Salkowski curves revisited: A family of curves with constant curvature and non-constant torsion

J. Monterde

Dep. de Geometria i Topologia, Universitat de València, Avd. Vicent Andrés Estellés, 1, E-46100-Burjassot (València), Spain

Abstract

In a paper by E. Salkowski ([10]) published one century ago, a family of curves with constant curvature but non-constant torsion was defined. We characterize them as space curves with constant curvature and whose normal vector makes a constant angle with a fixed line. The relation between these curves and rational curves with double Pythagorean hodograph is studied. A method to construct closed curves, including knotted curves, of constant curvature and continuous torsion using pieces of Salkowski curves is outlined.

Key words: Lancret's theorem, closed (composite) space curve, constant curvature, Salkowski curve.

1 Introduction

Circles and circular helices are curves with constant curvature and torsion. Salkowski curves are, to the best of the author's knowledge, the first known family of curves with constant curvature but non-constant torsion with an explicit parametrization. They were defined in an earlier paper [10] and retrieved, as an example of tangentially cubic curves, in a first version of [8], available on the Internet [9], but not included in the final published version.

We first obtain a geometric characterization of Salkowski curves. Among all the space curves with constant curvature, Salkowski curves are those for which the normal vector maintains a constant angle with a fixed direction in the space.

Email address: monterde@uv.es (J. Monterde).

Preprint submitted to Elsevier Science October 14, 2008

The reader will undoubtedly recall the analogous defining condition of general helices and Lancret's theorem (see Chapter 23 of [2]).

In some recent papers, [6,7], arcs of circular helices have been used to build closed curves or to show how to realize all knot types as C^2 smooth curves of constant curvature. Nevertheless, in both papers, the torsion function of the curves constructed by gluing arcs of circular helices (including circular arcs as a special case) is a discontinuous function. Introducing Salkowski curves allows enough freedom to solve this problem. The solution is based on the fact that the torsion of Salkowski curves is not constant, in fact, it is a monotone function. So, they can be used to join arcs of circular helices with different torsions. We will construct some examples of closed curves with constant curvature and continuous torsion.

Finally, some Salkowski curves can be reparametrized so that the resulting curve is a rational curve with a double Pythagorean hodograph (see [1,3] for the definition of polynomial curves with a double Pythagorean hodograph and Def. 2 for the extension of this notion to the rational case). This result is partially related with what happens with general polynomial helices: any polynomial curve whose tangent vector makes a constant angle with a fixed line is a Pythagorean curve.

A last appendix is devoted to the definition of the reciprocal class of space curves with constant torsion. We have called them, anti-Salkowski curves.

2 Salkowski curves

First of all, note that in the study of curves with constant curvature, thanks to a change of scale, we can suppose that $\kappa \equiv 1$.

Second, our convention for the sign of the torsion is $\tau = < \overrightarrow{\mathbf{b}}', \overrightarrow{\mathbf{n}} >$.

In a paper published one century ago, [10], on the transformation of space curves, the author devotes a section to the study of the particular case of space curves with constant curvature:

Definition 1 *Salkowski curves. (See [10] p. 538.)*

For any $m \in \mathbb{R}$ with $m \neq \pm\frac{1}{\sqrt{3}}, 0$, let us define the space curve

$$\begin{aligned}\gamma_m(t) = \tfrac{1}{\sqrt{1+m^2}} \Big(& -\tfrac{1-n}{4(1+2n)} \sin((1+2n)t) - \tfrac{1+n}{4(1-2n)} \sin((1-2n)t) - \tfrac{1}{2}\sin t, \\ & \tfrac{1-n}{4(1+2n)} \cos((1+2n)t) + \tfrac{1+n}{4(1-2n)} \cos((1-2n)t) + \tfrac{1}{2}\cos t, \\ & \tfrac{1}{4m}\cos(2nt)\Big), \end{aligned} \tag{1}$$

with $n = \frac{m}{\sqrt{1+m^2}}$.

