proofs, following these techniques, of some already known facts.

2. Poisson-Nijenhuis Structures

Let M be an n-dimensional differentiable manifold. Let $\mathcal{D}_*(M) = \sum_{i=0}^n \mathcal{D}_i(M)$ be the exterior algebra of multivector fields on M. The Schouten-Nijenhuis bracket, denoted by $[\ ,\ ]_{SN}$, gives a graded Lie algebra structure on $\mathcal{D}_*(M)$.

A bivector P is called a *Poisson bivector* on M if $[P,P]_{SN} = 0$. The pair (M,P), where P is a Poisson bivector, is called a *Poisson manifold*. Let us denote by $\mathbf{P}$ the linear mapping from T^*M to TM, defined by P as follows:

$$\alpha(\mathbf{P}\beta) = P(\alpha,\beta) \text{ for all } \alpha,\beta \in T^*M.$$

Let $\Omega(M) = \sum_{i=0}^n \Omega^i(M)$ be the exterior algebra of differential forms on M and let $\Omega(M;TM) = \sum_{i=0}^n \Omega^i(M;TM)$ be the space of vector valued forms endowed with the Frölicher-Nijenhuis bracket, denoted by $[\ ,\ ]_{FN}$.

Let N be a vector valued 1-form. We shall denote by $\mathbf{N}$ and $\mathbf{N}^*$ the linear mappings defined by N from TM to itself, and from T^*M to itself, respectively. A vector valued 1-form, N, is called a *Nijenhuis tensor* if $[N,N]_{FN} = 0$.

The following tensor fields defined by means of a bivector P and a vector valued 1-form N will be needed for the definition of the notion of Poisson-Nijenhuis structure:

$$\begin{aligned} NP(\alpha,\beta) &= \alpha(\mathbf{NP}\beta) = P(\mathbf{N}^*\alpha,\beta) \\ PN(\alpha,\beta) &= \alpha(\mathbf{PN}^*\beta) = P(\alpha,\mathbf{N}^*\beta) \end{aligned}$$

When these two tensor fields PN and NP are equal, then $PN = NP$ is a bivector and the following map

$$\begin{aligned} C(P,N)(\alpha,\beta) = {} & \mathcal{L}_{\mathbf{P}\alpha}(\mathbf{N}^*\beta) - \mathcal{L}_{\mathbf{P}\beta}(\mathbf{N}^*\alpha) + \mathbf{N}^*\mathcal{L}_{\mathbf{P}\beta}(\alpha) \\ & -\mathbf{N}^*\mathcal{L}_{\mathbf{P}\alpha}(\beta) + dP(\mathbf{N}^*\alpha,\beta) - \mathbf{N}^*dP(\alpha,\beta) \end{aligned}$$

defines a 1-form valued bivector, $C(P,N)$.

Definition 1. ([5]) A *Poisson-Nijenhuis structure* on M is a pair (P,N), where P is a Poisson bivector and N is a Nijenhuis tensor, such that the following compatibility conditions hold

(i) NP is a bivector.
(ii) The 1-form-valued bivector $C(P,N)$ vanishes.

It was shown in [5] that when these two conditions hold, the tensor field NP is a Poisson bivector and $[P,NP]_{SN} = 0$.

3. Decompositions of Differential Operators of Low Order

The aim of this section is to obtain a decomposition of differential operators of low order in terms of basic operators in a way analogous to that of Frölicher-Nijenhuis, [4], for the derivations of the algebra of differential forms.

When dealing with graded objects, we will denote by $|\alpha|$ the degree of a homogeneous element α. In order to simplify the notations related to signs, we adopt the notation that the symbol of a graded object used as exponent of (-1) denotes the degree of that object, mod 2. Unless otherwise stated, *linear* will mean $\mathbb{R}$-linear.

A linear operator $D : \Omega(M) \longrightarrow \Omega(M)$ is said to be *of degree* r if $D(\alpha) \in \Omega^{|\alpha|+r}(M)$, for all $\alpha \in \Omega^{|\alpha|}(M)$.

Let F, G be two graded maps of $\Omega(M)$ into itself. The graded commutator $[F,G]$ of the two graded maps F, G is defined by $[F,G] = F \circ G - (-1)^{FG} G \circ F$.

A differential form $\alpha \in \Omega^{|\alpha|}(M)$ can be interpreted as the graded map of degree $|\alpha|$, $\beta \mapsto \alpha \wedge \beta$, for all $\beta \in \Omega(M)$.

Let $\mathit{Diff}^r(M)$ be the set of all differential operators of degree r, and let $\mathit{Diff}(M)$ be the graded algebra of all differential operators acting on $\Omega(M)$, endowed with the bracket defined by the graded commutator.

A linear graded operator $D : \Omega(M) \longrightarrow \Omega(M)$ is said to be *differential of order* $\leq k$ if, $[\ldots[[D,\alpha_0],\alpha_1]\ldots,\alpha_k] = 0$, for all $\alpha_0,\alpha_1,\ldots,\alpha_k \in \Omega(M)$.

Let us denote by $\mathcal{D}_k^r$ the set of all differential operators of order $\leq k$ and degree r. The following properties are satisfied ([6]),

$$\mathcal{D}_r^p \mathcal{D}_s^q \subset \mathcal{D}_{r+s}^{p+q} \quad ; \quad [\mathcal{D}_r^p, \mathcal{D}_s^q] \subset \mathcal{D}_{r+s-1}^{p+q}.$$

A graded operator D is called *tensorial* if, $[D,f](\alpha) = D(f\alpha) - fD(\alpha) = 0$, for all $f \in \Omega^0(M)$ and $\alpha \in \Omega(M)$.

The differential operators are local operators, i.e., if $\alpha = \beta$ in a neighborhood of $p \in M$, then $(D\alpha)(p) = (D\beta)(p)$. This fact implies that any differential operator is totally determined by its action on differential forms of degree less than or equal to its order. If D is of order k, then, for each decomposable form α of degree $\geq k+1$, $D(\alpha)$ can be expressed in terms of D acting on products of ℓ forms of degree 1, where $\ell \leq k$. This result follows by writing explicitly the definition of the order of a differential operator with α_i the factors of the decomposable form (see [6]).

