Jacobi-Nijenhuis manifolds

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Abstract

We propose, thanks to a new characterization of Jacobi structures by means of a Lie algebroid bracket, a definition of Jacobi-Nijenhuis structures, that includes the Poisson-Nijenhuis structures as a particular case. The existence of a hierarchy of compatible Jacobi structures on a Jacobi-Nijenhuis manifold is also obtained.

1 Introduction

The aim of this work is to build up in the case of Jacobi manifolds a notion of structure analogous to that of Poisson-Nijenhuis. Let us recall that a Poisson-Nijenhuis structure is a pair (P, N) given by a Poisson structure on a manifold M, and a recursion operator, or Nijenhuis tensor field, N, which satisfy some compatibility conditions [7]. The compatibility conditions are posed in order to assure the following fact: a new tensor field defined by means of P and N, and denoted by NP, is again a Poisson tensor field and, moreover, it is compatible with the previous one, P, this is, $P + NP$ is a Poisson tensor field.

The notion of compatibility of Jacobi structures has been previously defined and partially studied in [4] and [13]. Therefore, we give in this note something that can be seen as a second step in the study of compatibility of Jacobi structures, i.e., the definition of recursion operators in the Jacobi setting, and the characterization of the compatibility conditions between a Jacobi structure and a recursion operator.

The first attempt of definition, and possibly the more natural, goes in the following direction: It is well known that given a Jacobi structure on a manifold M, one can associate a unique homogeneous Poisson structure on the manifold

$M \times \mathbb{R}$. Something similar can be done with recursion operators. Let us call these processes under the name of Poissonization. The first tentative definition states that a Jacobi structure and a recursion operator are compatible if their Poissonizations are compatible in the sense of Poisson-Nijenhuis structures.

Such an attempt does not work basically because, whereas Poisson structures are a subclass of Jacobi structures, Poisson-Nijenhuis structures do not fulfill the previous tentative definition. The hey point of this drawback is that compatibility conditions between Poisson and Nijenhuis structures imply the possibility of building up a hierarchy of compatible Poisson tensor fields, but they are not the sufficient and necessary conditions to assure this.

The second and definitive attempt, presented here, studies, directly, the sufficient and necessary conditions that a Jacobi structure and a recursion operator must fulfil in order to assure the existence of a hierarchy of compatible Jacobi structures.

But the same can be done in the Poisson setting, and in fact, it can been found in [12]. The resulting structures are called weak Poisson-Nijenhuis structures. Using this 'weak' version of Poisson-Nijenhuis structures, the tentative attempt through the Poissonization process now works.

A short version of this paper has been published in [11].

2 The Lie algebra $\mathfrak{X}(M) \times C^\infty(M)$

All along this paper $C^\infty(M)$ denotes the algebra of C^∞ real-valued functions on a manifold M, $\Omega^1(M)$ the space of 1-forms, $\mathfrak{X}(M)$ the Lie algebra of vector fields and $[\ ,\ ]$ the Lie bracket of vector fields.

On the $C^\infty(M)$-module $\mathfrak{X}(M) \times C^\infty(M)$ we consider the Lie bracket defined by

$$[(X,f),(Y,g)] = ([X,Y], X(g) - Y(f)), \tag{1}$$

for all $(X,f),(Y,g) \in \mathfrak{X}(M) \times C^\infty(M)$. Note that if t is the usual coordinate on $\mathbb{R}$, $\mathcal{L}$ is the Lie derivative operator on $M \times \mathbb{R}$ and

$$\Phi : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}_I(M \times \mathbb{R}) = \{\tilde{X} \in \mathfrak{X}(M \times \mathbb{R}) / \mathcal{L}_{\partial_t} \tilde{X} = 0\},$$

is the isomorphism of $C^\infty(M)$-modules given by

$$\Phi(X,f) = X + f\partial_t, \tag{2}$$

then $\Phi[(X,f),(Y,g)] = [\Phi(X,f), \Phi(Y,g)]$. Thus, the Lie algebras $(\mathfrak{X}(M) \times C^\infty(M), [\ ,\ ])$ and $(\mathfrak{X}_I(M \times \mathbb{R}), [\ ,\ ])$ are isomorphic.

Let $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ be a $C^\infty(M)$-linear map. The set of $C^\infty(M)$-linear maps of the space $\mathfrak{X}(M) \times C^\infty(M)$ onto $C^\infty(M)$ can be identified with $\Omega^1(M) \times C^\infty(M)$ in such a way that

$$(\alpha, f)((X, g)) = \alpha(X) + fg,$$

for all $(\alpha, f) \in \Omega^1(M) \times C^\infty(M)$ and for all $(X, g) \in \mathfrak{X}(M) \times C^\infty(M)$.