The geometric elements of the Salkowski curve γ_m are the following (see [10], Appendix I.):

(1) $||\gamma_m'(t)|| = \frac{\cos(nt)}{\sqrt{1+m^2}}$, so the curve is regular in $]-\frac{\pi}{2n}, \frac{\pi}{2n}[$,

(2) $\kappa(t) \equiv 1$ and $\tau(t) = \tan(nt)$,

and the Frenet's frame is

$$\begin{aligned} \overrightarrow{t}(t) &= -\Big(\cos(t)\cos(nt) + n\sin(t)\sin(nt), \cos(nt)\sin(t) - n\cos(t)\sin(nt), \tfrac{n}{m}\sin(nt)\Big), \\ \overrightarrow{n}(t) &= n\left(\tfrac{\sin(t)}{m}, -\tfrac{\cos(t)}{m}, -1\right), \\ \overrightarrow{b}(t) &= \Big(n\cos(nt)\sin(t) - \cos(t)\sin(nt), -n\cos(t)\cos(nt) - \sin(t)\sin(nt), \tfrac{n}{m}\cos(nt)\Big). \end{aligned} \tag{2}$$

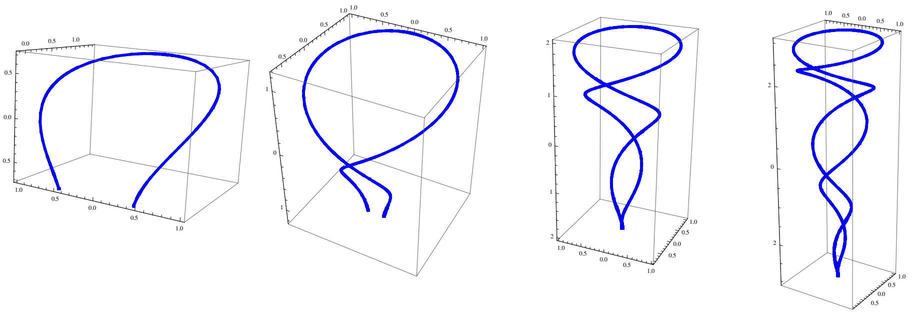

Figure 1. Some Salkowski curves with $\kappa \equiv 1$ for $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$, plotted for $t \in [-\frac{\pi}{2n}, \frac{\pi}{2n}]$.

3 A characterization of Salkowski curves

From the expression of the normal vector, see Eqs. (2), one can see that the normal indicatrix, or *nortrix*, of a Salkowski curve describes a parallel of the unit sphere. The angle between the normal vector and the fixed vector $(0,0,-1)$ is $\phi = \arccos n$.

This fact is reminiscent of what happens with another important class of curves, the general helices. Let us recall that general helices are those curves whose tangent vectors make a constant angle with a fixed line in space. Such a condition implies that the tangent indicatrix, or *tantrix*, describes a parallel in the unit sphere and that the quotient $\frac{\tau}{\kappa}$ is constant (Lancret's Theorem).

Therefore, it is natural to study a statement like Lancret's theorem, where the tantrix has been substituted by the nortrix. The analogous to Lancret's Theorem would give a relation between curvature and torsion. In our situation, we do not need such a general result. We will particularize the statement to the subclass of curves with constant curvature.

Lemma 1 *Let $\alpha : I \to \mathbb{R}^3$ be a curve that is parametrized by arc-length with $\kappa \equiv 1$. Its normal vectors make a constant angle, ϕ, with a fixed line in space if and only if $\tau(s) = \pm \frac{s}{\sqrt{\tan^2 \phi - s^2}}$.*

Remark 1 *If the angle, ϕ, is zero, then the normal vector is constant and the curve is a straight line, in contradiction with the fact that $\kappa \equiv 1$. If the angle is $\frac{\pi}{2}$, then the torsion is constant, and since $\kappa \equiv 1$, the curve is a circular helix.*

Proof: ($\Rightarrow$) Let $\overrightarrow{d}$ be the unitary fixed direction which makes a constant angle, $\phi \in]0, \frac{\pi}{2}[$, with the normal vector $\overrightarrow{n}$. Therefore

$$\overrightarrow{n} \cdot \overrightarrow{d} = \cos \phi. \tag{3}$$

Deriving Eq. (3) and using Frenet's formulae, we get

$$(-\overrightarrow{t} - \tau \overrightarrow{b}) \cdot \overrightarrow{d} = 0. \tag{4}$$

Therefore,

$$\overrightarrow{t} \cdot \overrightarrow{d} = -\tau \overrightarrow{b} \cdot \overrightarrow{d}.$$

If we put $b = \overrightarrow{b} \cdot \overrightarrow{d}$, we can write

$$\overrightarrow{d} = -\tau b \overrightarrow{t} + \cos \phi \overrightarrow{n} + b \overrightarrow{b}.$$