Let us mention some operators that will be needed later,

(i) A differential form $\alpha \in \Omega(M)$, as an operator, $\beta \mapsto \alpha \wedge \beta$, $\beta \in \Omega(M)$ is the unique differential operator of order zero, and degree $|\alpha|$ on $\Omega(M)$.
(ii) If $\mathcal{Z} = X_1 \wedge \ldots \wedge X_k$, then we define the differential operator $i_{\mathcal{Z}}$ by

$$i_{\mathcal{Z}}(\beta) = i_{X_1}(\ldots i_{X_k}(\beta)\ldots),\ \beta \in \Omega(M).$$

If $\mathcal{Z}$ is a k-multivector field, then we define $i_{\mathcal{Z}}$ by its linear extension. The differential operator $i_{\mathcal{Z}} \in \mathcal{D}_k^{-k}$. Moreover, every differential operator of this type is of the form $i_{\mathcal{Z}}$ with $\mathcal{Z}$ a k-multivector field.

(iii) If $K = \alpha \otimes Z$ with $\alpha \in \Omega^{|\alpha|}(M)$ and $Z \in \mathcal{D}_k(M)$ we define the differential operator $i_K \in \mathcal{D}_k^{|\alpha|-k}$, by

$$i_K(\beta) = (\alpha \otimes Z)(\beta) = \alpha \wedge i_Z(\beta)$$

If $K \in \Gamma(\Lambda T^*M \otimes \Lambda TM)$, we define i_K by its linear extension.

(iv) The exterior derivative, d, is a differential operator of order 1 and degree 1. If $K \in \Gamma(\Lambda T^*M \otimes \Lambda TM)$, we denote by $\mathcal{L}_K$ the commutator $[i_K, \mathrm{d}]$.

3.1. Decomposition of Operators of Order 1 and Degree p

It is easy to verify that

$$\mathcal{D}_1^p = Der_p\Omega(M) \oplus \mathcal{D}_0^p$$

where $Der_p\Omega(M)$ is the Lie algebra of derivations of degree p on $\Omega(M)$ and the splitting, is given by

$$D = (D - D(1)) + D(1) \in Der_p\Omega(M) \oplus \mathcal{D}_0^p$$

where 1 denotes the constant function $1(m) = 1$ for all $m \in M$. Therefore,

$$Der_p\Omega(M) = \{D \in \mathcal{D}_1^p \text{ with } D(1) = 0\}$$

i.e., derivations are exactly differential operators of order 1 vanishing on constants.

It is well known, [4], that any derivation $D \in Der_p\Omega(M)$ can be uniquely written as the sum of a derivation that commutes with the exterior derivative and a tensorial derivation,

$$D = \mathcal{L}_Q + i_L, \text{ for } Q \in \Omega^p(M;TM)\ ,\ L \in \Omega^{p+1}(M;TM)$$

3.2. Decomposition of Operators of Order 2 and Degree -1

In this subsection we will present an analysis of these operators to obtain a decomposition of them as a sum of basic operators of the same type defined by means of some fields. As usual we shall start with the class of tensorial operators.

Lemma 3.1. *Let $D \in \mathcal{D}_2^{-1}$ be a tensorial operator, then there are unique fields $X \in \mathcal{X}(M)$ and $C \in \Gamma(\Lambda^2TM \otimes T^*M)$ such that $D = i_X + i_C$.*

Proof. The map $D|_{\Omega^1(M)} : \Omega^1(M) \longrightarrow C^\infty(M)$ is $C^\infty(M)$-linear because D is tensorial, then it defines a vector field X. Now, the operator $D - i_X$ is a tensorial operator of order 2 and degree -1 that acts trivially on $\Omega^0(M)$ and on $\Omega^1(M)$. The map,

$$\begin{array}{rcl} \bar{C} : \Omega^1(M) \times \Omega^1(M) \times \mathcal{X}(M) & \longrightarrow & C^\infty(M) \\ (\alpha, \beta, Y) & \hookrightarrow & \bar{C}(\alpha, \beta, Y) = ((D - i_X)(\alpha \wedge \beta))(Y) \end{array}$$

is $C^\infty(M)$-linear, thus it defines a $(2,1)$ tensor field C that satisfies $i_C|_{\Omega^2(M)} = D - i_X|_{\Omega^2(M)}$. Therefore, the operators i_C and $D - i_X$ agree when acting on $\Omega(M)$, because they agree on $\Omega^1(M)$ and $\Omega^2(M)$. □

Now, we will study the first class of nontensorial basic operators given by operators defined by means of a bivector. Let P be a bivector, and let i_P be the operator of order 2 and degree -2 defined by P. The commutator of the exterior derivative with the differential operator i_P,

$$\mathcal{L}_P = [i_P, \mathrm{d}]$$

is an example of a nontensorial operator of order 2 and degree -1. Nevertheless, not all operators of order 2 and degree -1 can be written as a sum of a tensorial operator and an operator of the type $\mathcal{L}_P$ with P a bivector. It is necessary to introduce a new basic operator. To do this, we need to use a Riemannian metric. Let g be a Riemannian metric on M, then g determines an isomorphism, that we will also denote by g, between TM and T^*M by $(g(X))(Y) = g(X,Y)$. We call grad$f$ the vector field obtained by $g^{-1}(\mathrm{d}f)$, where g^{-1} is the inverse of the isomorphism g.

The codifferential δ associated to the Riemannian metric, see e.g. [12], is an operator of order 2 and degree -1. This operator will allow us to define a new class of basic operators.

Let K be a vector valued 1-form, then the operator $i_K \in \mathcal{D}_1^0$ is a derivation of degree 0. Let us denote by δ_K the commutator $[i_K, \delta]$, which is an operator of order 2 and degree -1. We can now state the following

Proposition 3.2. *Let g be a Riemannian metric on M and $D \in \mathcal{D}_2^{-1}$; then, there are unique fields $U \in \Gamma(\Lambda^2TM), K \in \Gamma(TM \otimes T^*M), X \in \mathcal{X}(M), C \in \Gamma(\Lambda^2TM \otimes T^*M)$ such that*

$$D = \mathcal{L}_U + \delta_K + i_X + i_C.$$

Proof. First, we are going to get the nontensorial part of the differential operator. Let $f \in C^\infty(M)$, then the operator $[D, f]$ is a differential operator of order 1 and degree -1, so it is a derivation of degree -1, and there exists a vector field H_f such that $[D, f] = i_{H_f}$. Note that the tensorial part of D does not appear in $[D, f]$.

