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Variational Problems on Graded Manifolds

J. MONTERDE AND J. MUÑOZ MASQUÉ

ABSTRACT. A geometric formulation of the variational calculus on a fibred graded manifold is presented, both for Berezinian Lagrangian densities and for graded Lagrangian densities. We prove that to every Berezinian Lagrangian density of order r we can associate a class of equivalent graded Lagrangian densities of order $n + r$ with the same first variation (n being the odd dimension of the base manifold). The theory is applied to several graded variational problems (scalar superfields, scalar supercurvature, supergeodesics and supermechanics). A Hamiltonian formalism for Berezinian Lagrangian densities in $(1, 1)$-graded mechanics is developed.

Introduction

For the past fifteen years action principles have been avalaible, invariantly formulated both in supergravity and supersymmetric gauge theories (see e.g. [**ANZ**], [**Mn**], [**CM**]). The interest of such theories has nothing but increased at the same time as the differential geometry of graded manifolds (or supermanifolds in the russian author's terminology) has developed extraordinarily ([**Ko**], [**Le**], [**Ma**], [**Lo**]). It then seems interesting to develop a general theory of calculus of variations for arbitrary graded submersions, which allow us both a Lagrangian and Hamiltonian formulation of variational problems on graded fibred manifolds in a fashion analogous to the Hamilton-Cartan theory in classical field theory.

Within the framework of graded manifolds two essentially different notions of Lagrangian density may be given, according to the notions of integration considered. For the first of them, a density is a graded differential m-form ω on a graded manifold $(M, \mathcal{A})$ of dimension (m, n) with coefficients in any graded

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jet-bundle $J^r(\mathcal{B}/\mathcal{A})$ associated to a graded submerssion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$. The action functional associates to each section s of p the integral on M of the ordinary m-form $((j^r s)^*(\omega))^\sim$. Such densities will from this point on be called Graded Lagrangian densities.

On the other hand, for the second notion a density is a global section ξ of the Berezinian sheaf of the graded manifold $(M, \mathcal{A})$ with coefficients in any $J^r(\mathcal{B}/\mathcal{A})$ and the action functional associates to each section s of p the Berezin integral [**Be**] of the section $(j^r s)^*(\xi) \in \mathrm{Ber}(M, \mathcal{A})$. Such densities will from this point on be called Berezinian Lagrangian densities.

In theoretical physics all variational problems are formulated by means of the second notion of density. In fact, physicists consider the Berezin integral to be the standard theory in graded manifolds. But, at least from a mathematical point of view, it is obvious that both Lagrangian and Hamiltonian formalisms can better be developed within the framework of graded differential forms, where a Cartan differential calculus which verifies the most important usual properties is available.

This difficulty may be overcome in two stages. In the first instance, the Euler-Lagrange equations are deduced for an arbitrary Graded Lagrangian density. These equations are not standard: only a reduced group of them adopts the traditional form of the Euler-Lagrange equations; contrarywise, others may be interpreted as constraints on the Lagrangian. In the second instance, it is proved that to every Berezinian Lagrangian density ξ we can canonically associate an equivalent class of Graded Lagrangian densities $\{\omega\}$ so that ξ and $\{\omega\}$ define the same "first variation" at every section s (and consequently, they have the same critical sections). Furthermore, for the Graded Lagrangian densities which come from Berezinian Lagrangian densities, the Euler-Lagrange equations are expressed in the usual way as equations in the graded ring $\mathcal{A}$ of the graded manifold $(M, \mathcal{A})$: these are the graded Euler-Lagrange equations that could be expected.

In this paper we complete some aspects of this theory that were not considered in [**Mo**], and we also include a brief résumé of it, basic to the understanding of what follows. Afterwards the theory is applied to several graded variational problems (scalar superfields, scalar supercurvature, supergeodesics and supermechanics) which are classics in the ungraded case. For those graded problems that have already been dealt with by other methods, the theory leads to the same results obtained by other authors, but it also allows the deduction of new results. Thus, for example, to every Berezinian Lagrangian density in $(1,1)$-graded mechanics a Poincaré-Cartan form can be associated, which allows us the development of a Hamiltonian formalism that is consistent for such problems.

2. Preliminaries and notations

2.1. Graded jet bundles. We shall always work in the category of C^∞ real graded manifolds. Definitions and notations for this category have been taken from [**Ko**].

As is well known, for every graded submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ we can construct the graded r-jet bundle of local sections of p, which will be denoted by $(J^r(\mathcal{B}/\mathcal{A}), \mathcal{A}^r)$. For the details see [**HM1**]. We shall also work with the inverse limit of these bundles ([**Mo**]), $(J^\infty(\mathcal{B}/\mathcal{A}) = \varprojlim J^r(\mathcal{B}/\mathcal{A}), \mathcal{A}^\infty = \varinjlim \mathcal{A}^r)$.

Let us recall some notations for graded coordinates. Usually, we shall work with positive indices for even coordinates and negative indices for odd coordinates: $(x_i; -n \leq i \leq -1, 1 \leq i \leq m)$ for a graded manifold $(M, \mathcal{A})$ of graded dimension (m, n). This is the notation used in DeWitt's book for supermanifolds [**dW**].

In some particular cases, however, we shall also use the notation $(x_i; 1 \leq i \leq m)$ for the even coordinates and $(s_J; 1 \leq J \leq n)$ for the odd coordinates. This is the standard notation in the standard setting; see e.g. [**Ko**], [**Le**], [**Ma**]. The notions of graded fibred coordinates for a graded submersion and the corresponding coordinate systems induced in $J^r(\mathcal{B}/\mathcal{A})$ can be seen in [**HM1**], [**HM2**] and [**Mo**].

2.2. Graded infinitesimal contact transformations. The notion of graded infinitesimal contact transformations of arbitrary order is introduced in [**HM3**], and it is proved that for every graded vector field X on $(N, \mathcal{B})$ a unique infinitesimal contact transformation $X_{(r)}$ on $(J^r(\mathcal{B}/\mathcal{A}), \mathcal{A}^r)$ exists, projecting onto X. Moreover, $X \to X_{(r)}$ is an injection of graded Lie algebras.

We note that $X_{(r+1)}$ projects onto $X_{(r)}$, for every $n \in \mathbb{N}$; hence each graded vector field X on $(N, \mathcal{B})$ induces an infinitesimal contact transformation $X_{(\infty)}$ on $(J^\infty(\mathcal{B}/\mathcal{A}), \mathcal{A}^\infty)$ ([**Mo**]).

2.3. Horizontal lifting. Given a graded vector field X on $(M, \mathcal{A})$ there exists a unique graded vector field X^H on $(J^\infty(\mathcal{B}/\mathcal{A}), \mathcal{A}^\infty)$, mapping $\mathcal{A}^r$ into $\mathcal{A}^{r+1}$ for every $r \in \mathbb{N}$, such that

$$(j^{r+1}s)^*(X^H f) = X((j^r s)^*(f)),$$

for every local section s of p.

X^H is called the total, formal or horizontal lifting of X. We note that $X \to X^H$ is a $\mathcal{A}$-linear injection of Lie algebras ([**Mo**]). A vector field X in $\mathcal{A}^\infty$ is called horizontal if vector fields $X_1, \ldots, X_k$ in $(M, \mathcal{A})$ and functions $f_1, \ldots, f_k \in \mathcal{A}^\infty$ exist such that $X = f_1 X_1^H + \cdots + f_k X_k^H$. We also set $\frac{d}{dx_i} = (\frac{\partial}{\partial x_i})^H, -n \leq i \leq -1, 1 \leq i \leq m$.

Let $\Omega^r = \Omega^r_{\mathcal{A}^\infty}$ be the sheaf of $\mathcal{A}^\infty$-modules of graded differential forms of degree r on $(J^\infty(\mathcal{B}/\mathcal{A}), \mathcal{A}^\infty)$, and let Ω^v be the subsheaf of $\Omega = \Omega^1$ determined by the linear forms ω such that $\omega(X^H) = 0$ for every $X \in \mathrm{Der}(\mathcal{A})$. We have a canonical decomposition

$$\Omega = (\mathcal{A}^\infty \otimes_{\mathcal{A}} \Omega_{\mathcal{A}}) \oplus \Omega^v,$$

and taking the r^{th} exterior power, we obtain: $\Omega^r = \bigoplus_{k+\ell=r} \Omega^\ell_k$, where $\Omega^\ell_k = \Omega^k_{\mathcal{A}} \otimes_{\mathcal{A}} \Lambda^\ell(\Omega^v)$ is the sheaf of $(k+\ell)$-forms k times horizontal and ℓ times vertical.