From $||\overrightarrow{d}|| = 1$ we get $b = \pm \frac{\sin \phi}{\sqrt{1+\tau^2}}$. Therefore, vector $\overrightarrow{d}$ can be written as

$$\overrightarrow{d} = \mp \frac{\tau}{\sqrt{1+\tau^2}} \sin \phi \overrightarrow{t} + \cos \phi \overrightarrow{n} \pm \frac{\sin \phi}{\sqrt{1+\tau^2}} \overrightarrow{b}. \tag{5}$$

If we derive Eq. (4) again, we obtain

$$\left(\dot{\tau}\overrightarrow{b} + (1+\tau^2)\overrightarrow{n}\right) \cdot \overrightarrow{d} = 0.$$

Using Eq. (5) we get the differential equation

$$\pm \tan\phi \frac{\dot{\tau}}{(1+\tau^2)^{\frac{3}{2}}} + 1 = 0. \tag{6}$$

By integration we get

$$\pm \tan\phi \frac{\tau}{(1+\tau^2)^{\frac{1}{2}}} + s + c = 0, \tag{7}$$

where c is an integration constant. The integration constant can be subsumed thanks to a parameter change $s \rightarrow s - c$. Finally, to solve Eq. (7) with τ as unknown let us write it as

$$\pm \tan\phi \frac{\tau}{(1+\tau^2)^{\frac{1}{2}}} = -s. \tag{8}$$

If we take squares at both sides of Eq. (8), the resulting equation can be easily solved in τ and we get the desired result.

($\Leftarrow$) Suppose that $\tau = \pm\frac{s}{\sqrt{\tan^2\phi - s^2}}$ and let us put

$$b = \mp\frac{\sin\phi}{\sqrt{1+\tau^2}} = \mp\frac{\sin\phi}{\sqrt{1+\frac{s^2}{\tan^2\phi - s^2}}} = \mp\cos\phi\sqrt{\tan^2\phi - s^2},$$

where we are assuming that when τ has the positive (negative) sign, then b gets the negative (positive) sign. Thus, $\tau b = -s\ \cos\phi$.

We will prove that the expression

$$\overrightarrow{d} = -\tau b\overrightarrow{t} + \cos\phi\overrightarrow{n} + b\overrightarrow{b} = \cos\phi\left(s\overrightarrow{t} + \overrightarrow{n} \mp \sqrt{\tan^2\phi - s^2}\overrightarrow{b}\right)$$

defines a constant vector. Indeed, applying Frenet's formulae

$$\dot{\overrightarrow{d}} = \cos\phi\left(\overrightarrow{t} + s\overrightarrow{n} - \overrightarrow{t} \mp \frac{s}{\sqrt{\tan^2\phi - s^2}}\overrightarrow{b} \pm \frac{s}{\sqrt{\tan^2\phi - s^2}}\overrightarrow{b} \mp (\pm s)\overrightarrow{n}\right) = 0.$$

Therefore, $\overrightarrow{d}$ is constant and $\overrightarrow{d} \cdot \overrightarrow{n} = \cos\phi$. $\square$

Once the intrinsic or natural equations of a curve have been determined, the next step is to integrate Frenet's formulae with $\kappa(s) \equiv 1$ and

$$\tau(s) = \pm \frac{s}{\sqrt{\tan^2\phi - s^2}} = \pm \frac{\frac{s}{\tan\phi}}{\sqrt{1 - (\frac{s}{\tan\phi})^2}} = \pm \tan(\arcsin(\frac{s}{\tan\phi})). \qquad (9)$$

Theorem 1 *The space curves with $\kappa \equiv 1$ and such that their normal vectors make a constant angle with a fixed line are, up to rigid movements in space or up to the antipodal map, Salkowski curves (see Def. 1).*

Proof: As it has been said after Def. 1, the arc-length parameter of Salkowski curves is $s = \int_0^t ||\gamma'(u)||\, du = \frac{1}{m}\sin(nt)$. Therefore, $t = \frac{1}{n}\arcsin(ms)$. In terms of the arc-length curvature and torsion are then

$$\kappa(s) \equiv 1, \qquad \tau(s) = \tan(\arcsin(ms)),$$

the same intrinsic equations, with $m = \frac{1}{\tan\phi}$ and $n = \cos\phi$ (compare with the positive case in Eq. (9)), as the ones shown in Lemma 1.

For the negative case in Eq. (9), let us recall that if a curve α has torsion τ^{α}, then the curve $\beta(t) = -\alpha(t)$ has as torsion $\tau^{\beta}(t) = -\tau^{\alpha}(t)$, whereas curvature is preserved.