Now, we shall prove that the mapping $H : C^\infty(M) \longrightarrow \mathcal{X}(M), f \mapsto H_f$ where H_f is the unique vector field such that $[D, f] = i_{H_f}$, is a derivation. Indeed,

$$\begin{aligned} i_{H_{fg}}(\alpha) &= [D, fg](\alpha) = D(fg\alpha) - fgD(\alpha) = g[D, f](\alpha) + f[D, g](\alpha) \\ &= (gi_{H_f} + fi_{H_g})(\alpha) \end{aligned}$$

where the following identity for operators of order 2, obtained from the condition $[[[D, f], g], \alpha] = 0$, has been used $D(fg\alpha) = fD(g\alpha) + gD(f\alpha) - fgD(\alpha)$.

We can associate a tensor field $Q \in \Gamma(TM \otimes TM)$ to every derivation from $C^\infty(M)$ into $\mathcal{X}(M)$ in the following way:

Given $\alpha \in \Omega^1(M)$, we define the mapping $T_\alpha : C^\infty(M) \longrightarrow C^\infty(M)$ by $T_\alpha(f) = \alpha(H_f)$. Thus, T_α is a derivation on $C^\infty(M)$ because H_f is, therefore T_α is a vector field. Now, the mapping, $T : \Omega^1(M) \longrightarrow \mathcal{X}(M)$ defined by $T(\alpha) = T_\alpha$, is $C^\infty(M)$-linear, so it determines a tensor field $Q \in \Gamma(TM \otimes TM)$, defined by $Q(\alpha, \beta) = \alpha(T_\beta)$.

We will split the tensor Q into its skew-symmetric and symmetric parts, $Q = Q_a + Q_s$, where $Q_a \in \Gamma(\Lambda^2TM)$ and $Q_s \in \Gamma(S^2TM)$. Now, let us take as the bivector U the skew-symmetric part of Q, Q_a; let us define the tensor field $K \in \Gamma(TM \otimes T^*M)$ by $\alpha(K(X)) = Q_s(g(X), \alpha)$, for all $X \in \mathcal{X}(M)$ and all $\alpha \in \Omega^1(M)$.

Next we show that the operator $D - \mathcal{L}_U - \delta_K \in \mathcal{D}_2^{-1}$ is tensorial. Let f and h be two differentiable functions, then

$$\begin{aligned}[D - \mathcal{L}_U - \delta_K, f](dh) &= [D, f](dh) - [\mathcal{L}_U, f](dh) - [\delta_K, f](dh) \\ &= i_{H_f}(dh) + i_U(df \wedge dh) - i_{\text{grad}f}(K dh) = 0\end{aligned}$$

because $i_{H_f}(dh) = Q(df, dh)$, $i_U(df \wedge dh) = Q_a(df, dh)$ and, by the definition of the tensor field K, we have that $i_{\text{grad}f}(K dh) = Q_s(df, dh)$.

Now, the result follows by application of lemma 3.1 to the tensorial operator $D - \mathcal{L}_U - \delta_K$. □

Remark. This decomposition depends on the Riemannian metric g. The introduction of this tensor, initially external to the problem, has been necessary in order to obtain the decomposition of all operators of degree -1 and order 2. This metric is not needed for the operators studied in the next section, but we have introduced it for the sake of completeness.

4. The Vinogradov Bracket of a Poisson Bivector and a Nijenhuis Tensor

It is well known that a Poisson bivector, P, is a bivector such that the Schouten-Nijenhuis bracket, $[P, P]_{SN}$ vanishes and a Nijenhuis tensor, N, is a vector valued differential 1-form such that the Frölicher-Nijenhuis bracket, $[N, N]_{FN}$ vanishes, or equivalently, the Nijenhuis torsion of N is equal to zero. In this section we will first recall the notion of what we have called Vinogradov bracket, and second we will use it to obtain a characterization of the Poisson-Nijenhuis structures. In particular, we will show that the vanishing of the higher order part of the Vinogradov bracket of the operators defined by a Poisson tensor, P and a Nijenhuis tensor, N, gives the compatibility conditions that (P, N) must satisfy in order to define a Poisson-Nijenhuis structure.

Whereas the Schouten and Frölicher-Nijenhuis brackets are defined on certain kinds of tensor fields, the Vinogradov bracket is defined on differential operators. However, when the operators are those defined by means of suitable tensor fields, then the Vinogradov bracket is related with the previous brackets (see [3], [11] for details).

Let F be a differential operator on $\Omega(M)$. The *Lievization* of F is defined by $L_F = F \circ d - (-1)^F d \circ F = [F, d]$.

Definition 2. Let F, G be two differential operators on $\Omega(M)$. The L-commutator of F, G is defined as

$$[F, G]_* = [L_F, G] + L_\Delta, \text{ where } \Delta = -\frac{1}{2}(-1)^G[F, G].$$

Let $\mathcal{N}(M) = \frac{Diff(M)}{L(Diff(M))}$ with the grading $\mathcal{N}^k(M) = \frac{Diff^k(M)}{L(Diff^{k-1}(M))}$.

Theorem 4.1. (Vinogradov unification theorem) *The quotient $\mathcal{N}(M)$ equipped with the bracket operation, $[\ ,\ ]_V$, induced from the L-commutator is a graded Lie algebra, and the compositions:*

$$\mathcal{D}_*(M) \hookrightarrow Diff(M) \to \mathcal{N}(M) \quad ; \quad \Omega(M; TM) \hookrightarrow Diff(M) \to \mathcal{N}(M)$$

are imbeddings of graded Lie algebras, considering $\mathcal{D}_*(M)$ equipped with the Schouten-Nijenhuis bracket, and $\Omega(M; TM)$ with the Frölicher-Nijenhuis bracket.