Therefore, the adjoint operator of N, ${}^tN$, can be seen as the $C^\infty(M)$-linear map

$${}^tN : \Omega^1(M) \times C^\infty(M) \to \Omega^1(M) \times C^\infty(M), \qquad (\alpha, f) \mapsto {}^tN(\alpha, f),$$

where ${}^tN(\alpha, f)$ is defined by $({}^tN(\alpha, f))(X, g) = (\alpha, f)(N(X, g))$, for all $(X, g) \in \mathfrak{X}(M) \times C^\infty(M)$.

Since $\mathfrak{X}(M) \times C^\infty(M)$ is a real Lie algebra, one can define, in a natural way, *the Nijenhuis torsion* of N as the $C^\infty(M)$-bilinear map $[N, N] : (\mathfrak{X}(M) \times C^\infty(M))^2 \to \mathfrak{X}(M) \times C^\infty(M)$ given by

$$\begin{array}{rcl} [N, N]((X, f), (Y, g)) & = & [N(X, f), N(Y, g)] - N[N(X, f), (Y, g)] \\ & & -N[(X, f), N(Y, g)] + N^2[(X, f), (Y, g)]. \end{array} \tag{3}$$

3 Jacobi and Poisson manifolds

A *Jacobi structure* on M is a pair (Λ, E), where Λ is a 2-vector and E a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda]_{SN} = 2E \wedge \Lambda, \quad [E, \Lambda]_{SN} = 0\,. \tag{4}$$

Here, $[\ ,\ ]_{SN}$ denotes the Schouten-Nijenhuis bracket (see [1, 10, 14]). The manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. A bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\}_{(\Lambda, E)} = \Lambda(df, dg) + fE(g) - gE(f)\,, \quad \text{for all} \quad f, g \in C^\infty(M). \tag{5}$$

The space $C^\infty(M)$ endowed with the Jacobi bracket is *a local Lie algebra* in the sense of Kirillov (see [6]). Conversely, a structure of local Lie algebra on $C^\infty(M)$ defines a Jacobi structure on M (see [3, 6]). If the vector field E identically vanishes then (M, Λ) is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz ([8, 9]; see also [1, 2, 10, 14]).

Other interesting examples of Jacobi manifolds, which are not Poisson manifolds, are the contact manifolds (see for example, [9] and [10]). A contact manifold is a $2m+1$-dimensional manifold endowed with a 1-form η on M such that $\eta \wedge (d\eta)^m \neq 0$ at every point. The Jacobi structure (Λ, E) associated with the contact 1-form η is given by

$$\Lambda(\alpha, \beta) = d\eta(\flat^{-1}(\alpha), \flat^{-1}(\beta)), \qquad E = \flat^{-1}(\eta), \tag{6}$$

for all $\alpha, \beta \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is the isomorphism of $C^\infty(M)$-modules defined by $\flat(X) = i_X d\eta + \eta(X)\eta$. Note that E is the *Reeb vector field* of M which is characterized by the relations $i_E\eta = 1$ and $i_E d\eta = 0$ (see [9]).

Let Λ be a 2-vector and E a vector field on M. Define the homomorphisms of $C^\infty(M)$-modules $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ and $\tilde{\#}_{(\Lambda,E)} : \Omega^1(M) \times C^\infty(M) \to \mathfrak{X}(M)$ as follows,

$$\beta(\#_\Lambda(\alpha)) = \Lambda(\beta, \alpha), \qquad \tilde{\#}_{(\Lambda,E)}(\alpha, f) = \#_\Lambda(\alpha) - fE,$$

for $\alpha, \beta \in \Omega^1(M)$ and $f \in C^\infty(M)$.

If (M, Λ, E) is a Jacobi manifold, then for all $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M)$,

$$[\tilde{\#}_{(\Lambda,E)}(\alpha, f), \tilde{\#}_{(\Lambda,E)}(\beta, g)] = \tilde{\#}_{(\Lambda,E)}(\gamma, h),$$

with $(\gamma, h) \in \Omega^1(M) \times C^\infty(M)$ given by

$$\begin{array}{rcl} \gamma & = & \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha + d(\Lambda(\alpha, \beta)) - f\mathcal{L}_E\beta + g\mathcal{L}_E\alpha + i_E(\alpha \wedge \beta), \\ h & = & \Lambda(\alpha, \beta) + \tilde{\#}_{(\Lambda,E)}(\alpha, f)(g) - \tilde{\#}_{(\Lambda,E)}(\beta, g)(f), \end{array} \tag{7}$$

$\mathcal{L}$ being the Lie derivative operator (see [5]).