Therefore, the Fundamental Theorem of curves in space states in our situation that, up to rigid movements or up to the antipodal map, $p \to -p$, the curves we are looking for are Salkowski curves. $\square$

4 Salkowski curves and rational double Pythagorean hodograph curves

The notion of a Pythagorean hodograph (PH) curve has been extensively studied in recent years. Its relation with generalized helices has been established by Farouki and co-workers (see [2]). Double PH curves (see [1,3,4]) are those polynomial curves for which not only the tangent vector is again a rational vectorial function but the whole Frenet's frame is rational.

The generalization of the PH condition to the rational case can be found in Chapter 20 of [2]. The extension to the definition of rational double PH curves is easy:

Definition 2 *A rational curve $\alpha : I \to \mathbb{R}^3$ is said to be a rational double Pythagorean hodograph curve (rational DPH curve) if both $||\alpha'||$ and $||\alpha' \wedge \alpha''||$ are rational functions.*

Salkowski curves provide a class of examples of rational DPH curves which are not generalized helices. For some values of the parameter m and under suitable reparametrization, the corresponding Salkowski curve admits a rational expression which is a rational double Pythagorean hodograph curve.

Lemma 2 *Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a rational PH curve. If, in addition, its normal vectors make a constant angle with a fixed line, then the curve is a rational DPH.*

Proof: As is well known,

$$\overrightarrow{t} = \frac{\alpha'}{||\alpha'||}, \qquad \overrightarrow{b} = \frac{\alpha' \wedge \alpha''}{||\alpha' \wedge \alpha''||},$$

and $\overrightarrow{n} = \overrightarrow{b} \wedge \overrightarrow{t}$. If for some unitary vector, $\overrightarrow{d}$, we have $\overrightarrow{n} \cdot \overrightarrow{d} = a$, a being constant, then

$$||\alpha' \wedge \alpha''|| = \frac{\det(\alpha' \wedge \alpha'', \alpha', \overrightarrow{d})}{a||\alpha'||}.$$

Since, by hypothesis, $||\alpha'||$ is a rational function, so is $||\alpha' \wedge \alpha''||$. $\square$

The reciprocal is not true. In [4] (Example 7) one can find the following example of a non-helical double PH curve

$$(-t^2 - \frac{t^3}{3} - t^4 + \frac{11}{5}t^5 - \frac{5}{9}t^6 - \frac{22}{63}t^7, -2t - t^2 - 4t^3 + t^4 - \frac{26}{5}t^5 + \frac{34}{3}t^6 - \frac{124}{21}t^7, -\frac{2}{3}t^3 - \frac{4}{3}t^4 + \frac{2}{5}t^5 + 2t^6 - \frac{4}{3}t^7)$$

whose nortrix does not describe a parallel in the unit sphere.

Proposition 1 *For each $a \in \mathbb{Z}$ with $|a| > 2$, let $m_a = \frac{1}{\sqrt{a^2-1}}$. Then the Salkowski curve $\beta_a(t) := \gamma_{m_a}(2a\arctan(t))$ is a rational DPH curve.*

Proof: First, we have to show that β_a is a rational curve. Note that since $\frac{\sqrt{1+m_a^2}}{m_a} = a$ then $n = \frac{1}{a}$ and $1 \pm 2n = 1 \pm \frac{2}{a}$. Therefore, all the trigonometric functions in the parametrization (1), when evaluated in $2a\arctan(t)$, become rational functions. Indeed, the three coordinates of $\beta_a(t)$ are combinations of

$$\cos(4\arctan(t)) = \frac{1 - 6t^2 + t^4}{(1+t^2)^2}, \quad \sin(4\arctan(t)) = -\frac{4t(t^2-1)}{(1+t^2)^2}$$

and of

$$\cos(2a\arctan(t)), \sin(2a\arctan(t))$$

which can also be written as rational functions when $a \in \mathbb{Z}$. In fact, suitable recursive trigonometric formulae allow us to reduce $\cos(a\ (2\arctan(t)))$ and $\sin(a\ (2\arctan(t)))$ into the simplest terms of the kind $\cos(2\arctan(t))$ and $\sin(2\arctan(t))$.