So, in a certain sense, the Vinogradov bracket is a generalization of both graded Lie brackets.

Now, let us consider the following situation: Let P be a Poisson tensor, and let $i_P \in \mathcal{D}_2^{-2}$ be the operator that it defines. Analogously, let N be a Nijenhuis tensor, and consider $i_N \in \mathcal{D}_1^0$. In order to simplify notations let us recall that the Lievization of i_K, $L_{i_K} = [i_K, d]$, with $K \in \Gamma(\Lambda T^*M \otimes \Lambda TM)$ has been previously denoted by $\mathcal{L}_K$.

The relation between the Vinogradov bracket and the Poisson-Nijenhuis structures is given by the next theorem.

Theorem 4.2. *Let P be a Poisson bivector and let N be a Nijenhuis tensor,*

(i) *The tensor fields PN and NP coincide if and only if $[i_N, i_P]_* = i_X + i_C$. In this case the vector field X is given by $X(f) = P(dN^*df)$, where f is a smooth function, and the tensor field C is nothing but $C(P, N)$, defined in Section 2.*

(ii) *The pair (P, N) is a Poisson-Nijenhuis structure (i.e., $NP = PN$, and $C(P, N) = 0$) if and only if $[i_N, i_P]_* \in \mathcal{D}_1^{-1}$, i.e., it is an insertion operator, i_X, for some vector field X. In this case $X = i_{-\frac{1}{2}P(d\,\mathrm{tr}\,N)}$.*

Proof. The operator $[i_N, i_P]_*$ is an operator of order 2 and degree -1. Let us decompose it as a sum of basic operators as in Proposition 3.2. To do this, we first must compute the tensor field Q that determines the nontensorial part. Let f and h be differentiable functions, then, by the definition of Q,

$$Q(df, dh) = [[i_N, i_P]_*, f](dh) = -\frac{1}{2}(P(N^*df, dh) - P(df, N^*dh))$$

This tensor field is symmetric, thus, the operator $[i_N, i_P]_*$ can be written, once a Riemannian metric on the manifold is given, in the form $[i_N, i_P]_* = \delta_K + i_X + i_C$, where $K \in \Gamma(TM \otimes T^*M)$ is the tensor field defined by $\alpha(K(Y)) = Q_s(g(Y), \alpha)$, for all $Y \in \mathcal{X}(M)$ and all $\alpha \in \Omega^1(M)$.

(i) Obviously $NP = PN$ if and only if the symmetric tensor field Q vanishes. Thus, in this case, the operator $[i_N, i_P]_*$ is just a tensorial operator now independent of the Riemannian metric used to decompose it. We are going to determine this tensorial operator. An easy computation shows that $[i_N, i_P]_*(df) = P(dN^*df)$ for all $f \in C^\infty(M)$, then the vector field X is determined by $X(f) = P(dN^*df)$. Now, $D = [i_N, i_P]_* - i_X$ is a tensorial operator without part of order 1. To get the part of order 2, let us compute its action on differential 2-forms. Computations show that

$$\begin{aligned}D(df \wedge dh) &= ([\mathcal{L}_N, i_P] - \frac{1}{2}L_{[i_N, i_P]})(df \wedge dh) - X(f)\,dh + X(h)\,df \\ &= N^*dP(df, dh) + i_P(dN^*df \wedge dh) - i_P(df \wedge N^*dh) \\ &\quad -dP(df, N^*dh) - P(dN^*df)dh + P(dN^*dh)df \\ &= N^*dP(df, dh) - i_{P_{dh}}(dN^*df) + i_{P_{df}}(dN^*dh) - dP(df, N^*dh)\end{aligned}$$

and this expression agrees with $i_{C(P,N)}(df \wedge dh)$.

(ii) Now, by the second compatibility condition of the Poisson-Nijenhuis structure, the tensor field $C(P,N)$ vanishes, and we have that

$$[i_N, i_P]_\bullet = i_X \in \mathcal{D}_1^{-1}.$$

Conversely, if the operator $[i_N, i_P]_\bullet$ is an operator of order 1 and degree -1, i.e., it is of the type i_X, where X is a vector field, then the tensor fields K and C must be equal to zero. $K = 0$ implies $Q = 0$, which is equivalent to the first compatibility condition. By (i), the tensor field C is equal to $C(P,N)$. Thus, (P,N) is a Poisson-Nijenhuis structure.

Let us determine the vector field X in this case. Let $m \in M$ such that there exists a neighborhood, V, of m where the rank of P is constant and equal to $2p$. Let $\{x^i\}_{i=1}^{2p}$, $2p \leq n$, be local adapted coordinates for the Poisson bivector P. Let us compute the trace of the linear mapping $\mathbf{C}(\alpha)$, $\alpha \in \Omega^1(M)$, defined by $(\mathbf{C}(\alpha))(\beta, Y) = (C(P,N)(\alpha,\beta))(Y)$, for $\beta \in \Omega^1(M)$ and $Y \in \mathcal{X}(M)$. For $\alpha = \mathrm{d}x^j$ we have that, after some computations,

$$\begin{aligned}
\operatorname{tr}\mathbf{C}(\mathrm{d}x^j) &= \sum_{i=1}^{2p}(C(P,N)(\mathrm{d}x^j, \mathrm{d}x^i))(\frac{\partial}{\partial x^i}) \\
&= \sum_{i=1}^{2p}\mathcal{L}_{\mathbf{P}\mathrm{d}x^j}((\mathbf{N}^*\mathrm{d}x^i)(\frac{\partial}{\partial x^i})) - (\mathrm{d}\mathbf{N}^*\mathrm{d}x^j)(\mathbf{P}\mathrm{d}x^i, \frac{\partial}{\partial x^i}) \\
&= P(\mathrm{d}\operatorname{tr}\mathbf{N}, \mathrm{d}x^j) - 2\mathrm{d}x^j(X)
\end{aligned}$$

By linearity, for all $\alpha \in \Omega^1(M)$, we have that $\operatorname{tr}\mathbf{C}(\alpha) = -2\alpha(X) - P(\alpha, \mathrm{d}\operatorname{tr}\mathbf{N})$. Now, as (P,N) is a Poisson-Nijenhuis structure we have that $C(P,N) = 0$, and then $X = -\frac{1}{2}\mathbf{P}(\mathrm{d}\operatorname{tr}\mathbf{N})$ in the coordinate open set where the adapted coordinates are defined.