These facts allows to introduce a Lie algebroid $(T^*M\times\mathbb{R}, [\![\, ,\,]\!]_{\nu(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)})$ over M, where T^*M is the cotangent bundle of M and $[\![\, ,\,]\!]_{\nu(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M))^2 \to \Omega^1(M) \times C^\infty(M)$ is the bracket on $\Omega^1(M) \times C^\infty(M)$ defined by

$$[\![(\alpha, f), (\beta, g)]\!]_{\nu(\Lambda,E)} = (\gamma, h), \tag{8}$$

(for more details, see [5]).

In the particular case when (M, Λ) is a Poisson manifold we recover, by projection, the Lie algebroid $(T^*M, [\![\, ,\,]\!]_{\nu(\Lambda)}, \#_\Lambda)$, where $[\![\, ,\,]\!]_{\nu(\Lambda)}$ is the bracket of 1-forms given by

$$[\![\alpha, \beta]\!]_{\nu(\Lambda)} = \mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha + d(\Lambda(\alpha, \beta)), \tag{9}$$

for $\alpha, \beta \in \Omega^1(M)$ (see [1, 14]).

4 Characterizations of Jacobi structures

Let Λ be a 2-vector and E a vector field on M. For a real C^∞-differentiable function f, *the hamiltonian vector field* associated with f is the vector field X_f given by $X_f = \tilde{\#}_{(\Lambda,E)}(df, f) = \#_\Lambda(df) - fE$.

As in the case when (Λ, E) is a Jacobi structure, we also can define the bracket $[\![\, ,\,]\!]_{\nu(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M))^2 \to \Omega^1(M) \times C^\infty(M)$ by (7) and (8). It is easy to check that this bracket is skew-symmetric and that

$$[\![(df, f), (dg, g)]\!]_{\nu(\Lambda,E)} = -(d\{f, g\}_{(\Lambda,E)}, \{f, g\}_{(\Lambda,E)}),$$

$$[\![(\alpha,f),h(\beta,g)]\!]_{\nu(\Lambda,E)} = h[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)} + \tilde{\#}_{(\Lambda,E)}(\alpha,f)(h)(\beta,g),$$

for all $\alpha,\beta\in\Omega^1(M)$ and $f,g\in C^\infty(M)$.

Using the above definitions, we derive the following characterizations of Jacobi manifolds (see [3, 5, 9]).

THEOREM 4.1 *Let M be a differentiable manifold, Λ a 2-vector and E a vector field on M. The following statements are equivalent:*

1. *(M,Λ,E) is a Jacobi manifold.*
2. *The bracket $\{\ ,\ \}_{(\Lambda,E)}$ satisfies the Jacobi identity.*
3. *For all $f,g\in C^\infty(M)$, $[X_f,X_g]=-X_{\{f,g\}_{(\Lambda,E)}}$.*
4. *For all $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$*

$$[\tilde{\#}_{(\Lambda,E)}(\alpha,f),\tilde{\#}_{(\Lambda,E)}(\beta,g)] = \tilde{\#}_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}. \tag{10}$$

Besides this we can give another characterization of a Jacobi structure in terms of the Lie bracket $[\![\ ,\ ]\!]$ defined in (1) and a new homomorphism $\#_{(\Lambda,E)}$ associated with a pair (Λ,E), where Λ is a 2-vector and E is a vector field on M. The homomorphism of $C^\infty(M)$-modules $\#_{(\Lambda,E)}:\Omega^1(M)\times C^\infty(M)\to\mathfrak{X}(M)\times C^\infty(M)$ is given by

$$\#_{(\Lambda,E)}(\alpha,f) = (\tilde{\#}_{(\Lambda,E)}(\alpha,f),\alpha(E)) = (\#_\Lambda(\alpha)-fE,\alpha(E)). \tag{11}$$

THEOREM 4.2 *Let Λ be a 2-vector on M and E be a vector field on M. Then (M,Λ,E) is a Jacobi manifold if and only if for all $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$,*

$$[\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)] = \#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}.$$

Proof. Let us suppose first that (M,Λ,E) is a Jacobi manifold and write $[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}=(\gamma,h)$. Then,

$$\#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)} = (\tilde{\#}_{(\Lambda,E)}(\gamma,h),\gamma(E)).$$

On the other hand, using (7), (10) and (11),

$$\begin{aligned}[\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)] &= (\tilde{\#}_{(\Lambda,E)}(\gamma,h),\tilde{\#}_{(\Lambda,E)}(\alpha,f)(\beta(E))\\ &\quad -\tilde{\#}_{(\Lambda,E)}(\beta,g)(\alpha(E))).\end{aligned}$$