Let us check that the curve β_a is a curve with a Pythagorean hodograph. Since

$||\gamma_m'(u)|| = \frac{\cos(nu)}{\sqrt{1+m^2}}$ then under a change of the kind $u = u(t) = 2a\arctan(t)$, with $u'(t) = \frac{2a}{1+t^2}$ we will have

$$||\beta_a'(t)|| = -\frac{\sqrt{a^2-1}}{a}\frac{t^2-1}{t^2+1}u'(t).$$

Therefore, the curve β_a is PH. By Lemma 2, the curve is double PH. $\square$

For example, for $a = 3$, $m_a = \frac{1}{2\sqrt{2}}$ and the new parametrization of the Salkowski curve is

$$\left(-\frac{4\sqrt{2}t(15-20t^2+58t^4-20t^6+15t^8)}{15(1+t^2)^5}, -\frac{16\sqrt{2}(-1+t^2)^5}{15(1+t^2)^5}, \frac{2(1-6t^2+t^4)}{3(1+t^2)^2}\right).$$

5 Closed space curves of constant curvature consisting of arcs of Salkowski curves and of circular helices

In some recent papers arcs of curves with constant curvature and constant torsion, that is, of circular helices, have been used to build closed curves [6] or to show how to realize all knot types as C^2 smooth curves of constant curvature [7].

Nevertheless, in both these papers, the torsion function of the curves constructed by gluing arcs of circular helices (including circular arcs as a special case) is a discontinuous function. The introduction of Salkowski curves allows enough freedom to solve this problem. We will see four examples of closed curves, one of them knotted, with constant curvature and continuous torsion.

The curves we will construct throughout this section are made by gluing pieces of curves in such a way that, at the junction points, the Frenet's elements (curvature, torsion and frame) will be the same. Therefore, the resulting curves are not directly C^3 smooth curves but of geometric continuity of order 3 (G^3). To get C^3 smooth curves, all we have to do is to reparameterize all the pieces by arc length. Recall that the change of parameter in Salkowski curves is $t = \frac{1}{n}\arcsin(ms)$. Thus, any of the curves so constructed can be easily reparameterized to get a C^3 smooth curve.

Finally, it should be mentioned that in [5] it is shown that any space curve can be approximated by a C^∞ curve of constant curvature.

5.1 Closed curves

In paper [6] it is shown how to build closed curves from pieces of circular helices. The method shown in the cited paper uses the property verified by circular helices: their tangent vector makes a constant angle with a fixed line. So, the tantrix of a circular helix describes a parallel in the unit sphere. Therefore, we can choose a closed curve in the unit sphere made with pieces of parallels as a tantrix of a possible closed curve in space made by pieces of circular helices.

The same method is useful when we replace circular helices by Salkowski curves, and the tantrix by the nortrix.

The intersection between each face of a regular polyhedron and the sphere defines a parallel. Thanks to a spatial rotation, we can suppose that such a parallel is horizontal. Therefore there is a Salkowski curve γ_m whose nortrix describes the horizontal parallel.

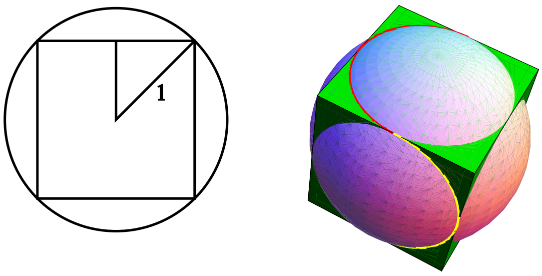

Figure 2. The intersection between a sphere and a cube makes it possible to select a continuous nortrix.

We will work with two simple constructions using a cube. The parallels defined as the intersection of the faces of the cube and the unit sphere are all of latitude $\frac{\pi}{4}$ (see Fig. 2). Therefore, let us consider the piece of the Salkowski curve γ_1 in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Note that the torsion of the Salkowski curve at $\frac{\pi}{2}$ is $\tau_0 := \tau(\frac{\pi}{2}) = \tan(\frac{\pi}{2\sqrt{2}})$ and, analogously, $\tau(-\frac{\pi}{2}) = -\tau_0$.

Let $a = \gamma_1|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ and $b = \gamma_{-1}|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$. It is possible to join four basic pieces, $abab$, to form a closed curve (see Fig. 3, left). The junction point in ab has τ_0 as the torsion, whereas the torsion at the junction point in ba is $-\tau_0$. Therefore, the torsion function of the closed curve is continuous. The Frenet's frames at the junction points are the same. Moreover, since the curvature is constant, then its derivative vanishes. Therefore the closed curve has geometric continuity of order 3.