Let us denote by $d : \Omega^r \to \Omega^{r+1}$ the graded exterior differential. There exist unique $\mathbb{R}$–linear operators

$$D : \Omega^\ell_k \to \Omega^\ell_{k+1}, \quad \partial : \Omega^\ell_k \to \Omega^{\ell+1}_k$$

defined for every $k, \ell \in \mathbb{N}$, such that $d\omega = D\omega + \partial\omega$ for every $\omega \in \Omega^\ell_k$.

2.4. The splitting produced by a section. As is well known ([**Ba**]), the structure morphism $\mathcal{A} \to C^\infty_M$, $f \to \tilde{f}$, always admits a global section $\sigma : C^\infty_M \to \mathcal{A}$. Once a section σ has been fixed, we have a bi-graduation of each Ω^ℓ_k,

$$\Omega^\ell_k = \bigoplus_{p+q=k} \Omega^\ell_{p,q},$$

defined as follows: The structure morphism induces a homomorphism of sheaves of $\mathcal{A}$-modules

$$\kappa_\sigma : \Omega_\mathcal{A} \to \mathcal{A} \otimes_{C^\infty_M} \Omega_M,$$

where the tensor product is taken with respect to σ, and σ induces a splitting of the exact sequence

$$0 \to \ker \kappa_\sigma \to \Omega_\mathcal{A} \to \mathcal{A} \otimes_{C^\infty_M} \Omega_M \to 0.$$

Hence,

$$\Omega^k_\mathcal{A} = \bigoplus_{p+q=k} (\mathcal{A} \otimes_{C^\infty_M} \Omega_M) \otimes \Lambda^q(\ker \kappa_\sigma).$$

We set :

$$\Omega^\ell_{p,q} = (\mathcal{A} \otimes_{C^\infty_M} \Omega_M) \otimes \Lambda^q(\ker \kappa_\sigma) \otimes \Lambda^\ell(\Omega^v).$$

2.5. Two kinds of integration. Let $(M, \mathcal{A})$ be a graded manifold of dimension (m, n). Assume the underlying manifold M is oriented. For every graded m-form ω on $(M, \mathcal{A})$ with compact support we set:

$$\int_{(M,\mathcal{A})} \omega = \int_M \tilde{\omega},$$

where $\tilde{\omega}$ is the image of ω in the canonical homomorphism $\Omega^m_\mathcal{A}(M) \to \Omega^m(M)$.

Moreover, for every section $\xi \in \mathrm{Ber}(M, \mathcal{A})$ of the Berezinian sheaf of $(M, \mathcal{A})$ with compact support, we denote by

$$\int_{\mathrm{Ber}} \xi$$

the integral of ξ (over M) in the sense of Berezin.

We recall ([**HM4**], [**Mo**]) the following construction of $\mathrm{Ber}(M, \mathcal{A})$.

Let $P^k(\mathcal{A})$ be the sheaf of differential operators on $\mathcal{A}$ of order $\leq k$. We have,

$$\mathrm{Ber}(M, \mathcal{A}) = \Omega^m_\mathcal{A} \otimes_\mathcal{A} P^n(\mathcal{A})/K_n,$$

where K_n is the subsheaf of right $\mathcal{A}$-modules of the operators P such that for every $f \in \mathcal{A}(U)$ with compact support, there exists an ordinary $(m-1)$-form with compact support $\omega \in \Omega^{m-1}(U)$ which satisfies $d\omega = \widetilde{P(f)}$, U being an arbitrary open subset of M. We denote by $[P]$ the coset of a differential operator $P \in \Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} P^n(\mathcal{A})$ in the Berezinian sheaf. Then,

$$[dx_1 \wedge \cdots \wedge dx_m \otimes \frac{\partial}{\partial x_{-1}} \circ \cdots \circ \frac{\partial}{\partial x_{-n}}]$$

is a basis of $\mathrm{Ber}(\mathbb{R}^{m|n})$. Moreover, we have:

$$\int_{\mathrm{Ber}} [P] = \int_M P(1).$$

Consequently, for every graded function $f = \sum_\beta f_\beta x^{-\beta} \in C^\infty(\mathbb{R}^m) \otimes \Lambda(\mathbb{R}^n)$ with compact support, we have:

$$\int_{\mathbb{R}^{m|n}} dx_1 \wedge \cdots \wedge dx_m . f = \int_{\mathbb{R}^m} f_0 d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m.$$

$$\int_{\mathrm{Ber}} [dx_1 \wedge \cdots \wedge dx_m \otimes \frac{\partial}{\partial x_{-1}} \circ \cdots \circ \frac{\partial}{\partial x_{-n}}] f$$
$$= (-1)^{\binom{n}{2}} \int_{\mathbb{R}^m} f_{(1,2,\ldots,n)} d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m.$$

This shows that $\int_{(M,\mathcal{A})}$ integrates on the first component of f, while $\int_{\mathrm{Ber}}$ integrates on the last component of f.

2.6.- $\mathrm{Ber}^\infty(M, \mathcal{A})$. (cf. [Mo]) Given a section P of $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} P^n(\mathcal{A})$, we call the differential operator $P^H : \mathcal{A}^\infty \to \Omega^0_m$ uniquely determined by

$$j^\infty(s)^* = (P^H(f)) = P(j^\infty(s)^*(f))$$

for every $f \in \mathcal{A}^\infty$ and every local section s of p, the horizontal lifting of p.

Let $PH^n(\mathcal{A}^\infty)$ be the right $\mathcal{A}^\infty$-module generated by the horizontal liftings P^H in $P^n(\mathcal{A}^\infty)$, and let $K_n H(\mathcal{A}^\infty)$ be the right $\mathcal{A}^\infty$-module generated by the horizontal liftings of elements of K_n.

We set:

$$\mathrm{Ber}^\infty(M, \mathcal{A}) = PH^n(\mathcal{A}^\infty)/K_n H(\mathcal{A}^\infty).$$

Locally

$$[dx_1 \wedge \cdots \wedge dx_m \otimes \frac{d}{dx_{-1}} \circ \cdots \circ \frac{d}{dx_{-n}}]$$

is a basis of $\mathrm{Ber}^\infty(M, \mathcal{A})$.

Given a section s of $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ and a differential operator $P : \mathcal{A}^\infty \to \Omega^0_m$ of order $\leq n$, we define $(j^\infty s)^*(P) : \mathcal{A} \to \Omega^m_{\mathcal{A}}$ as follows: $(j^\infty s)^*(P)(f) = (j^\infty s)^*(P(f))$.

It follows from the very definitions that

LEMMA 2.1. *For every section s of p we have $(j^\infty s)^*(K_n H(\mathcal{A}^\infty)) \subset K_n$.*

Accordingly, there exists a unique morphism

$$(j^\infty s)^* : \mathrm{Ber}^\infty(M, \mathcal{A}) \to \mathrm{Ber}(M, \mathcal{A})$$

such that $(j^\infty s)^*[P] = [(j^\infty s)^*(P)]$.

Moreover, there exists a unique isomorphism of $\mathcal{A}^\infty$-modules

$$\phi : \mathrm{Ber}(M, \mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^\infty \to \mathrm{Ber}^\infty(M, \mathcal{A})$$

such that $\phi([P] \otimes 1) = [P^H]$. Consequentely, we can define the Lie derivative of a section $\xi = [P^H \circ f]$ with respect to a vector field X on $\mathcal{A}^\infty$, vertical over $(M, \mathcal{A})$, by imposing that

$$\mathcal{L}_X(\xi) = (-1)^{|P||X|}[P^H \circ X(f)]$$

for homogeneous P and X.

3. Graded Lagrangian densities

Assume M is an oriented differentiable manifold.

Every global section ω of the sheaf $\Omega^0_m = \Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{A}^\infty$ gives rise to a functional $\mathbb{L}^\omega$ defined by the formula

$$\mathbb{L}^\omega(s) = \int_{(M,\mathcal{A})} (j^\infty s)^* \omega$$

on the space of the sections of $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ for the which the above integral converges.

Given a section s, we can define a linear functional called the *first variation of* $\mathbb{L}^\omega$ *at* s,

$$\delta_s \mathbb{L}^\omega : \mathrm{Der}_c(\mathcal{B}/\mathcal{A}) \to \mathbb{R}$$

as follows:

$$\delta_s \mathbb{L}^\omega(X) = \int_{(M,\mathcal{A})} (j^\infty s)^*(\mathcal{L}_{X_{(\infty)}} \omega),$$

where $\mathrm{Der}_c(\mathcal{B}/\mathcal{A})$ is the ideal of vector fields with compact support.