Therefore, we only have to prove that

$$\gamma(E) = \tilde{\#}_{(\Lambda,E)}(\alpha,f)(\beta(E)) - \tilde{\#}_{(\Lambda,E)}(\beta,g)(\alpha(E)).$$

Indeed, from (7), we deduce that

$$\left.\begin{array}{rcl}\gamma(E) & = & (\mathcal{L}_{\#_\Lambda(\alpha)}\beta)(E) - (\mathcal{L}_{\#_\Lambda(\beta)}\alpha)(E) + E(\Lambda(\alpha,\beta)) \\ & & +gE(\alpha(E)) - fE(\beta(E)) \\ & = & \tilde{\#}_{(\Lambda,E)}(\alpha,f)(\beta(E)) - \tilde{\#}_{(\Lambda,E)}(\beta,g)(\alpha(E)) \\ & & -\beta(\mathcal{L}_{\#_\Lambda(\alpha)}E) + \alpha(\mathcal{L}_{\#_\Lambda(\beta)}E) + E(\Lambda(\alpha,\beta)).\end{array}\right\} \quad (12)$$

On the other hand , since $\mathcal{L}_E\Lambda = 0$ (see (4)) it follows that $\mathcal{L}_E \circ \#_\Lambda = \#_\Lambda \circ \mathcal{L}_E$. Applying this we have that the last three terms of (12) vanish.

The converse statement follows from Theorem 4.1. ■

In the particular case of a contact manifold we get

PROPOSITION 4.3 *Let (M,Λ,E) be a Jacobi manifold. Then (M,Λ,E) is a contact manifold if and only if the mapping*

$$\#_{(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M), [\![\, ,\,]\!]_{\nu(\Lambda,E)}) \to (\mathfrak{X}(M) \times C^\infty(M), [\, ,\,])$$

is an isomorphism of Lie algebras.

Proof. Suppose that (M,η) is a contact manifold and that (Λ,E) is its associated Jacobi structure. From Theorem 4.2 we deduce that $\#_{(\Lambda,E)}$: $(\Omega^1(M) \times C^\infty(M), [\![\, ,\,]\!]_{\nu(\Lambda,E)}) \to (\mathfrak{X}(M) \times C^\infty(M), [\, ,\,])$ is a homomorphism of Lie algebras.

On the other hand, from (6), (11) and since $\#_\Lambda(\alpha) = \flat^{-1}(\alpha) - \alpha(E)E$, the homomorphism of $C^\infty(M)$-modules $\flat_{(\Lambda,E)} : \mathfrak{X}(M) \times C^\infty(M) \to \Omega^1(M) \times C^\infty(M)$ given by

$$\flat_{(\Lambda,E)}(X,f) = (i_X(d\eta) + f\eta, -\eta(X)), \qquad \text{for } (X,f) \in \mathfrak{X}(M) \times C^\infty(M),$$

is the inverse homomorphism of $\#_{(\Lambda,E)}$.

Conversely, assume that (M,Λ,E) is a Jacobi manifold and that the mapping $\#_{(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M), [\, ,\,]_{\nu(\Lambda,E)}) \to (\mathfrak{X}(M) \times C^\infty(M), [\, ,\,])$ is an isomorphism of Lie algebras. We will prove that for all $x \in M$,

$$T_xM = (\#_\Lambda)_x(T_x^*M) + < E_x > .$$

Let v be a tangent vector at a point x of M. Since the mapping $\#_{(\Lambda,E)}$ is $C^\infty(M)$-linear, it induces an isomorphism between the vector bundles $T^*M \times \mathbb{R}$ and $TM \times \mathbb{R}$ which we also denote by $\#_{(\Lambda,E)}$. In fact, the restriction of this isomorphism to $T_x^*M \times \mathbb{R}$ is defined by

$$(\#_{(\Lambda,E)})_x : T_x^*M \times \mathbb{R} \to T_xM \times \mathbb{R}, \qquad (\alpha_x, t) \to ((\#_\Lambda)_x(\alpha_x) - tE_x, \alpha_x(E_x)).$$

Thus, there exists $(\alpha_x, t) \in T_x^*M \times \mathbb{R}$ such that

$$(v, 0) = (\#_{(\Lambda,E)})_x(\alpha_x, t) = ((\#_\Lambda)_x(\alpha_x) - tE_x, \alpha_x(E_x)).$$

Therefore, v belongs to the subspace $(\#_\Lambda)_x(T_x^*M) + < E_x >$. On the other hand, $E_x \notin (\#_\Lambda)_x(T_x^*M)$. Indeed, if we suppose that there exists $w_x \in T_x^*M$ such that $E_x = (\#_\Lambda)_x(w_x)$ then $(\#_{(\Lambda,E)})_x(w_x, 0) = (\#_{(\Lambda,E)})_x(0, -1)$. But this is not possible, since the mapping $(\#_{(\Lambda,E)})_x$ is a linear isomorphism.