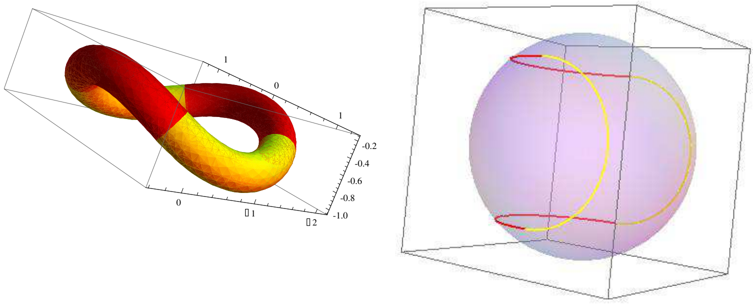

Figure 3. Left, a closed curve with $\kappa = 1$ and continuous torsion represented as a tubular surface to help the reader to visualize it. Right, its nortrix. Note that the nortrix is made of pieces of circular arcs.

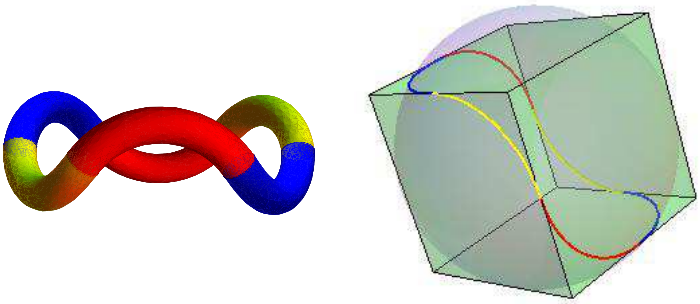

Figure 4. Left, another example of a closed curve with $\kappa = 1$ and continuous torsion also represented as a tubular surface. Right, its nortrix, a part of the intersection between a cube and the unit sphere.

In the previous example the junction points are points with non-vanishing torsion. An alternative construction consists in using the points with vanishing torsion as some of the junction points. Let $c = \gamma_1|_{[0,\frac{\pi}{2}]}$ and $d = \gamma_{-1}|_{[-\frac{\pi}{2},0]}$. Then $cdcdcd$ (see Figure 4) is again a closed curve. Note that the union cd is at a point with torsion τ_0 whereas the union dc is at a point with vanishing torsion.

5.2 A constant curvature knot

In paper [7] curves are defined by joining pieces of circular helices or of circles and maintaining the same Frenet's frame at junction points. In that work this type of procedure is called "splicing".

Here we will repeat the same procedure but now substituting pieces of Salkowski curves for circular pieces, so that we can continuously join two pieces of circular helices of different torsion.

Let us recall the explicit parametrization of a circular helix. Given $a, b \in \mathbb{R}$, with $a > 0$,

$$\alpha(t) = (a \cos t, a \sin t, bt), \qquad t \in \mathbb{R}$$

is the usual parametrization of a circular helix with constant curvature $\kappa \equiv \frac{a}{a^2+b^2}$ and constant torsion $\tau \equiv -\frac{b}{a^2+b^2}$. Therefore, given any $r \in \mathbb{R}$, the circular helix defined by the parameters $a = \frac{1}{1+r^2}$ and $b = \frac{r}{1+r^2}$ is of constant curvature 1 and torsion r.

As before, let $a = \gamma_1|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$ and $b = \gamma_{-1}|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$. Moreover, let ℓ_p (r_p) be the circular helix with curvature 1, and torsion τ_0 ($-\tau_0$) defined for $t \in [0, 2p\pi]$.

In Fig. 5, left, the closed curve is $a\ell_{\frac{1}{2}}br_{\frac{1}{2}}a\ell_{\frac{1}{2}}br_{\frac{1}{2}}$. In Fig. 5, right, the curve is

$$a\ell_7 br_6 a\ell_6 br_4 a\ell_2 br_4 a\ell_5 br_4 a\ell_3 br_2 a\ell_1 br_4.$$