The subset $W = \{m \in M$ such that there exists V neighborhood of m where the rank of P is constant$\}$ is dense. By continuity, we have that $X = -\frac{1}{2}\mathbf{P}(\mathrm{d}\operatorname{tr}\mathbf{N})$ in M. □

Remark. Theorem 4.2 shows that a Poisson-Nijenhuis structure is a pair (P,N) such that $[i_P, i_P]_V = 0$, $[i_N, i_N]_V = 0$ and such that the operator $[i_N, i_P]_V$, which is a priori of order 2, actually is of order 1, but not necessarily equal to zero. Note that $[i_N, i_P]_\bullet$ has no Lievization term (see proof of Theorem 4.2), thus the condition "$[i_N, i_P]_\bullet$ is of order 1" is equivalent to "$[i_N, i_P]_V$ is of order 1".

Next, we will characterize the Poisson-Nijenhuis structures such that not only the bracket $[i_N, i_P]_V$ is an operator of order 1, but is actually zero.

Remark. In particular, $[i_N, i_P]_\bullet = 0$ if and only if the trace of N is a Casimir function for the Poisson tensor P. If, in addition, P is everywhere nondegenerated, then $[i_N, i_P]_\bullet = 0$ if and only if the trace of N is a constant function.

An example of Poisson-Nijenhuis structure such that $[i_N, i_P]_\bullet = 0$ with trace of N a nonconstant Casimir function is the following:

Example. Let $\{x^1, x^2, y^1, y^2\}$ be a system of local coordinates on $\mathbb{R}^4$. Let us consider the Poisson bivector, $P = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2}$ and the Nijenhuis tensor $N = f(y^1)\,\mathrm{d}y^1 \otimes \frac{\partial}{\partial y^1}$, where f is a nonconstant smooth function. It is easy to prove that the pair (P,N) is a Poisson-Nijenhuis structure and that the trace of the Nijenhuis tensor is a nonconstant Casimir function.

Some examples of Poisson-Nijenhuis structures with $[i_N, i_P]_\bullet$ not equal to zero can be found at the end of Section 6.

5. A Differential Bicomplex for Poisson-Nijenhuis Manifolds

It is well known that given a Nijenhuis tensor, N, it is possible to define a cochain complex on the module of differential forms where its differential is the derivation of degree 1, $\mathcal{L}_N : \Omega^k(M) \to \Omega^{k+1}(M)$. It is also well known, see [2], [6], that given a Poisson bivector, P, it is possible to define a chain complex on the module of differential forms, which is called the canonical complex associated to the Poisson structure, where its differential is the operator of degree -1, $\mathcal{L}_P : \Omega^k(M) \to \Omega^{k-1}(M)$.

In the previous sections, we have studied Poisson-Nijenhuis structures in terms of the Vinogradov bracket. Our aim in this section is to study them in terms of the two differentials $\mathcal{L}_N$ and $\mathcal{L}_P$.

First let us recall the following property of the Vinogradov bracket.

Proposition 5.1. ([3]) *Given F and G graded differential operators, then*

$$\mathcal{L}_{[F,G]_\bullet} = [\mathcal{L}_F, \mathcal{L}_G].$$

An easy application of this proposition gives the following alternative characterization of Poisson-Nijenhuis structures.

Proposition 5.2. *Let P be a Poisson bivector and let N be a Nijenhuis tensor, then (P,N) is a Poisson-Nijenhuis structure if and only if $[\mathcal{L}_N, \mathcal{L}_P] \in \mathcal{D}_1^0$, i.e., it is a Lie derivative with respect to a vector field X, $\mathcal{L}_X$. In this case $X = -\frac{1}{2}\mathbf{P}(\mathrm{d}\operatorname{tr}\mathbf{N})$.*

Remark. The bracket $[\mathcal{L}_N, \mathcal{L}_P]$ is just the graded commutator of the two differentials, this is, $\mathcal{L}_N \circ \mathcal{L}_P + \mathcal{L}_P \circ \mathcal{L}_N$.

Proof. By Proposition 5.1, we have $[\mathcal{L}_N, \mathcal{L}_P] = [[i_N, i_P]_\bullet, \mathrm{d}]$. If (P,N) is a Poisson-Nijenhuis structure then, by Theorem 4.2, $[i_N, i_P]_\bullet \in \mathcal{D}_1^{-1}$ and then $[\mathcal{L}_N, \mathcal{L}_P] \in \mathcal{D}_1^0$.

Conversely, let us suppose that $[\mathcal{L}_N, \mathcal{L}_P] \in \mathcal{D}_1^0$. Let us recall that $[i_N, i_P]_\bullet = \delta_K + i_C + i_X$ for some tensor fields K, C, X as in Proposition 3.2. Therefore, the operator $[\delta_K + i_C, \mathrm{d}]$, which is a priori of order 2, must be of order 1. Now, we will prove that this condition implies that the tensor fields K and C must vanish.

The technique is similar to the proof of Theorem 4.2, first, we reduce the order of the operator $[\delta_K + i_C, \mathrm{d}]$ by taking its commutator with a differentiable function. Once we have an operator of order 1, we will apply the decomposition of Subsection 3.1.

Let f be a differentiable function and consider $[[\delta_K + i_C, \mathrm{d}], f]$. By hypothesis, it belongs to $\mathcal{D}_0^0$, this is, it is just the multiplication by a smooth function. This function is determined by the action of the operator on the constant function 1, and then the operator $[[\delta_K + i_C, \mathrm{d}], f] - [[\delta_K + i_C, \mathrm{d}], f](1)$ is equal to zero.