The above facts imply that (M, Λ, E) is a transitive Jacobi manifold and that the dimension of M is odd, that is, (M, Λ, E) is a contact manifold (see [2, 6, 9]). ∎

5 Recursion operators and Jacobi-Nijenhuis manifolds

Let M be a differentiable manifold. Suppose that Λ (respectively, E) is a 2-vector (respectively, a vector field) on M and that $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ is a $C^\infty(M)$-linear map.

We can consider the tensor field Λ_1 of type $(2, 0)$ and the vector field E_1 characterized by

$$\#_{(\Lambda_1,E_1)} = N \circ \#_{(\Lambda,E)}. \tag{13}$$

A direct computation shows that Λ_1 is a 2-vector if and only if $N \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ {}^tN$, where ${}^tN : \Omega^1(M) \times C^\infty(M) \to \Omega^1(M) \times C^\infty(M)$ is the adjoint operator of N.

Now, we will study the following problem:

Given a Jacobi structure (Λ, E) on M and a $C^\infty(M)$-linear map $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ satisfying $N \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ {}^tN$, which are the necessary and sufficient conditions for assuring that the pair (Λ_1, E_1) given by (13) is a Jacobi structure compatible with (Λ, E).

We recall that two Jacobi structures (Λ, E) and (Λ_1, E_1) on M are compatible if $(\Lambda + \Lambda_1, E + E_1)$ is a Jacobi structure (see [4, 13]).

In order to solve the above problem, we define the deformed bracket $[\![\, , \,]\!]_{{}^tN.\nu(\Lambda,E)} : (\Omega^1(M) \times C^\infty(M))^2 \to \Omega^1(M) \times C^\infty(M)$ as follows

$$\begin{aligned}[\![(\alpha, f), (\beta, g)]\!]_{{}^tN.\nu(\Lambda,E)} = & \; [\![{}^tN(\alpha, f), (\beta, g)]\!]_{\nu(\Lambda,E)} + [\![(\alpha, f), {}^tN(\beta, g)]\!]_{\nu(\Lambda,E)} \\ & - {}^tN[\![(\alpha, f), (\beta, g)]\!]_{\nu(\Lambda,E)}.\end{aligned} \tag{14}$$

First we characterize when (Λ_1, E_1) defines a Jacobi structure in the following result.

PROPOSITION 5.1 *Let (M,Λ,E) be a Jacobi manifold and let $N:\mathfrak{X}(M)\times C^\infty(M)\to\mathfrak{X}(M)\times C^\infty(M)$ be a $C^\infty(M)$-linear map such that $N\circ\#_{(\Lambda,E)}=\#_{(\Lambda,E)}\circ{}^tN$. Then the pair (Λ_1,E_1) is a Jacobi structure on M if and only if, for all $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$,*

$$\begin{aligned}[N,N](\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)) &= N(\#_{(\Lambda,E)}([\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)}\\ &\quad -[\![(\alpha,f),(\beta,g)]\!]_{{}^tN.\nu(\Lambda,E)})).\end{aligned}$$

Proof. Using (3), (13) and Theorem 4.2, we have that (Λ_1,E_1) is a Jacobi structure if and only if

$$\begin{aligned}&[N,N](\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g))+N([N\circ\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)]\\ &+[\#_{(\Lambda,E)}(\alpha,f),N\circ\#_{(\Lambda,E)}(\beta,g)]-N\circ\#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}\\ &-\#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)})=0\end{aligned} \tag{15}$$

for all $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$.