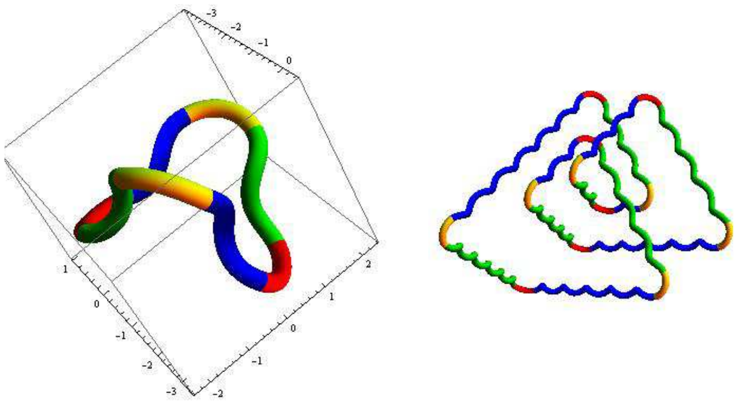

Figure 5. Left: Another curve of constant curvature built from the curve shown in Fig. 4 but with the addition of some pieces of circular helices. Right: A C^3 model of a knot as a curve of constant curvature. Note that after a 'red' piece of a Salkowski curve, the circular helix, in 'blue', is of negative (and constant) torsion. Reciprocally, after a 'yellow' piece, the circular helix, in 'green', has a positive value.

Both curves are also represented as tubular surfaces to help the reader to visualize the pieces they are made of and the crossings of the knot curve.

6 Conclusions

The closed curves we have seen in the previous sections, including the knotted one, are just examples of the possibilities offered by the use of Salkowski curves. A systematic study of the construction of closed curves and of models of knotted curves will be developed in the future.

We have shown that a reparametrization of some Salkowski curves gives rise to double PH curves. Therefore we have a new kind of double PH curves defined by geometric conditions, that is, constant curvature and normal vector making a constant angle with a fixed direction.

It has to be said that, in spite of their clear geometrical interpretation, the use of rational Salkowski curves in practical applications could be difficult because these curves depend on a parameter a (see Prop. 1), which can only vary in $\mathbb{Z}$.

Acknowledgement. The author wishes to express his gratitude to the referees. Their insightful comments and suggestions have contributed to improve the contents of the paper and to correct some mistakes in the original version.

7 Appendix. A family of curves with constant torsion and non-constant curvature

As an additional material we will show in this appendix how to build, from a curve of constant curvature, another curve of constant torsion.

Let us recall that a curve $\alpha :]a, b[\to \mathbb{R}^3$, is 2-regular at a point t_0 if $\alpha'(t_0) \neq 0$ and if $\kappa^\alpha(t_0) \neq 0$.

Lemma 3 *Let $\alpha : I \to \mathbb{R}^3$ be a regular curve parametrized by arc-length, with curvature κ^α, torsion τ^α, and Frenet's frame $\{\overrightarrow{t}^\alpha, \overrightarrow{n}^\alpha, \overrightarrow{b}^\alpha\}$. Let us consider the curve $\beta(s) = \int_{s_0}^{s} \overrightarrow{b}^\alpha(u)\,du$. Then at a parameter $s \in I$ such that $\tau^\alpha(s) \neq 0$, the curve β is 2-regular at s and*

$$\kappa^\beta = |\tau^\alpha(s)|, \quad \tau^\beta(s) = \kappa^\alpha(s),$$

$$\overrightarrow{t}^\beta = \overrightarrow{b}^\alpha, \quad \overrightarrow{n}^\beta = \overrightarrow{n}^\alpha, \quad \overrightarrow{b}^\beta = -\overrightarrow{t}^\alpha.$$

Proof: In order to obtain the tangent vector of β let us compute $\dot{\beta}(s) = \overrightarrow{b}^\alpha(s)$. Since $||\overrightarrow{b}^\alpha(s)|| = 1$, then β is a curve parametrized by arc-length and $\overrightarrow{t}^\beta = \overrightarrow{b}^\alpha$.

If we derive the tangent vector we obtain $\dot{\overrightarrow{t}}^{\beta} = \dot{\overrightarrow{b}}^{\alpha} = \tau^{\alpha}\overrightarrow{n}^{\alpha}$. Therefore $\kappa^{\beta}(s) = ||\dot{\overrightarrow{t}}^{\beta}(s)|| = |\tau^{\alpha}(s)|$ and $\overrightarrow{n}^{\beta} = \overrightarrow{n}^{\alpha}$.