A section s is said to be a *graded critical section* for the functional $\mathbb{L}^\omega$ if $\delta_s \mathbb{L}^\omega = 0$; i.e., if the first variation of $\mathbb{L}^\omega$ vanishes at s.

DEFINITION. Two m-forms ω, ω' in Ω^0_m are said to be *equivalent* if $\delta_s \mathbb{L}^\omega = \delta_s \mathbb{L}^{\omega'}$ for every section s of p.

REMARK. Note that $\mathbb{L}^{\omega - \omega'} = \mathbb{L}^\omega - \mathbb{L}^{\omega'}$; hence ω is equivalent to ω' (in short, $\omega \sim \omega'$) if and only if $\delta_s \mathbb{L}^{\omega - \omega'} = 0$ for every section.

NOTATION. Let us denote by $\mathcal{N}$ the set of m-forms ω in Ω^0_m such that $\delta_s \mathbb{L}^\omega = 0$ for every section s. That is, the elements of $\mathcal{N}$ define trivial variational problems in the sense that all the sections are critical. Note that $\mathcal{N}$ is just a real vector space.

We are basically interested in the quotient space $\Omega^0_m(J^\infty)/\mathcal{N}$.

PROPOSITION 3.1. *Let* ν *be a volume form on the underlying base manifold* M, *and let* $\sigma : C^\infty_M \to \mathcal{A}$ *be a global section of the structure morphism. Each*

m-form ω in Ω^0_m is equivalent to a form of the type $L\sigma^*(\nu)$ where L is a graded function in $\mathcal{A}^\infty$.

PROOF. By means of σ, we have a decomposition $\omega = \sum_{p+q=m} \omega^0_{p,q}$, where $\omega^0_{p,q}$ is a section of the sheaf $\Omega^0_{p,q}$. Hence, for every $X \in \mathrm{Der}_c(\mathcal{B}/\mathcal{A})$ and every section s, we have $((j^\infty s)^*(\mathcal{L}_{X_{(\infty)}}\omega_{p,q}))^\sim = 0$, whenever $q > 0$.

Consequently, ω is equivalent to $\omega^0_{m,0}$, thus finishing the proof. □

According to the above proposition, in order to deduce the Euler-Lagrange equations for a graded Lagrangian density we can confine ourselves to the Lagrangians of the type $L\sigma^*(\nu)$, or in local coordinates

$$Ldx_1 \wedge \cdots \wedge dx_m.$$

THEOREM 3.1. *Assume L is of order r globally; that is, L factors through $p_{\infty,r} : J^\infty(\mathcal{B}/\mathcal{A}) \to J^r(\mathcal{B}/\mathcal{A})$. Let $(x_j, y_i), -n \le j \le m, j \ne 0; -n_1 \le i \le m_1, i \ne 0$, be a fibred graded coordinate system for the submersion $p : (N, \mathcal{B}) \to (M, \mathcal{A})$, and let us denote by (y^i_α) the induced graded coordinate system in $J^\infty(\mathcal{B}/\mathcal{A})$, where α stands for a multiindex $\alpha = (\alpha^+, \alpha^-) \in \mathbb{N}^m \times \{0,1\}^n$, with $|\alpha| = |\alpha^+| + |\alpha^-|$, $|\alpha^+| = \alpha^+_1 + \cdots + \alpha^+_m$ and $|\alpha^-|$ is the number of ones in α^-.*

With the above notations, a section s is critical for the graded Lagrangian density $Ldx_1 \wedge \cdots \wedge dx_m$ if and only if it satisfies the following equations:

$$\left[(j^{2r}s)^* \left(\sum_{|\alpha^+|=0}^{r-|\alpha^-|} (-1)^{|\alpha^+|} \frac{d^{|\alpha^+|}}{dx^{\alpha^+}} \left(\frac{\partial L}{\partial y^i_{(\alpha^+,\alpha^-)}} \right) \right) \right]^\sim = 0,$$

for every i, α^- such that $-n_1 \le i \le m_1, i \ne 0, 0 \le |\alpha^-| \le \min\{r, n\}$.

For the proof of this theorem see **[Mo]**, Th. 4.2.

REMARKS.

(1) The number of the above equations is $(m_1 + n_1)\sum_{h=0}^{\min\{r,n\}} \binom{n}{h}$. That is, $(m_1 + n_1)2^n$ if $n \le r$, and $(m_1 + n_1)\sum_{h=0}^{r} \binom{n}{h}$ if $r \le n$. Note that in the case $r \le n$ the number of equations depends on the order of the problem.

(2) For $\alpha^- = 0$, we obtain $m_1 + n_1$ "standard" Euler–Lagrange equations for the even coordinates:

$$\left[(j^{2r}s)^* \left(\sum_{|\alpha^+|=0}^{r} (-1)^{|\alpha^+|} \frac{d^{|\alpha^+|}}{dx^{\alpha^+}} \left(\frac{\partial L}{\partial y^i_{(\alpha^+,0)}} \right) \right) \right]^\sim = 0.$$

Nevertheless, the rest of equations for $|\alpha^-| > 0$ have no counterpart in the ungraded theory.

(3) In the case $r \le n$, the equations corresponding to the indices α^- of length $|\alpha^-| = r$ can be explicitly written down as follows:

$$\left[(j^r s)^* \left(\frac{\partial L}{\partial y^i_{(0,\alpha^-)}} \right) \right]^\sim = 0, \quad -n_1 \le i \le m_1, i \ne 0; |\alpha^-| = r,$$

thus showing that in this case the above equations can be understood as previous constraints on the Lagrangian density (for $r = 1$ see [**HM2**]).

EXAMPLES.

(1) (1, 1)-Graded Mechanics. Assume $(M, \mathcal{A}) = \mathbb{R}^{1|1}$ with coordinates (t, s), $|t| = 0$, $|s| = 1$, and $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ is the canonical projection of $\mathbb{R}^{1|1} \times (Q, \mathcal{C})$ onto its first factor. We denote by (q_i), $-n_1 \leq i \leq m_1, i \neq 0$, the graded coordinates in Q.

Then, the above equations become

$$\begin{cases} \left[(j^{2r}s)^* \sum_{k=0}^{r} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^i_{(k,0)}} \right) \right]^{\sim} = 0, \\ \left[(j^{2r}s)^* \sum_{k=0}^{r-1} (-1)^k \frac{d^k}{dt^k} \left(\frac{\partial L}{\partial q^i_{(k,1)}} \right) \right]^{\sim} = 0. \end{cases}$$

These are $2(m_1 + n_1)$ equations.

(2) Assume $n = 0$, i.e., $\mathcal{A} = C^\infty_M$. Then, the equations are:

$$\left[(j^{2r}s)^* \left(\sum_{|\alpha^+|=0}^{r} (-1)^{|\alpha^+|} \frac{d^{|\alpha^+|}}{dx^{\alpha^+}} \left(\frac{\partial L}{\partial y^i_{\alpha^+}} \right) \right) \right]^{\sim} = 0.$$

4. Berezinian Lagrangian densities

Again, let us assume that M is oriented. Every global section ξ of $\mathrm{Ber}^\infty(M, \mathcal{A})$ gives rise to a functional $\mathbb{L}^\xi$ defined by the formula

$$\mathbb{L}^\xi(s) = \int_{\mathrm{Ber}} (j^\infty s)^* \xi \qquad \text{(cf. Lemma 2.1)}$$

on the space of the sections of $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ for which the above integral converges.

Given a section s, we can define a linear functional, called the *first variation of $\mathbb{L}^\xi$ at s*,

$$\delta_s \mathbb{L}^\xi : \mathrm{Der}_c(\mathcal{B}/\mathcal{A}) \to \mathbb{R}$$

as follows

$$\delta_s \mathbb{L}^\xi(X) = \int_{\mathrm{Ber}} (j^\infty s)^* (\mathcal{L}_{X_{(\infty)}} \xi).$$

A section s is said to be a *Berezinian critical section* for the functional $\mathbb{L}^\xi$ if $\delta_s \mathbb{L}^\xi = 0$.