But, initially this operator belongs to $\mathcal{D}_1^0$. Then, if we compute the action of $[[\delta_K + i_C, \mathrm{d}], f] - [[\delta_K + i_C, \mathrm{d}], f](1)$ on another smooth function h, we have

$$\begin{aligned} 0 &= ([[\delta_K + i_C, \mathrm{d}], f] - [[\delta_K + i_C, \mathrm{d}], f](1))(h) \\ &= \delta_K(h\mathrm{d}f) + \delta_K(f\mathrm{d}h) - f\delta_K(\mathrm{d}h) - h\delta_K(\mathrm{d}f) \\ &= -\delta(hK\mathrm{d}f) - \delta(fK\mathrm{d}h) + f\delta(K\mathrm{d}h) + h\delta(K\mathrm{d}f) \\ &= g^{-1}(\mathrm{d}f, K\mathrm{d}h) + g^{-1}(\mathrm{d}h, K\mathrm{d}f) \end{aligned}$$

where the following property of the divergence operator, $\delta(f\mathrm{d}h) = f\delta(\mathrm{d}h) - g^{-1}(\mathrm{d}f, \mathrm{d}h)$, has been used. This implies that the tensor field K must be skew-symmetric, but, by definition, it is symmetric. (Recall that $K(Y;\alpha) = Q_s(g(Y), \alpha)$ where $Q_s \in \Gamma(S^2TM)$). Thus, the tensor K vanishes.

Now, if we compute the action of the same operator on an exact 1-form, $\mathrm{d}h$, we have

$$0 = ([[i_C, \mathrm{d}], f] - [[i_C, \mathrm{d}], f](1))(\mathrm{d}h) = i_C(\mathrm{d}f \wedge \mathrm{d}h) = C(\mathrm{d}f, \mathrm{d}h;).$$

Therefore, the tensor field C must be equal to zero.

Consequently, $[i_N, i_P]_\bullet = i_X$ for some vector field and then, by Theorem 4.2, (P, N) is a Poisson-Nijenhuis structure. □

The next step is to glue together, in a single condition, the four conditions that a vector valued differential 1-form, N, and a bivector field, P, must fulfil in order to be a Poisson-Nijenhuis structure. This can be done by studying the sum of the two operators $\mathcal{L}_N$ and $\mathcal{L}_P$.

Theorem 5.3. *Let N be a vector valued differential 1-form and let P be a bivector field, then (P, N) is a Poisson-Nijenhuis structure if and only if $[\mathcal{L}_N + \mathcal{L}_P, \mathcal{L}_N + \mathcal{L}_P] \in \mathcal{D}_1^0$, i.e., it is a Lie derivative with respect to a vector field X, $\mathcal{L}_X$. In this case $X = -\frac{1}{2}\mathbf{P}(\mathrm{d}\,\mathrm{tr}\,\mathbf{N})$.*

Proof. First, let us compute

$$\begin{aligned} [\mathcal{L}_N + \mathcal{L}_P, \mathcal{L}_N + \mathcal{L}_P] &= [\mathcal{L}_N, \mathcal{L}_N] + [\mathcal{L}_P, \mathcal{L}_P] + 2[\mathcal{L}_N, \mathcal{L}_P] \\ &= \mathcal{L}_{[N,N]_{FN}} - \mathcal{L}_{[P,P]_{SN}} + 2[\mathcal{L}_N, \mathcal{L}_P]. \end{aligned}$$

We get the desired result as a consequence of Proposition 5.2, the definition and properties of the Schouten-Nijenhuis and the Frölicher-Nijenhuis brackets and the fact that the operators $\mathcal{L}_{[N,N]_{FN}}$ and $\mathcal{L}_{[P,P]_{SN}}$ belongs to $\mathcal{D}_1^2$ and to $\mathcal{D}_3^{-2}$, respectively. □

Let us express this result in algebraic terms. It is natural in our context to try to introduce a double complex like in [2]. In fact, the trivial case of Poisson-Nijenhuis manifold where the Nijenhuis tensor is given by the identity map of the tangent bundle, $\mathbf{N} = \mathbf{Id}$, is the case studied in [2]. To do that, let us define $\mathcal{C}^{k,\ell}(M) = \Omega^{\ell-k}(M)$ for all $k, \ell \in \mathbb{Z}$. Given a vector valued differential 1-form, N, and a bivector field, P, let us consider the horizontal operator $\mathcal{L}_N : \mathcal{C}^{k,\ell}(M) \to \mathcal{C}^{k,\ell+1}(M)$ and the vertical operator $\mathcal{L}_P : \mathcal{C}^{k,\ell}(M) \to \mathcal{C}^{k+1,\ell}(M)$.

With this data, only a subclass of Poisson-Nijenhuis manifolds will provide us with a differential bicomplex.

Corollary 5.4. *Let N be a vector valued differential 1-form and let P be a bivector field, then $(\mathcal{C}^{\bullet,\bullet}(M), \mathcal{L}_N, \mathcal{L}_P)$ is a differential bicomplex if and only if (P, N) is a Poisson-Nijenhuis manifold and $\mathbf{P}(\mathrm{d}\,\mathrm{tr}\,\mathbf{N}) = 0$.*

Proof. The definition and properties of the Frölicher-Nijenhuis and the Schouten-Nijenhuis brackets give us the following relations,

$$\mathcal{L}_N \circ \mathcal{L}_N = \frac{1}{2}[\mathcal{L}_N, \mathcal{L}_N] = \frac{1}{2}\mathcal{L}_{[N,N]_{FN}}, \quad \mathcal{L}_P \circ \mathcal{L}_P = \frac{1}{2}[\mathcal{L}_P, \mathcal{L}_P] = -\frac{1}{2}\mathcal{L}_{[P,P]_{SN}}$$

and $\mathcal{L}_N \circ \mathcal{L}_P + \mathcal{L}_P \circ \mathcal{L}_N = [\mathcal{L}_N, \mathcal{L}_P]$.

Obviously, if (P, N) is a Poisson-Nijenhuis structure and $\mathbf{P}(\mathrm{d}\,\mathrm{tr}\,\mathbf{N}) = 0$ then $(\mathcal{C}^{\bullet,\bullet}(M), \mathcal{L}_N, \mathcal{L}_P)$ is a differential bicomplex.