The remaining part of the proof can be easily deduced using the same techniques. □

Let us apply the previous result to the Salkowski curve γ_m defined in Eq. (1). From the expression of the binormal vector in Eq. (2) and from the fact that

$$\int \overrightarrow{b}^{\gamma_m}(s)\ ds = \int \overrightarrow{b}^{\gamma_m}(t)||\gamma_m'(t)||\ dt,$$

we have

$$\begin{aligned}\beta_m(t) = \Big(&\tfrac{1}{2(4n^2-1)m}(n(1-4n^2+3\cos(2nt))\cos(t) + (2n^2+1)\sin(t)\sin(2nt),\\ &\tfrac{1}{2(4n^2-1)m}(n(1-4n^2+3\cos(2nt))\sin(t) - (2n^2+1)\cos(t)\sin(2nt),\\ &\tfrac{n^2-1}{4n^2}(2nt+\sin(2nt))\Big),\end{aligned} \tag{10}$$

where, as for Salkowski curves, $n = \frac{m}{\sqrt{1+m^2}}$. Let us call these curves by the name anti-Salkowski curves. The presence of the non-trigonometric term $2nt$ in the third component of $\beta_m(t)$ makes that the change of variable studied in Section 3 for Salkowski curves does not work for anti-Salkowski curves.

Applying Lemma 3 we get the following result:

Proposition 2 *The curves β_m in Eq. (10) are curves of constant torsion equal to* 1.

Theorem 2 *The space curves with $\tau \equiv 1$ and such that their normal vectors make a constant angle with a fixed line are the anti-Salkowski curves defined in Eq. (10).*

Proof: Let α be a curve with $\tau \equiv 1$ and let $\beta(s) = \int_{s_0}^{s} \overrightarrow{b}^{\alpha}(u)\ du$. By Lemma 3, β is a curve with constant curvature $\kappa \equiv 1$ and with the same normal vector. Therefore, β is a Salkowski curve and α is an anti-Salkowski curve. □

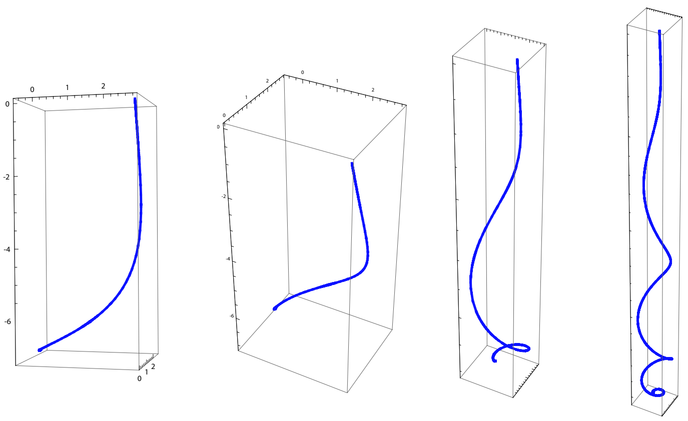

Figure 6. Some anti-Salkowski curves curve with $\tau \equiv 1$ for $m = \frac{1}{3}, \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ and plotted for $t \in \; [0, \frac{\pi}{2n}]$.

References

[1] Beltran, J.V. and Monterde, J., *A characterization of quintic helices*, Journal of Computational and Applied Mathematics, **206** (1) 116–121 (2007).

[2] Farouki, R. T., *Pythagorean-Hodograph Curves. Algebra and Geometry inseparable*, Springer, Berlin (2008).

[3] Farouki, R. T., Giannelli, C. and Sestini, A., *Helical polynomial curves and double Pythagorean hodographs I. Quaternion and Hopf map representations*, Journal of Symbolic Computation, to appear.

[4] Farouki, R. T., Giannelli, C. and Sestini, A., *Helical polynomial curves and double Pythagorean hodographs II. Enumeration of low-degree curves*, Journal of Symbolic Computation, to appear.

[5] Ghomi, M., *H-Principles for curves and knots of constant curvature* Geometriae Dedicata **127** (1), 19–35 (2007).

[6] Koch, R. and Engelhardt, C. *Closed space curves of constant curvature consisting of arcs of circular helices*, J. for Geometry and Graphics, **2** (1998), **1**, 17–31.

[7] McAtee Ganatra, J.M., *Knots of constant curvature*, Journal of Knot Theory and its Ramifications **16** (4), 461–470 (2007).

[8] Pottmann, H. and Hofer, M., *A variational approach to spline curves on surfaces*, Computer Aided Geometric Design, **22** (2005) 693–709.

[9] Pottmann, H. and Hofer, M., previous version of *A variational approach to spline curves on surfaces*, in *http://dmg.tuwien.ac.at/geom/ig/papers/vscs.pdf.*

[10] Salkowski, E. , *Zur Transformation von Raumkurven*, Mathematische Annalen, **66** (4), 517–557 (1909).