COMPARISON THEOREM. *Let $\xi = [P]$ be a global section of* $\mathrm{Ber}^\infty(M, \mathcal{A})$. *We have:*

$$\delta_s \mathbb{L}^\xi = \delta_s \mathbb{L}^{P(1)} \qquad \textit{for every section } s.$$

Consequently, we can associate in a canonical way to each Berezinian Lagrangian density $\xi = [P]$ a coset of equivalent Graded Lagrangian densities $P(1) + \mathcal{N}$ in $\Omega^0_m(J^\infty)/\mathcal{N}$.

PROOF. We have $P = P_i^H \circ f^i$ for some P_i in $\Omega^m_{\mathcal{A}} \otimes_{\mathcal{A}} P^n(\Omega)$ and some functions f^i in $\mathcal{A}^\infty$. Hence for every $X \in \mathrm{Der}(\mathcal{B}/\mathcal{A})$ we have:

$$\begin{aligned}\delta_s \mathbb{L}^\xi(X) &= \int_{\mathrm{Ber}} (j^\infty s)^*(\mathcal{L}_{X_{(\infty)}}[P_i^H \circ f^i]) \\ &= (-1)^{|P_i||X|} \int_M \left((j^\infty s)^* P_i^H(X_{(\infty)} f^i)\right)^\sim \\ &= \int_M \left((j^\infty s)^* \mathcal{L}_{X_{(\infty)}}(P_i^H(f^i))\right)^\sim \\ &= \int_{(M,\mathcal{A})} (j^\infty s)^* \mathcal{L}_{X_{(\infty)}}(P(1)) = \delta_s \mathbb{L}^{P(1)}(X). \quad \square\end{aligned}$$

REMARKS.

(1) Locally, if $\xi = [dx_1 \wedge \cdots \wedge dx_m \otimes \frac{d}{dx_{-1}} \circ \cdots \circ \frac{d}{dx_{-n}}].f$ then $P(1) = (dx_1 \wedge \cdots \wedge dx_m)L$, with

$$L = \frac{d^n f}{dx_1 \ldots dx_{-n}}, \quad \text{(cf. [Mo],Th. 5.2).}$$

The above formula shows that the order of the Graded Lagrangian density associated to an r-order Berezinian Lagrangian density is $r + n$. In other words, the mapping $\xi = [P] \to P(1) + \mathcal{N}$ increases in general by n the order of the variational problem. Consequently, although we start with considering first order Berezinian Lagrangian densities exclusively, the associated Graded Lagrangian densities can be of higher order (actually, of order $\leq n+1$). This fact also explains the role of higher order Graded Lagrangians in the theory.

(2) The mapping $\mathrm{Ber}^\infty(M, \mathcal{A}) \to \Omega^0_m/\mathcal{N}$ is neither injective nor surjective. It remains as an open problem to characterize its image.

COROLLARY 4.1. *Let ξ be a Berezinian Lagrangian density. Assume that $\xi = [dx_1 \wedge \cdots \wedge dx_m \otimes \frac{d}{dx_{-1}} \circ \cdots \circ \frac{d}{dx_{-n}}].f$, locally, for some graded function $f \in \mathcal{A}^r$. Then, a section s is critical for the Berezinian density ξ if and only if it satisfies the following equations:*

$$(j^{2r} s)^* \left(\sum_{|\alpha|=0}^{r} (-1)^{I(\alpha,i)} \frac{d^{|\alpha|}}{dx^\alpha} \left(\frac{\partial L}{\partial y^i_\alpha} \right) \right) = 0,$$

where $I(\alpha, i) = |\alpha^+| + (1 + |y_i|)|\alpha^-|)$, for $-n_1 \leq i \leq m_1, i \neq 0$, where we have used the same notations as in Theorem 4.1.

For the proof see [Mo], Th. 6.3.

REMARKS.

(1) Unlike the Euler-Lagrange equations of a Graded Lagrangian density, we note that the above equations are equations in the graded ring $\mathcal{A}$ of the base manifold.

(2) Also note that these equations are the standard graded Euler-Lagrange equations that one could expect in the graded setting.

(3) The number of these equations is $m_1 + n_1$, but each of them gives rise to 2^n scalar equations.

5. Scalars superfields

The graded manifold $\mathbb{R}^{1|1} = (\mathbb{R}, C^\infty(\mathbb{R}) \otimes \Lambda \mathbb{R})$ plays the role of the scalars in the graded setting. Actually, if $(M, \mathcal{A})$ is an arbitrary graded manifold we have

$$\mathrm{Mor}((M,\mathcal{A}), \mathbb{R}^{1|1}) = \Gamma(M, \mathcal{A}),$$

where $\mathrm{Mor}((M,\mathcal{A}),(N,\mathcal{B}))$ stands for the set of morphisms of the graded manifold $(M,\mathcal{A})$ to the graded manifold $(N,\mathcal{B})$ (cf. [**SV**]).

Let $p : (M,\mathcal{A}) \times \mathbb{R}^{1|1} \to (M,\mathcal{A})$ be the canonical projection onto the first factor. There is a natural bijection between the sections of p and the graded functions on M; i.e., $\Gamma(p) = \Gamma(M,\mathcal{A})$.

Let us assume $(M,\mathcal{A})$ is endowed with a non-singular graded symmetric metric g of degree zero. This forces the odd dimension of $(M,\mathcal{A})$ to be even; i.e., $\dim(M,\mathcal{A}) = (m, 2n)$. The metric g induces a pseudo-Riemannian metric $\tilde{g}$ on the underlying manifold M.

A local basis $(X_1, \dots, X_m, Y_1, \bar{Y}_1, \dots, Y_n, \bar{Y}_n), |X_i| = 0, |Y_J| = |\bar{Y}_J| = 1$, of graded vector fields is *graded-orthonormal* for g if the matrix of g in this basis is as follows

$$\begin{pmatrix} Id_{m^+} & 0 & 0 & 0 & \dots & 0 \\ 0 & -Id_{m^-} & 0 & 0 & \dots & 0 \\ 0 & 0 & J_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & J_n \end{pmatrix}$$

where Id_r stands for the $r \times r$ identity matrix and $J_1 = \dots = J_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Note that (m^+, m^-) coincides with the signature of $\tilde{g}$, so that $m = m^+ + m^-$.

Let us put $((-1)^{m^-} \mathrm{Ber}(G))^{\frac{1}{2}} = |G|$.

Locally, graded-orthonormal bases always exist.

PROPOSITION 5.1. *Let $(X_i, Y_J, \bar{Y}_J)$ be a graded orthonormal basis for g with dual basis $(\omega_i, \eta_J, \bar{\eta}_J)$ on the graded manifold $(M,\mathcal{A})$. Assume M is oriented and $\tilde{\omega}_1 \wedge \dots \wedge \tilde{\omega}_m$ belongs to the orientation of M. The coset defined by*

$$\omega_1 \wedge \dots \wedge \omega_m \otimes Y_1 \circ \bar{Y}_1 \circ \dots \circ Y_n \circ \bar{Y}_n$$

in the Berezinian sheaf does not depend on the graded orthonormal basis chosen, thus defining a global section ξ_g of $\mathrm{Ber}(M, \mathcal{A})$ *which will be called the Berezinian volume element associated to g.*

If $\{x_i\}, -2n \leq i \leq m, i \neq 0$, is an arbitrary graded coordinate system such that $d\tilde{x}_1 \wedge \cdots \wedge d\tilde{x}_m$ belongs to the orientation of M, then

$$\xi_g = [dx_1 \wedge \cdots \wedge dx_m \otimes \frac{\partial}{\partial x_{-1}} \circ \cdots \circ \frac{\partial}{\partial x_{-n}}]|G|$$

where G is the matrix of g with respect to $(\frac{\partial}{\partial x_i})$. (cf. **[Ma]**, chapter 4, Lemma 7.7)

Assume $n > 1$. We can define a unique global section L of the sheaf $\mathcal{A}^1$ of graded functions on the 1-jet bundle of local section of the projection p, by imposing the following two conditions:

(1) L is quadratic with respect to the graded vector bundle structure $J^1((M, \mathcal{A}), \mathbb{R}^{1|1}) \to (M, \mathcal{A}) \times \mathbb{R}^{1|1}$.

(2) $j^1(1_M, f)^*(L) = \frac{1}{2} g^2(df, df)$ for every local section f of $\mathcal{A}$, where g^2 stands for the contravariant metric induced by $g = g_2$ on the graded cotangent bundle of $(M, \mathcal{A})$.