On the other hand, using (14), Theorem 4.2 and the equality $N\circ\#_{(\Lambda,E)}=\#_{(\Lambda,E)}\circ{}^tN$, we obtain that

$$\begin{aligned}&[N\circ\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)]+[\#_{(\Lambda,E)}(\alpha,f),N\circ\#_{(\Lambda,E)}(\beta,g)]\\ &-N\circ\#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}-\#_{(\Lambda,E)}[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)}\\ &=\#_{(\Lambda,E)}([\![{}^tN(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}+[\![(\alpha,f),{}^tN(\beta,g)]\!]_{\nu(\Lambda,E)}\\ &-{}^tN[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)}-[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)})\\ &=\#_{(\Lambda,E)}([\![(\alpha,f),(\beta,g)]\!]_{{}^tN.\nu(\Lambda,E)}-[\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)}).\end{aligned}$$

Therefore, substituting in (15) we deduce that (Λ_1,E_1) is a Jacobi structure on M if and only if

$$\begin{aligned}[N,N](\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g)) &= N(\#_{(\Lambda,E)}([\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)}\\ &\quad -[\![(\alpha,f),(\beta,g)]\!]_{{}^tN.\nu(\Lambda,E)})).\end{aligned}$$

■

If the $C^\infty(M)$-trilinear map $\#^*_{(\Lambda,E)}\circ[N,N]:(\Omega^1(M)\times C^\infty(M))^3\to C^\infty(M)$ given by

$$\#^*_{(\Lambda,E)}\circ[N,N]((\alpha,f),(\beta,g),(\gamma,h))=(\gamma,h)([N,N](\#_{(\Lambda,E)}(\alpha,f),\#_{(\Lambda,E)}(\beta,g))),$$

vanishes then N is called *a recursion operator* of (Λ,E).

Under the same hypotheses as in Proposition 5.1, one can define the $C^\infty(M)$-trilinear map $\#_{(\Lambda,E)}\circ C((\Lambda,E),N):(\Omega^1(M)\times C^\infty(M))^3\to C^\infty(M)$ given by

$$\begin{aligned}&(\#_{(\Lambda,E)}\circ C((\Lambda,E),N))((\alpha,f),(\beta,g),(\gamma,h)) =\\ &\quad(\gamma,h)(\#_{(\Lambda,E)}([\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda_1,E_1)}-[\![(\alpha,f),(\beta,g)]\!]_{{}^tN.\nu(\Lambda,E)})).\end{aligned} \tag{16}$$

Now, we give in the following result an answer for the problem posed at the beginning in this Section.

THEOREM 5.2 *Let (M, Λ, E) be a Jacobi manifold and let $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ be a $C^\infty(M)$-linear map such that $N \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ {}^tN$. Then (Λ_1, E_1) is a Jacobi structure compatible with (Λ, E) if and only if N is a recursion operator of (Λ, E) and the $C^\infty(M)$-trilinear map $\#_{(\Lambda,E)} \circ C((\Lambda, E), N)$ vanishes.*

Proof. From Theorem 4.2, we deduce that two Jacobi structures (Λ', E') and (Λ'_1, E'_1) on a manifold M' are compatible if and only if, for all $(\alpha, f), (\beta, g) \in \Omega^1(M') \times C^\infty(M')$,

$$\begin{aligned}&\#_{(\Lambda',E')}[\![(\alpha, f), (\beta, g)]\!]_{\nu(\Lambda'_1,E'_1)} + \#_{(\Lambda'_1,E'_1)}[\![(\alpha, f), (\beta, g)]\!]_{\nu(\Lambda',E')} = \\ &= [\#_{(\Lambda',E')}(\alpha, f), \#_{(\Lambda'_1,E'_1)}(\beta, g)] + [\#_{(\Lambda'_1,E'_1)}(\alpha, f), \#_{(\Lambda',E')}(\beta, g)].\end{aligned} \tag{17}$$

Using this fact, (13), (14), (16), Theorem 4.2 and Proposition 5.1, we prove the result. ■

Theorem 5.2 suggests us to introduce the following definition.

DEFINITION 5.3 *Let (M, Λ, E) be a Jacobi manifold and let $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ be a $C^\infty(M)$-linear map. The triple (Λ, E, N) is said to be a Jacobi-Nijenhuis structure if $N \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ {}^tN$, N is a recursion operator of (Λ, E) and $\#_{(\Lambda,E)} \circ C((\Lambda, E), N)$ vanishes.*

EXAMPLES 5.4 1. Weak Poisson-Nijenhuis manifolds: Let P be a Poisson 2-vector on M and $\tilde{N} : \mathfrak{X}(M) \to \mathfrak{X}(M)$ be a $(1, 1)$-tensor field on M. Consider the tensor field of type $(2, 0)$ $P_1 = \tilde{N}P$ given by $P_1(\alpha, \beta) = P({}^t\tilde{N}\alpha, \beta)$, for all $\alpha, \beta \in \Omega^1(M)$, where ${}^t\tilde{N} : \Omega^1(M) \to \Omega^1(M)$ is the adjoint operator of $\tilde{N}$. The pair $(P, \tilde{N})$ is said to be a *weak Poisson-Nijenhuis structure* on M if P_1 is a Poisson structure compatible with P (see [12]).