Conversely, if $(\mathcal{C}^{\bullet,\bullet}(M), \mathcal{L}_N, \mathcal{L}_P)$ is a differential bicomplex then $[\mathcal{L}_N, \mathcal{L}_N] = 0$, i.e., N is a Nijenhuis tensor, $[\mathcal{L}_P, \mathcal{L}_P] = 0$, i.e., P is a Poisson tensor and $[\mathcal{L}_N, \mathcal{L}_P] = 0$, i.e. (P, N) is a Poisson-Nijenhuis structure with $\mathbf{P}(\mathrm{dtr}\mathbf{N}) = 0$. □

In order to associate a differential bicomplex to any Poisson-Nijenhuis manifold, it is necessary to reduce the algebra $\Omega(M)$ in the following way: Let (P, N) be a Poisson-Nijenhuis structure with $X = -\frac{1}{2}\mathbf{P}(\mathrm{d}\,\mathrm{tr}\,\mathbf{N})$. Let $\Omega_X(M)$ be the algebra of differential forms invariant by X, i.e., $\alpha \in \Omega_X(M)$ if and only if $\mathcal{L}_X\alpha = 0$, and let $\mathcal{C}_X^{k,\ell}(M) = \Omega_X^{\ell-k}(M)$ for all $k, \ell \in \mathbb{Z}$.

Corollary 5.5. *Let (P, N) be a Poisson-Nijenhuis structure, then the differentials $\mathcal{L}_N$ and $\mathcal{L}_P$ can be restricted to $\Omega_X(M)$, and $(\mathcal{C}_X^{\bullet,\bullet}(M), \mathcal{L}_N, \mathcal{L}_P)$ is a differential bicomplex.*

Proof. First let us note that, by application of the graded Jacobi identity, we have

$$[[\mathcal{L}_N, \mathcal{L}_P], \mathcal{L}_N] = 0, \qquad [[\mathcal{L}_N, \mathcal{L}_P], \mathcal{L}_P] = 0.$$

Now, let $\alpha \in \Omega_X^k(M)$, we are going to show that $\mathcal{L}_N\alpha \in \Omega_X^{k+1}(M)$ and $\mathcal{L}_P\alpha \in \Omega_X^{k-1}(M)$.

$$\begin{aligned} \mathcal{L}_X\mathcal{L}_N\alpha &= (\mathcal{L}_X\mathcal{L}_N - \mathcal{L}_N\mathcal{L}_X)\alpha = [\mathcal{L}_X, \mathcal{L}_N]\alpha = [[\mathcal{L}_N, \mathcal{L}_P], \mathcal{L}_N]\alpha = 0. \\ \mathcal{L}_X\mathcal{L}_P\alpha &= (\mathcal{L}_X\mathcal{L}_P - \mathcal{L}_P\mathcal{L}_X)\alpha = [\mathcal{L}_X, \mathcal{L}_P]\alpha = [[\mathcal{L}_N, \mathcal{L}_P], \mathcal{L}_P]\alpha = 0. \end{aligned}$$

So, the differentials $\mathcal{L}_N, \mathcal{L}_P$ can be restricted to $\Omega_X(M)$, and then the triplet $(\mathcal{C}_X^{\bullet,\bullet}(M), \mathcal{L}_N, \mathcal{L}_P)$ is a differential bicomplex, because now $[\mathcal{L}_N, \mathcal{L}_P]|_{\Omega_X(M)} = 0$. □

6. Bihamiltonian Manifolds

If P, Q are Poisson bivectors on M and $[P, Q]_{SN} = 0$ then, (M, P, Q) is called a *bihamiltonian manifold*. It is well known that in this case (P, PQ^{-1}) is a Poisson-Nijenhuis structure on M ([7]). In this section we will prove this fact using the previous techniques.

First let us define some differentiable forms that will be used later. Let P and

Q be bivectors on a manifold M and assume that Q is invertible as a mapping $\mathbf{Q}: T^*M \longrightarrow TM$. We can define two 2-forms, Ω and Φ, as follows:

$$\Omega(X,Y) = Q(\mathbf{Q}^{-1}X, \mathbf{Q}^{-1}Y) = (\mathbf{Q}^{-1}X)(Y), \quad \Phi(X,Y) = P(\mathbf{Q}^{-1}X, \mathbf{Q}^{-1}Y).$$

Let PQ^{-1} be the vector valued 1-form defined by the composition of the mappings $\mathbf{P}$ and $\mathbf{Q}^{-1}$. Note that $(\mathbf{PQ}^{-1})^* = \mathbf{Q}^{-1}\mathbf{P}$ is the composition of the same mappings in the reverse order.

Lemma 6.1. *Let P, Q be bivectors such that Q is invertible, and let Ω and Φ be the 2-forms defined above, then*

(i) $[\Omega, i_P] = i_{PQ^{-1}} - P(\Omega)$.
(ii) $[\Omega, \mathcal{L}_P] = \mathcal{L}_{PQ^{-1}} + \mathrm{d}P(\Omega) - [i_P, \mathrm{d}\Omega]$.
(iii) $[\Omega, i_{PQ^{-1}}] = -2\Phi$.
(iv) $[\Omega, \mathcal{L}_{PQ^{-1}}] = 2\mathrm{d}\Phi - [i_{PQ^{-1}}, \mathrm{d}\Omega]$.

Proof. It is easy to check that both sides of part (i) coincide acting on 0- and 1-forms. To prove (ii) consider the commutator of the left hand side of (i) with the exterior derivative and then apply the graded Jacobi identity. For (iii), note that the 2-form $i_{PQ^{-1}}\Omega$ is equal to 2Φ. A straightforward computation shows (iv):

$$\begin{aligned}[\Omega, \mathcal{L}_{PQ^{-1}}] &= [\Omega, [i_{PQ^{-1}}, \mathrm{d}]] = [[\Omega, i_{PQ^{-1}}], \mathrm{d}] + [i_{PQ^{-1}}, [\Omega, \mathrm{d}]] \\ &= 2\mathrm{d}\Phi - [i_{PQ^{-1}}, \mathrm{d}\Omega].\end{aligned}$$

□

Let us recall the following equivalence between the Schouten-Nijenhuis brackets of Q and $P+Q$ and the closedness of the 2-forms Ω and Φ, respectively: Condition $[Q,Q]_{SN} = 0$ is equivalent to $\mathrm{d}\Omega = 0$ and condition $[P+Q, P+Q]_{SN} = 0$ is equivalent to $\mathrm{d}\Phi = 0$.