THEOREM 5.1. *With the above notations, the Euler-Lagrange equation of the Berezinian Lagrangian density $\xi_g.L$ is given by*

$$\Delta(f) = 0,$$

where $\Delta(f) = \mathrm{div}(\mathrm{grad}\, f)$, *the gradient of a graded function being defined as usual, and the divergence (with respect to ξ_g) of an arbitrary graded vector field X by means of the formula*

$$\mathcal{L}_X(\xi_g) = \xi_g . \mathrm{div}(X).$$

(Recall that the odd dimension of $(M, \mathcal{A})$ is even; cf. **[HM3]**, 2.4; **[Le]** 2.4.6.)

PROOF. Let us choice $\{y, z\}, |y| = 0, |z| = 1$ as coordinates for $\mathbb{R}^{1|1}$, and $\{x_i\} - 2n \leq i \leq m, i \neq 0$ as coordinates for $(M, \mathcal{A})$.

After corollary 4.1, the Euler-Lagrange equations for a Berezinian Lagrangian density

$$[dx_1 \wedge \cdots \wedge dx_m \otimes \frac{d}{dx_{-1}} \circ \cdots \circ \frac{d}{dx_{-2n}}].F,$$

where $F \in \mathcal{A}^1$, are

$$\frac{\partial F}{\partial y} - \sum_{i=1}^{m} \frac{d}{dx_i} \frac{\partial F}{\partial y_i} - \sum_{i=-2n}^{-1} \frac{d}{dx_i} \frac{\partial F}{\partial y_i} = 0,$$

$$\frac{\partial F}{\partial z} - \sum_{i=1}^{m} \frac{d}{dx_i} \frac{\partial F}{\partial z_i} + \sum_{i=-2n}^{-1} \frac{d}{dx_i} \frac{\partial F}{\partial z_i} = 0.$$

In this case

$$F = |G|L = \frac{1}{2}|G| \sum_{i,j=-2n}^{m} (-1)^{|x_i||x_j|}(y_i y_j + y_i z_j + (-1)^{|x_j|} z_i y_j + (-1)^{|x_j|} z_i z_j).$$

Thus, the equations are reduced to

$$\sum_{k=-2n}^{m} \frac{d}{dx_k}\frac{\partial F}{\partial y_k} = 0, \qquad \sum_{k=-2n}^{m} (-1)^{|x_k|}\frac{d}{dx_k}\frac{\partial F}{\partial z_k} = 0.$$

An easy computation shows that

$$\frac{\partial L}{\partial y_k} = (-1)^{|x_k|} \sum_{i=-2n}^{m} g^{ki}(y_i + z_i) \text{ and } \frac{\partial L}{\partial z_k} = \sum_{i=-2n}^{m} g^{ki} y_i.$$

Then, the two variational equations are

$$\sum_{k=-2n}^{m} (-1)^{|x_k|}\frac{d}{dx_k}(g^{ki}(y_i + z_i)|G|) = 0, \qquad \sum_{k=-2n}^{m} (-1)^{|x_k|}\frac{d}{dx_k}(g^{ki} y_i |G|) = 0.$$

Note that the second equation is the even part of the first one, then the variational equations are reduced to the first equation. Now, let us check that this equation is, up to factors, the expression in local coordinates of the graded Laplacian.

If $f \in \mathcal{A}$, then $\operatorname{grad}(f)$ is the graded vector field on $(M, \mathcal{A})$ defined by $g(\operatorname{grad}(f), Z) = df(Z)$ for every graded vector field on $(M, \mathcal{A})$, Z. It is easy to check that

$$\operatorname{grad}(f) = \sum_{i,k=-2n}^{m} (-1)^{|x_i||f|} g^{ik} \frac{\partial f}{\partial x_k}\frac{\partial}{\partial x_i}.$$

If Z is a graded vector field on $(M, \mathcal{A})$ the divergence of Z, $\operatorname{div} Z \in \mathcal{A}$ is defined by

$$\mathcal{L}_Z \xi_g = \xi_g . \operatorname{div} Z,$$

where ξ_g is the Berezinian volume element associated to g and $\mathcal{L}_Z$ is the Lie derivative of the Berezinian sheaf (see **[HM3]**). It is just a matter of computation to show that if $Z = Z^i \frac{\partial}{\partial x_i}$, then

$$\operatorname{div} Z = \frac{1}{|G|} \sum_{i=-2n}^{m} (-1)^{|x_i|(|Z|+1)} \frac{\partial}{\partial x_i} \left(|G| Z^i\right).$$

Now, it is easy to check that the Euler-Lagrange equations are $\Delta(f) = 0$. □

REMARK. In the graded coordinates introduced above, $\Delta(f)$ takes the following local expression

$$\Delta(f) = \frac{1}{|G|} \sum_{i=-2n}^{m} (-1)^i \frac{\partial}{\partial x_i} \left(g^{ij} \frac{\partial}{\partial x_j}(f.|G|) \right).$$

EXAMPLE. Let $(M, \mathcal{A})$ be $\mathbb{R}^{4|4}$ with coordinates (x_i), $-4 \le i \le 4$, $i \ne 0$, and with the metric g of matrix

$$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 & & & & \\ 0 & \epsilon_2 & 0 & 0 & & & & \\ 0 & 0 & \epsilon_3 & 0 & & 0 & & \\ 0 & 0 & 0 & \epsilon_4 & & & & \\ & & & & 0 & -1 & 0 & 0 \\ & & & & 1 & 0 & 0 & 0 \\ & & 0 & & 0 & 0 & 0 & -1 \\ & & & & 0 & 0 & 1 & 0 \end{pmatrix}$$

with $\epsilon_i^2 = 1$, $i = 1, \dots, 4$. Assume $f = f_0 + f_i x_{-i} + f_{ij} x_{-i} x_{-j} + \dots$ is the expansion of $f \in C^\infty(\mathbb{R}^4) \otimes \lambda(\mathbb{R}^4)$. Then, $\Delta f = 0$ if and only if the following equations hold

$$\tilde{\Delta} f_0 = -2(f_{12} + f_{34}),$$
$$\tilde{\Delta} f_1 = -2f_{134},\ \tilde{\Delta} f_2 = -2f_{234},\ \tilde{\Delta} f_3 = -2f_{123},\ \tilde{\Delta} f_4 = -2f_{124},$$
$$\tilde{\Delta} f_{12} = \tilde{\Delta} f_{34} = -f_{1234},\ \tilde{\Delta} f_{ij} = 0 \text{ for } (i,j) \ne (1,2), (3,4),$$
$$\tilde{\Delta} f_{ijk} = 0 \text{ for all } i < j < k,$$
$$\tilde{\Delta} f_{1234} = 0,$$

where $\tilde{\Delta}$ is the ungraded Laplacian associated to the standard metric $\tilde{g} = \sum_{i=1}^4 \epsilon_i dx_i^2$.

COMPARISON REMARK. This is the first of a series of remarks showing, for each application of this theory, that the choice of graded Lagrangian densities instead of Berezinian ones leads us to very different results. The use of graded Lagrangian densities will not produce true graded equations where even and odd parts of the problem interact, but the classical equations of the underlying manifold plus equations affecting only the odd part.

In the case of scalar fields, after theorem 3.1, the Euler-Lagrange equations for a graded Lagrangian density $F dx_1 \wedge \cdots \wedge dx_m$, where $F \in \mathcal{A}^1$, are

$$\left[j^1(1_M, f)^* \left(\frac{\partial F}{\partial y} - \sum_{i=1}^m \frac{d}{dx_i} \frac{\partial F}{\partial y_i} \right) \right]^\sim = 0,$$

$$\left[j^1(1_M, f)^* \left(\frac{\partial F}{\partial z} - \sum_{i=1}^m \frac{d}{dx_i} \frac{\partial F}{\partial z_i} \right) \right]^\sim = 0,$$

$$\left[j^1(1_M, f)^* \left(\frac{\partial F}{\partial y_i} \right) \right]^\sim = 0, \quad \left[j^1(1_M, f)^* \left(\frac{\partial F}{\partial z_i} \right) \right]^\sim = 0,\ i = -2n, \dots, -1.$$

In the same situation of the previous example, the variational equations now are $\tilde{\Delta} f_0 = 0$ and $f_i = 0$, $i = 1, \dots, 4$. Thus, the critical sections are given by harmonic functions of the underlying manifold plus arbitrary terms of degree greater than one in the odd coordinates. Nothing new appears.