A direct computation, using (13), Theorem 5.2 and Definition 5.3, proves that $(P, \tilde{N})$ is a weak Poisson-Nijenhuis structure on M if and only if the triple $(P, 0, N)$ is a Jacobi-Nijenhuis structure, where $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ is the $C^\infty(M)$-linear map given by $N(X, f) = (\tilde{N}X, 0)$.

Thus, from Definition 5.3, it follows that $(P, \tilde{N})$ is a weak Poisson-Nijenhuis structure if and only if $\tilde{N} \circ \#_P = \#_P \circ {}^t\tilde{N}$ and

$$[\tilde{N}, \tilde{N}]_{FN}(\#_P(\alpha), \#_P(\beta)) = 0, \qquad \#_P(C(P, \tilde{N})(\alpha, \beta)) = 0,$$

where $[\ ,\ ]_{FN}$ is the Frölicher-Nijenhuis bracket and $C(P, \tilde{N})$ is the concomitant of P and $\tilde{N}$.

On the other hand, if P is a Poisson structure on M and $\tilde{N}$ is a Nijenhuis operator ($[\tilde{N},\tilde{N}]_{FN}=0$) then the pair $(P,\tilde{N})$ is said to be a *Poisson-Nijenhuis structure* if $\tilde{N}\circ\#_P=\#_P\circ\,{}^t\tilde{N}$ and the concomitant $C(P,\tilde{N})$ vanishes (see [7]). It is clear that a Poisson-Nijenhuis manifold is always a weak Poisson-Nijenhuis manifold. However, the converse is not true as shows the following simple example:

Let $P=fP_0$ be the Poisson bivector on a manifold M defined by a Poisson structure P_0 and by a Casimir $f:M\to\mathbb{R}$ of P_0, this is, f satisfies $\#_{P_0}(df)=0$. Now, consider the Nijenhuis tensor $\tilde{N}=fId$, where Id denotes the identity on $\mathfrak{X}(M)$. Then, $\tilde{N}\circ\#_P=\#_P\circ\,{}^t\tilde{N}$. Moreover, $\tilde{N}P=f^2P_0$ is again a Poisson structure which is compatible with P. However, the concomitant $C(P,\tilde{N})$ does not vanish in general. In fact, $C(P,\tilde{N})=-P\otimes df$.

2. Contact manifolds: Suppose that (M,η) is a contact manifold with associated Jacobi structure (Λ,E) and that (Λ_1,E_1) is a Jacobi structure on M compatible with (Λ,E). In such a case, the homomorphism $\#_{(\Lambda,E)}$ is an isomorphism (see Proposition 4.3). Thus, we can consider the $C^\infty(M)$-linear map $N=\#_{(\Lambda_1,E_1)}\circ\#^{-1}_{(\Lambda,E)}$. Since Λ_1 is a 2-vector it follows that $N\circ\#_{(\Lambda,E)}=\#_{(\Lambda,E)}\circ\,{}^tN$. Therefore, from Theorem 5.2 and Definition 5.3, we deduce that (Λ,E,N) is a Jacobi-Nijenhuis structure.

6 The hierarchy of Jacobi-Nijenhuis structures on a Jacobi-Nijenhuis manifold

Let Λ be a 2-vector and let E be a vector field on M. Denote by P the homogeneous 2-vector on $M\times\mathbb{R}$ defined by $P=e^{-t}(\Lambda+\partial_t\wedge E)$, where t is the usual coordinate on $\mathbb{R}$.

Consider the bracket $[\![\ ,\ ]\!]_{\nu(\Lambda,E)}$ (respectively, $[\![\ ,\ ]\!]_{\nu(P)}$) on the space $\Omega^1(M)\times C^\infty(M)$ (respectively, $\Omega^1(M\times\mathbb{R})$) given by (7) and (8) (respectively, (9)) and the $C^\infty(M)$-module $\Omega^1_c(M\times\mathbb{R})$ defined by

$$\Omega^1_c(M\times\mathbb{R})=\{\tilde{\alpha}\in\Omega^1(M\times\mathbb{R})/\mathcal{L}_{\partial_t}\tilde{\alpha}=\tilde{\alpha}\}.$$

The isomorphism of $C^\infty(M)$-modules

$$\psi:\Omega^1(M)\times C^\infty(M)\to\Omega^1_c(M\times\mathbb{R}),\qquad(\alpha,f)\to e^t(\alpha+fdt)\tag{18}$$

satisfies the condition $[\![\psi(\alpha,f),\psi(\beta,g)]\!]_{\nu(P)}=\psi([\![(\alpha,f),(\beta,g)]\!]_{\nu(\Lambda,E)})$, for $(\alpha,f),(\beta,g)\in\Omega^1(M)\times C^\infty(M)$ (see [15]). Moreover, a direct computation proves that