Proposition 6.2. *If P, Q are Poisson bivectors then*

$$[\mathcal{L}_{PQ^{-1}}, \mathcal{L}_P] = -[\mathrm{d}P(\Omega), \mathcal{L}_P] \in \mathcal{D}_1^0.$$

Moreover if $[P,Q]_{SN} = 0$ then (P, PQ^{-1}) is a Poisson-Nijenhuis structure.

Proof. The result follows applying Lemma 6.1.

$$\begin{aligned}[\mathcal{L}_{PQ^{-1}}, \mathcal{L}_P] &= [[\Omega, \mathcal{L}_P], \mathcal{L}_P] - [\mathrm{d}P(\Omega), \mathcal{L}_P] = \frac{1}{2}[\Omega, [\mathcal{L}_P, \mathcal{L}_P]] - [\mathrm{d}P(\Omega), \mathcal{L}_P] \\ &= -\frac{1}{2}[\Omega, \mathcal{L}_{[P,P]_{SN}}] - [\mathrm{d}P(\Omega), \mathcal{L}_P] = -[\mathrm{d}P(\Omega), \mathcal{L}_P] \in \mathcal{D}_1^0.\end{aligned}$$

Finally, let us prove that PQ^{-1} is a Nijenhuis tensor. Applying the graded Jacobi identity and Lemma 6.1 (ii) and (iv), we have

$$\begin{aligned}[\mathcal{L}_{PQ^{-1}}, \mathcal{L}_{PQ^{-1}}] &= [\mathcal{L}_{PQ^{-1}}, [\Omega, \mathcal{L}_P]] - [\mathcal{L}_{PQ^{-1}}, \mathrm{d}P(\Omega)] \\ &= [\Omega, [\mathcal{L}_{PQ^{-1}}, \mathcal{L}_P]] - [\mathcal{L}_{PQ^{-1}}, \mathrm{d}P(\Omega)].\end{aligned}$$

Applying twice again Lemma 6.1 (ii) and the fact that $[\mathrm{d}P(\Omega), \mathrm{d}P(\Omega)] = 0$ and $[\Omega, \mathrm{d}P(\Omega)] = 0$, we have

$$\begin{aligned}[\mathcal{L}_{PQ^{-1}}, \mathcal{L}_{PQ^{-1}}] &= [\Omega, [\mathrm{d}P(\Omega), \mathcal{L}_P]] - [[\Omega, \mathcal{L}_P], \mathrm{d}P(\Omega)] \\ &= -2[[\Omega, \mathcal{L}_P], \mathrm{d}P(\Omega)] \in \mathcal{D}_0^2.\end{aligned}$$

This last differential operator belongs to $\mathcal{D}_0^2$, but $[\mathcal{L}_{PQ^{-1}}, \mathcal{L}_{PQ^{-1}}]$ is a derivation of degree two, then both operators must be equal to zero. This implies that $[PQ^{-1}, PQ^{-1}]_{FN} = 0$, i.e., that PQ^{-1} is a Nijenhuis tensor. □

Remark. For this kind of Poisson-Nijenhuis structures it is easy to check that $\operatorname{tr}\mathbf{PQ}^{-1} = 2P(\Omega)$, thus $[\mathcal{L}_N, \mathcal{L}_P] = \mathcal{L}_{-\mathbf{P}(\mathrm{d}P(\Omega))} = \mathcal{L}_{[P,P(\Omega)]_{SN}}$.

Example. Let us consider the two particle Calogero system as in [8]. Let $\{x_1, x_2, p_1, p_2\}$ be a system of coordinates on $\mathbb{R}^4 - \{0\}$. It can be shown that the following tensor fields are Poisson tensors,

$$\begin{aligned}Q &= \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial p_2} \\ P &= \frac{2x_{12}}{\Delta} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + (p_1 + Q_{12}) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial p_1} - Q_{12} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial p_2} \\ &\quad + Q_{12} \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial p_1} + (p_2 - Q_{12}) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial p_2} + 2x_{12}^3 \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial p_2}\end{aligned}$$

where, $\Delta = 4x_{12}^2 + (p_1 - p_2)^2$ and $Q_{12} = \frac{x_{12}^2 (p_1 - p_2)}{\Delta}$.

We have that $[P,Q]_{SN} = 0$, and that Q is invertible, then (P, PQ^{-1}) is a Poisson-Nijenhuis structure. The function $K = p_1 + p_2$ is the total momentum of the system. Computations show that $\operatorname{tr}\mathbf{PQ}^{-1} = 2(p_1 + p_2) = 2K$ and $[i_P, i_{PQ^{-1}}]_V = i_X$, where,

$$X = \mathbf{P}\mathrm{d}K = p_1 \frac{\partial}{\partial x_1} + p_2 \frac{\partial}{\partial x_2} + 2x_{12}^3 \frac{\partial}{\partial p_1} - 2x_{12}^3 \frac{\partial}{\partial p_2}$$

Example. Finally, let us study an example of two compatible Poisson structures on $\mathbb{R}^4$ taken from [1] with $[i_P, i_N]_\bullet \neq 0$. Let $\{x^1, x^2, y^1, y^2\}$ be a system of coordinates on $\mathbb{R}^4$. Let us consider the following two Poisson bivectors,

$$\begin{aligned}P &= \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial y^2}, \\ Q &= (x^1 - 2) \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^1} + e^{x^2} \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial y^2} + \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial y^1}.\end{aligned}$$

We have that $[P,Q]_{SN} = 0$, and that Q is invertible, then (P, PQ^{-1}) is a Poisson-Nijenhuis structure.

For this Poisson-Nijenhuis structure we have that $\operatorname{tr}\mathbf{PQ}^{-1} = 2(2 - x^1)e^{-x^2}$ and $[i_P, i_{PQ^{-1}}]_V = i_X$, where

$$X(f) = -e^{-x^2} \frac{\partial}{\partial y^1} - (2 - x^1) e^{-x^2} \frac{\partial}{\partial y^2}.$$

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