6. The second order Berezinian Lagrangian density defined by the Scalar Supercurvature

Let $(M, \mathcal{A})$ be a graded manifold of dimension $(m, 2n)$. We can construct a graded fibre bundle $p : (N, \mathcal{B}) \to (M, \mathcal{A})$ in such a way that the sections of p over every open subset U of M coincide with the non-singular graded symmetric metrics of degree zero and signature (m^+, m^-) on $(U, \mathcal{A}_U)$. Actually, $(N, \mathcal{B})$ can be obtained as a (not vector) subbundle of $S^2T^*(M, \mathcal{A})$.

Assume $n > 1$. We can define a unique graded function R on $J^2(\mathcal{B}/\mathcal{A})$ by imposing the following two conditions:

(1) R is an affine function with respect to the affine bundle structure $J^2(\mathcal{B}/\mathcal{A}) \to J^1(\mathcal{B}/\mathcal{A})$.

(2) For each metric g in $(N, \mathcal{B})$, the pull-back $(j^2g)^*(R)$ coincides with the scalar supercurvature of g.

Note that the scalar supercurvature contains some terms with five factors, which are of the form

$$g^{ij} g^{lk} g^{vu} \frac{\partial g_{ui}}{\partial x_l} \frac{\partial g_{jk}}{\partial x_v}.$$

But all the factors of these terms can not be of degree 1.

Moreover, let us assume the underlying differentiable manifold M is oriented, and let ξ_g be the Berezinian volume element associated to a metric g in N (see proposition 5.1). Then, there exists a unique section ξ of $\mathrm{Ber}^2(M, \mathcal{A}) \subset \mathrm{Ber}^\infty(M, \mathcal{A})$ such that

$$(j^2g)^*(\xi) = \xi_g.$$

THEOREM 6.1. *A graded metric g of the above bundle $(N, \mathcal{B})$ is a Berezinian critical section for the second order Berezinian Lagrangian density $\xi.R$ if and only if* $\mathrm{Ricci}(g) = 0$ (cf. [**ANZ**]).

PROOF. This result can be obtained from the equations of the corollary 4.1 by a brute force computation for this particular case. Nevertheless, it is better to proceed directly using the method explained in the general case.

We also note that we treat $\xi.R$ as a second order density; in other words, we shall not consider the Palatini method.

First let us recall the local expression of the Scalar Supercurvature. Christoffel symbols are defined by

$$\Gamma_{ijk} = \frac{1}{2}[g_{ij,k} + (-1)^{|x_j||x_k|} g_{ik,j} - (-1)^{|x_i|(|x_j|+|x_k|)} g_{jk,i}] = (-1)^{|x_j||x_k|}\Gamma_{ikj}.$$

The components of the curvature tensor field

$$\begin{aligned} R^\ell_{ijk} &= -\Gamma^\ell_{ij;k} + (-1)^{|x_j||x_k|}\Gamma^\ell_{ik;j} = \\ &= -\Gamma^\ell_{ij,k} + (-1)^{|x_j||x_k|}\Gamma^\ell_{ik,j} + (-1)^{|x_j|(|x_m|+|x_i|)} \\ &\quad \times \Gamma^\ell_{mj}\Gamma^m_{ik} - (-1)^{|x_k|(|x_m|+|x_i|+|x_j|)}\Gamma^\ell_{mk}\Gamma^m_{i\ell}. \end{aligned}$$

The components of the Ricci tensor field

$$R_{ij} = (-1)^{|x_k|(|x_i|+1)} R^k_{ikj} = -(-1)^{|x_k|} \Gamma^k_{ki;j} + (-1)^{|x_k|(|x_i|+|x_j|+1)} \Gamma^k_{ij;k}.$$

And the curvature scalar field $R = R_{ij} g^{ij}$.

NOTE. As usual, the subindex $,k$ denotes partial differentiation with respect to x_k. And the subindex $;k$ denotes covariant differentiation with respect to x_k.

By the Comparison Theorem the Berezinian Lagrangian $\xi_g.R$ has the same critical sections than the graded Lagrangian

$$\frac{d}{dx_{-1}} \circ \dots \frac{d}{dx_{-n}} (|G|.R) dx^1 \wedge \dots \wedge dx^m.$$

Instead of compute the Euler-Lagrange equations for the graded Lagrangian density, we will borrow the deduction of the equations showed in **[Mo]**.

We will denote by g_{ij} the fibre coordinates. The jet coordinates are denoted by $g_{ij,\gamma}$. Note that $|g_{ij}| = (-1)^{|x_i|+|x_j|}$ and $|g_{ij,\gamma}| = (-1)^{|x_i|+|x_j|+|\gamma^-|}$.

The first step is to compute the exterior derivative:

$$\begin{aligned} d(\frac{d}{dx_{-1}} \circ \dots \frac{d}{dx_{-n}} (R.|G|)) &= d(\mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} (R.|G|)) \\ &= \mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} d(R.|G|). \end{aligned}$$

By the same reasonings of **[Wi]** page 115, we know that $d(R.|G|) =$

$$= |G|(-1)^{|x_i|} \left([-(-1)^{|x_k|} d\Gamma^k_{kj} g^{ji} + d\Gamma^i_{kj} g^{jk}]_{;i} - (R^{ij} - \frac{1}{2} g^{ij} R) dg_{ij} \right).$$

Let $V^i = [-(-1)^{|x_k|} d\Gamma^k_{kj} g^{ji} + d\Gamma^i_{kj} g^{jk}]$. Due to the definition of covariant differentiation and to the formula $\Gamma^i_{ij} = (\ln |G|)_j$ it is easy to see that $|G| V^i_{;i} = (|G| V^i)_i$. Thus

$$\mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} (|G|(-1)^i V^{|x_i|}_{;i}) \mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} (|G|(-1)^{|x_i|} V^i)_i.$$

The sum goes for indexes form $-2n$ to m, but the terms with a negative index vanish due to a repetition of the derivation with respect such coordinate. Therefore we get

$$\sum_{i=1}^{m} (\mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} \frac{d}{dx_i} (|G| V^i)) dx_1 \wedge \dots \wedge dx_m,$$

and this form does not contribute to the variational equations.

The term that will give rise to the variational equations is

$$\mathcal{L}_{\frac{d}{dx_{-1}}} \circ \dots \circ \mathcal{L}_{\frac{d}{dx_{-n}}} \left(|G|(-1)^{|x_i|} (R^{ij} - \frac{1}{2} g^{ij} R) \right) dg_{ij} \wedge dx_1 \wedge \dots \wedge dx_m.$$

Thus, the Euler-Lagrange equation is

$$\sum_{0 \le |\gamma| \le 2} (-1)^{I(\gamma, ij)} \frac{d^{|\gamma|}}{dx^\gamma} \frac{\partial L}{\partial g_{ij,\gamma}} = -(R^{ij} - \frac{1}{2} g^{ij} R) = 0,$$

for all $i, j = -2n, \ldots, m,\ i, j \neq 0$, where $I(\gamma, ij) = |\gamma^+| + |\gamma^-|(|x_i| + |x_j| + 1)$.

Note that $R^{ij} - \frac{1}{2}g^{ij}R = 0$ for all $i, j = -2n, \ldots, m,\ i, j \neq 0$ implies $R^{ij} = 0$. □

7. Supergeodesics

Let $(M, \mathcal{A})$ be a graded manifold of dimension $(m, 2n)$ endowed with a non-singular, graded symmetric metric g of degree zero, and let us denote by (t, s), $|t| = 0, |s| = 1$, the standard graded coordinates in $\mathbb{R}^{1|1}$.

We define below a Berezinian Lagrangian density on the graded fibred manifold given by the projection $p : \mathbb{R}^{1|1} \times (M, \mathcal{A}) \to \mathbb{R}^{1|1}$ onto the first factor.

We recall that $\frac{d}{dt} = (\frac{\partial}{\partial t})^H$ is a derivation mapping $\mathcal{A}^r$ into $\mathcal{A}^{r+1}$. In particular, we shall consider $\frac{d}{dt}$ from $\mathcal{A}^0$ into $\mathcal{A}^1$. Then, $(\frac{d}{dt})^v = \frac{d}{dt} - \frac{\partial}{\partial t}$ is a derivation from $\mathcal{A}^0$ into $\mathcal{A}^1$, vertical over $C^\infty(\mathbb{R}) \otimes \Lambda(\mathbb{R})$.