LEMMA 6.1 *If Λ (respectively, E) is a 2-vector (respectively, a vector field) on M and P is a 2-vector on $M \times \mathbb{R}$ then $P = e^{-t}(\Lambda + \partial_t \wedge E)$ if and only if $\Phi \circ \#_{(\Lambda,E)} = \#_P \circ \psi$, where $\Phi : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}_I(M \times \mathbb{R})$ is the isomorphism of $C^\infty(M)$-modules given by (2).*

Now, suppose that $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ is a $C^\infty(M)$-linear map. N induces a $(1,1)$-tensor field on $M \times \mathbb{R}$ as follows. Note first that $\Phi \circ N \circ \Phi^{-1} : \mathfrak{X}_I(M \times \mathbb{R}) \to \mathfrak{X}_I(M \times \mathbb{R})$ defines a $C^\infty(M)$-linear map on $\mathfrak{X}_I(M \times \mathbb{R})$. Thus, since $\mathfrak{X}(M \times \mathbb{R})$ is a $C^\infty(M \times \mathbb{R})$-module locally generated by $\mathfrak{X}_I(M \times \mathbb{R})$, this map can be extended to $\mathfrak{X}(M \times \mathbb{R})$ by linearity. Let us denote the extension by N^Φ. From (18), it follows that ${}^t(N^\Phi) \circ \psi = \psi \circ {}^tN$. Therefore, using Lemma 6.1, we have that

$$N \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ {}^tN \iff N^\Phi \circ \#_P = \#_P \circ {}^t(N^\Phi). \qquad (19)$$

On the other hand, it is well-known that the pair (Λ, E) defines a Jacobi structure on M if and only if the 2-vector P defines a Poisson structure on $M \times \mathbb{R}$ (see [8]). In fact, if (Λ, E) is a Jacobi structure on M, the 2-vector P is called the *Poissonization* of (Λ, E). Furthermore, in [4, 13] the authors prove that two Jacobi structures on M are compatible if and only if their Poissonizations on $M \times \mathbb{R}$ are compatible. Using these results, (19), Theorem 5.2 and Lemma 6.1, we deduce

PROPOSITION 6.2 *Let (Λ, E) be a Jacobi structure on M and $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ be a $C^\infty(M)$-linear map. The triple (Λ, E, N) is a Jacobi-Nijenhuis structure on M if and only if the pair (P, N^Φ) is a weak Poisson-Nijenhuis structure on $M \times \mathbb{R}$, where P is the Poissonization of the Jacobi structure (Λ, E) and N^Φ is the extension of N to $\mathfrak{X}(M \times \mathbb{R})$.*

Now, assume that (Λ, E, N) is a Jacobi-Nijenhuis structure on M and that P (respectively, N^Φ) is the Poissonization of (Λ, E) (respectively, the extension of N to $\mathfrak{X}(M \times \mathbb{R})$). Then, using Proposition 6.2 and the results of [12], we obtain that $(P_k = (N^\Phi)^k P, (N^\Phi)^\ell)$ is a weak Poisson-Nijenhuis structure, for all $k, \ell \in \mathbb{N}$.

On the other hand, it is easy to check that $(N^\Phi)^\ell = (N^\ell)^\Phi$ and, from Lemma 6.1, it follows that $(N^\Phi)^k P = e^{-t}(\Lambda_k + \partial_t \wedge E_k)$, where (Λ_k, E_k) is the pair characterized by the condition $\#_{(\Lambda_k,E_k)} = N^k \circ \#_{(\Lambda,E)}$. Consequently, using Proposition 6.2, we deduce the following:

THEOREM 6.3 *Let (Λ, E) be a Jacobi structure and $N : \mathfrak{X}(M) \times C^\infty(M) \to \mathfrak{X}(M) \times C^\infty(M)$ be a $C^\infty(M)$-linear map. If the triple (Λ, E, N) is a Jacobi-Nijenhuis structure then for any $k, \ell \in \mathbb{N}$, (Λ_k, E_k, N^ℓ) is a Jacobi-Nijenhuis structure on M. Thus, for any $k, \ell \in \mathbb{N}$, (Λ_k, E_k) and (Λ_ℓ, E_ℓ) are compatible Jacobi structures on M.*

Acknowledgments

Research partially supported by DGICYT grants PB97-1386 and PB97-1487.

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