Since $\mathrm{Der}_{C^\infty(\mathbb{R})\otimes\Lambda(\mathbb{R})}(\mathcal{A}^0, \mathcal{A}^1) = \mathrm{Der}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^1$, we can define a graded function L in $\mathcal{A}^1$ by setting

$$L = \frac{1}{2} g \left((\frac{d}{dt})^v, (\frac{d}{dt})^v \right).$$

Note that $|L| = 0$, because $|g| = 0$. Locally, we have

$$L = \frac{1}{2}(-1)^{|x_i|+|x_j|+|x_i||x_j|} g_{ij} x_t^i x_t^j.$$

REMARK. Unlike the previous cases (see sections 5,6), we can not define L by prescribing its values along the holonomic sections, because the degree of L (which is 2) is greater than the odd dimension of the base manifold (which is 1!).

THEOREM 7.1. *A curve $\gamma : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ is a Berezinian critical section for the Berezinian density $[dt \otimes \frac{d}{ds}].L$ if and only if γ satisfies the following equations:*

$$(j^2\gamma)^*(x_{tt}^k + x_t^i x_t^j \Gamma_{ij}^k) = 0, \quad -2n \leq k \leq m, k \neq 0,$$

where Γ_{ij}^k are the graded Christoffel symbols of g in the graded coordinate system $\{x_i\}$, $-2n \leq k \leq m, k \neq 0$.

PROOF. We have

$$2\frac{\partial L}{\partial x_t^k} = (-1)^{|x_j|} g_{kj} x_t^j + (-1)^{|x_i|+|x_i||x_j|} g_{ik} x_t^i,$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_t^k}\right) = (-1)^{|x_i|(|x_j|+|x_k|+1)+|x_j|} \frac{\partial g_{kj}}{\partial x_i} x_t^i x_t^j +$$

$$+(-1)^{|x_i|+|x_j|+|x_k|(|x_i|+|x_j|)} \frac{\partial g_{ik}}{\partial x_j} x_t^i x_t^j + (-1)^{|x_j|} g_{kj} x_t^j,$$

$$2\frac{\partial L}{\partial x_k} = (-1)^{|x_i|+|x_j|+|x_i||x_j|} \frac{\partial g_{ij}}{\partial x_k} x_t^i x_t^j.$$

Hence, the equation $\frac{\partial L}{\partial x_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_t^k}\right) = 0$ can be written as:

$$2x_{tt}^j g_{jk} + x_t^i x_t^j \left(\frac{\partial g_{ik}}{\partial x_j} + (-1)^{|x_i||x_j|}\frac{\partial g_{jk}}{\partial x_i} - (-1)^{|x_k|(|x_i|+|x_j|)}\frac{\partial g_{ij}}{\partial x_k}\right) = 0,$$

and we can conclude taking into account the following formulas ([**dW**])

$$2\Gamma_{ik}^{\ell} g_{\ell j} = \frac{\partial g_{ij}}{\partial x_k} + (-1)^{|x_i||x_j|}\frac{\partial g_{jk}}{\partial x_i} - (-1)^{|x_j|(|x_i|+|x_k|)}\frac{\partial g_{ki}}{\partial x_j}. \quad \square$$

COROLLARY 7.1. *The equations in the above theorem for a curve* $\gamma : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ *are equivalent to*

$$\nabla_T T = 0,$$

where $T = \gamma_*(\frac{\partial}{\partial t}) : \mathcal{A} \to C^\infty(\mathbb{R}) \otimes \Lambda(\mathbb{R})$ *is the derivation given by* $T(f) = \frac{\partial}{\partial t}(\gamma^*(f))$, *for every graded function* f *in* $\mathcal{A}$.

The proof is straightforward and thus it is omitted.

COMPARISON REMARK. The Euler-Lagrange equations for the graded Lagrangian density Ldt are

$$\left[(j^2\gamma)^*(x_{tt}^k + x_t^i x_t^j \Gamma_{ij}^k)\right]^\sim = 0, \quad -2n \leq k \leq m, k \neq 0.$$

For $k < 0$ the equations are reduced to a trivial identity due to the fact that $x_{tt}^k + x_t^i x_t^j \Gamma_{ij}^k$ is an odd function. For $k > 0$ the equations are the classical equations of the geodesics of the underlying manifold with the underlying metric. Thus, with this approach, graded geodesics are just the geodesics of the underlying manifold. This is an ungraded result coming out from a graded problem.

8. SuperMechanics

Let us consider variational problems on the fibred manifold $p : \mathbb{R}^{1|q} \times (M, \mathcal{A}) \to \mathbb{R}^{1|q}$, $q \geq 1$, where p stands for the canonical projection. Of course, the most interesting case is $q = 1$, to which we shall confine ourselves after giving the notion of regularity.

Let $(t; s_1, \ldots, s_q)$ be the standard coordinates in $\mathbb{R}^{1|q}$, and let $(x_i), -n \leq i \leq m, i \neq 0$, be a graded coordinate system in $(M, \mathcal{A})$.

The Euler-Lagrange equations for a first order Berezinian Lagrange density

$$[dt \otimes \frac{d}{ds_1} \circ \cdots \circ \frac{d}{ds_q}].L,$$

L being a graded function in $\mathcal{A}^1$, are the following:

$$(j^2\gamma)^*\left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x_t^i}\right) - (-1)^{|x_i|}\frac{d}{ds_J}\left(\frac{\partial L}{\partial x_J^i}\right)\right) = 0.$$

Developing, we obtain:

$$(j^2\gamma)^*\left(x^h_{tt}\frac{\partial^2 L}{\partial x^h_t \partial x^i_t} + \text{ terms involving derivatives of order ; 2 with respect to } t\right)$$
$$= 0.$$

Therefore, in order to be able to write down the above system in the form

$$(j^2\gamma)^*(x^h_{tt}) = F_h(\gamma^*(x_i), (j^1\gamma)^*(x^h_t))$$

we must impose the matrix $\left(\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right)$ to be non-singular. In this case, the problem is said to be *regular*. Hence, the Euler-Lagrange equations of a regular Lagrangian are equivalent to a system of ungraded ordinary differential equations.

Let us assume that L is homogeneous. If $|L| = 0$, the regularity condition means

$$\det\left(\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right)^{\sim} \neq 0, \text{ for } h > 0, i > 0, \text{ and } \det\left(\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right)^{\sim} \neq 0, \text{ or } h < 0, i < 0.$$

Note that the second matrix is skew-symmetric, thus forcing n to be even.

If $|L| = 1$, then $m = n$ necessarily and we must have

$$\det\left(\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right)^{\sim} \neq 0, \text{ for } h > 0, i < 0.$$

¿From now onwards, we assume $q = 1$.

THEOREM 8.1. *Let* $\Theta_0 = (dx_i - x^i_t dt - x^i_s ds)\frac{\partial L}{\partial x^i_t} + Ldt$ *be the Poincaré-Cartan form associated with the Graded Lagrangian density* Ldt. *Let* Θ *be the* 1*-form on* $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ *defined by* $\Theta = \mathcal{L}_{\frac{d}{ds}}\Theta_0$.

A curve $\gamma : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ *is a Berezinian critical section for the Berezinian Lagrangian density* $[dt \otimes \frac{d}{ds}].L$ *if and only if:*

$$(j^2\gamma)^*(i_X d\Theta) = 0$$

for every vector field X *on* $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ *vertical over* $\mathbb{R}^{1|1}$.

PROOF. A simple (but rather long) computation shows the following identity:

$$(j^2\gamma)^*(i_X d\Theta) = (j^2\gamma)^*\left(X(x^i_s)\Omega + (-1)^{|x_i|}X(x_i)(\mathcal{L}_{\frac{d}{ds}}\Omega)\right)$$

where

$$\Omega = \left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x^i_t}\right) - (-1)^{|x_i|}\frac{d}{ds_J}\left(\frac{\partial L}{\partial x^i_J}\right)\right) dt. \quad \square$$

REMARK. It follows from the above formula that in the statement of the theorem we can assume that the vector field X is vertical over $\mathbb{R}^{1|1} \times (M, \mathcal{A})$.

In order to develop a true Hamiltonian formalism we should first specify the appropiate fibre bundle on which the Poincaré-Cartan form is defined.

First of all, note that from the very definition of Θ we obtain,

$$\Theta = (dx_s^i - x_{st}^i dt)\frac{\partial L}{\partial x_t^i} + (-1)^{|x_i|}(dx_i - x_t^i dt - x_s^i ds)\frac{d}{ds}\left(\frac{\partial L}{\partial x_t^i}|\right) + \frac{dL}{ds}dt.$$

Hence Θ only depends on the coordinates $t, s, x_i, x_t^i, x_s^i, x_{st}^i$, that is, Θ does not depend on x_{tt}^i.

Let $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A})) \subset J^1(\mathbb{R}^{1|0}, J^1(\mathbb{R}^{0|1}, (M, \mathcal{A})))$ be the sub-bundle defined by $s_t = 0$.

Then, there exists a canonical submersion over $\mathbb{R}^{1|1}$,

$$\pi : J^2(\mathbb{R}^{1|1}, (M, \mathcal{A})) \to J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$$

defined as follows. Each morphism $f : \mathbb{R}^{1|1} \to (M, \mathcal{A})$ induces a family $f_t : \mathbb{R}^{0|1} \to (M, \mathcal{A}), t \in \mathbb{R}$, and, taking jets, $j^1(f_t) : \mathbb{R}^{0|1} \to J^1(\mathbb{R}^{0|1}, (M, \mathcal{A}))$. By composing $j^1(f_t)^*$ with the structure morphism $\Lambda(\mathbb{R}) \to \mathbb{R}$ we obtain $[j^1(f_t)]^* : \mathcal{A}_{J^1(\mathbb{R}^{0|1},(M,\mathcal{A}))} \to \mathbb{R}$.

Let $[j^1(f)]^* : \mathcal{A}_{J^1(\mathbb{R}^{0|1},(M,\mathcal{A}))} \to C^\infty(\mathbb{R})$ be the homomorphism $[j^1(f)]^*(a)(t) = [j^1(f_t)]^*(a)$, and let

$$[j^1(f)] : \mathbb{R}^{0|1} \to J^1(\mathbb{R}^{0|1}, (M, \mathcal{A}))$$

be the corresponding morphism. It is easy to see that the mapping $j^{1,1}(f) = j^1([j^1 f])$ takes values in $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, and also that

$$j^{1,1} : \mathrm{Mor}(\mathbb{R}^{1|1}, (M, \mathcal{A})) \to \Gamma(J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))/\mathbb{R}^{1|1})$$

is a differential operator of second order. Consequently, $j^{1,1}$ must factor through $J^2(\mathbb{R}^{1|1}, (M, \mathcal{A}))$, thus providing the desired submersion.

¿From the previous local expression for Θ, we can conclude that Θ is π-projectable. We also denote by Θ its projection.

THEOREM 8.2. *Let Θ be the graded 1-form associated to the Berezinian Lagrangian density $[dt \otimes \frac{d}{ds}].L$ introduced in the above theorem.*

(i) For every Berezinian critical section γ we have

$$(j^{1,1}\gamma)^*(i_X d\Theta) = 0$$

for every vector field X on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ vertical over $\mathbb{R}^{1|1}$.

(ii) Conversely, assume that L is regular and that $\bar{\gamma} \colon \mathbb{R}^{1|1} \to J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ is a section such that,

$$\bar{\gamma}^*(i_X d\Theta) = 0$$

for every vector field X on $J^{1,1}(\mathbb{R}^{1|1}, (M, \mathcal{A}))$ vertical over $\mathbb{R}^{1|1}$. Then, there exists a unique Berezinian critical section γ such that

$$\bar{\gamma} = j^{1,1}(\gamma).$$

PROOF. The first part of the statement follows taking into account that the formula in the proof of the previous theorem remains true for $j^{1,1}\gamma$.

For the second part, we just use the following equations and the fact that L is regular:

$$\gamma^*(i_{\frac{\partial}{\partial x^h_{st}}} d\Theta) = (-1)^{|x_h||x_i|}\bar{\gamma}^*\left((dx_i - x^i_t dt - x^i_s ds)\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right) = 0,$$

$$\gamma^*(i_{\frac{\partial}{\partial x^h_t}} d\Theta) = (-1)^{|x_h|(1+|x_i|)}\bar{\gamma}^*\left((dx^i_s - x^i_{st} dt)\frac{\partial^2 L}{\partial x^h_t \partial x^i_t} + (-1)^{|x_i|}(dx_i - x^i_t dt - x^i_s ds)\,\frac{d}{ds}\frac{\partial^2 L}{\partial x^h_t \partial x^i_t}\right) = 0. \quad \square$$

COMPARISON REMARK. The problem of the previous paragraph (the geodesics) is a particular case of this section. Thus, the reasons given there for the choice of Berezinian Lagrangian densities instead of graded ones will be also valid here.

In the general case, the Euler-Lagrange equations for the graded Lagrangian density Ldt, L being a graded function in $\mathcal{A}^1$, are the following:

$$\left[(j^2\gamma)^*\left(\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial x^i_t}\right)\right)\right]^\sim = 0, \quad \left[(j^2\gamma)^*\frac{\partial L}{\partial x^i_s}\right]^\sim = 0,$$

where $-n \leq i \leq m, i \neq 0$. Again, these last equations show that the odd part goes separately from the even part.

An study of the regularity conditions for this approach was done in **[Mu]**.

REFERENCES

[ANZ] Arnowitt, R., Nash, P. and Zumino, B., *Superfield densities and action principle in curved space*, Physics Letters **56** (1975), 81–84.

[Ba] Batchelor, M., *The structure of supermanifolds*, Trans. Amer. Math. Soc. **253** (1979), 329–338.

[Be] Berezin, F.A., *The method of second quantization*, Academic Press, New York, 1974.

[CM] Chang, L.N. and Mansouri, F., *Nonlinear gauge fields and the structure of gravity and supergravity theories*, Physical Review D **17** (1978), 3168–3178.

[HM1] Hernández Ruipérez, D. and Muñoz Masqué, J., *Global variational calculus on graded manifolds, I: Graded jet bundles, structure 1-form and graded infinitesimal contact transformations*, J. Math. pures et appl. **63** (1984), 283–309.

[HM2] ———, *Global variational calculus on graded manifolds, II*, J. Math. pures et appl. **63** (1985), 87–104.

[HM3] ———, *Variational Berezinian problems and their relationship with graded variational problems*, Lecture Notes in Math. (García,P. L. and Pérez–Rendón, A., eds.), Proc. Conf. on Diff. Geom. Methods in Math. Phys., Salamanca 1985., vol. 1251, Springer Verlag, Berlin and New York, 1987, pp. 137–149.

[HM4] ———, *Construction intrinsèque du faisceau de Berezin d'une variété graduée*, C. R. Acad. Sc. Paris t. 301, Série I, n. 20 (1985), 915–918.

[Ko] Kostant, B., *Graded manifolds, graded Lie theory and prequantization*, Lecture Notes in Math. (Bleuler, K. and Reetz, A, eds.), Proc. Conf. on Diff. Geom. Methods in Math. Phys., Bonn 1975., vol. 570, Springer Verlag, Berlin and New York, 1977, pp. 177–306.

[Le] Leites, D. A., *Introduction to the theory of Supermanifolds*, Russian Math Surveys 35:1 (1980), Uspehi Math. Nauk. **35** (1980), 1–64.

[Lo] Lott, J., *Torsion constraints in Supergeometry*, Commun. Math. Phys. **133** (1990), 563–615.

[Ma] Manin, Y., *Gauge field theory and complex geometry*, vol. 289, Springer Verlag, Berlin and New York, 1988.

[Mn] Mansouri, F., *Superunified theories based on the geometry of local (super-)gauge invariance*, Physical Review D **16** (1977), 2456–2467.

[Mo] Monterde, J., *Higher order graded and Berezinian Lagrangian densities and their Euler–Lagrange equations*, Ann. Inst. Henri Poincaré (to appear).

[Mu] Muñoz Masqué, J., *Some applications of graded variational calculus*, IV Meeting on Mathematical Physics, Coimbra (1984).

[SV] Sánchez Valenzuela, O.A., *Differential geometric approach to Supervector bundles*, Guanajuato, Gto. Méjico, CIMAT-preprint (1991).

[Wi] deWitt, B., *Supermanifolds*, Cambridge monographs on Math. Phys., Cambridge University Press, Cambridge, 1984.

J. MONTERDE. C.S.I.C, C/SERRANO 123, 28006–MADRID. SPAIN

Current address: Dept. de Geometría y Topología, Universitat de València, C/Dr. Moliner 50, 46100–Burjasot (Valencia), Spain.

E-mail address: MONTERDE@VM.CI.UV.ES

J. MUÑOZ MASQUÉ. C.S.I.C, C/SERRANO 123, 28006–MADRID. SPAIN

E-mail address: VCTQU01@CC.CSIC.ES