GENERALIZED SYMPLECTOMORPHISMS

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Introduction

In [4] P. Michor gives a generalization of Hamiltonian Mechanics extending the Poisson bracket to a graded Lie bracket in the space of differential forms modulo exact forms. Such extension is based on the well known Frölicher-Nijenhuis operators that commute with the exterior derivative. [2]

In the first part of this work we improve a result of [4] that allows us to check the goodness of the generalization: Let (M, ω) be a symplectic manifold. We prove that there is an exact sequence of graded Lie algebras

$$0 \longrightarrow H^*(M) \xrightarrow{i} \frac{A(M)}{B(M)} \xrightarrow{H} Lham(M) \xrightarrow{\gamma} H^{*+1}(M) \longrightarrow 0$$

where $A(M)$ is the algebra of differential forms, $B(M)$ is the subspace of exact forms, $H^k(M)$ is the k-th singular homology group and $LHam(M)$ is the space of TM-valued differential forms K such that $\mathcal{L}_K\omega = 0$ and $P(K) = 0$, where $\mathcal{L}_K$ is the Frölicher-Nijenhuis operator induced by K and P is the projector from the space of all TM-valued differential forms to the subspace of K such that $i(K)\omega = 0$.

On the other hand, A. Montesinos and myself [5] have found a method to compute the integral curves of derivations on the algebra of sections of a exterior bundle. In particular, thanks to this method we can integrate the Frölicher-Nijenhuis operators. These integral curves are the orbits in the Michor's generalization of Hamiltonian Mechanics and they can be used to find conserved quantities.

Finally, these results are applied to the study of the relation between symplectomorphisms ($S\omega = \omega$), Poisson automorphisms ($\{S\alpha, S\beta\} = S\{\alpha, \beta\}$) and the $\mathbb{Z}_2$-graded algebra automorphisms ($S(\alpha\wedge\beta) = S\alpha\wedge S\beta$) that commute with the exterior derivative.

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1.- Derivations on $A(M)$.

In the first two paragraphs, we recall some definitions and results taken from [4].

Let M be a manifold. Let $A(M) = \oplus_{0\leq k\leq n} A^k(M)$ be the graded algebra of differential forms and let $A(M;TM) = \oplus_{0\leq k\leq n} A^k(M;TM)$ be the graded-space of TM-valued differential forms over M. Given $K \in A^k(M;TM)$, let $i(K)$ be the algebraic derivation of degree $k-1$ defined by

$$i(K)\alpha = K(e^j;\ ) \wedge \alpha(e_j,\ ),$$

where $\alpha \in A(M)$, $\{e_j\}$ is a local frame and $\{e^j\}$ its dual.

The proper derivations are defined by $\mathcal{L}_K = [i(K), d]$, where d is the exterior derivative and $[\ ,\ ]$ is the graded bracket of derivations, which induces the Frölicher-Nijenhuis bracket on $A(M;TM)$, and we have [4]:

Proposition 1.1:

1.- If $K \in A^k(M;TM)$ and $L \in A^{l+1}(M;TM)$, then

$$[\mathcal{L}_K, i(L)] = i([K,L]) - (-1)^{kl}\mathcal{L}_{i(L)K}.$$

2.- If $K_j \in A^{k_j}(M;TM)$, for $j = 1,2$, $\alpha \in A^a(M)$; then

$$[\alpha \wedge K_1, K_2] = \alpha \wedge [K_1,K_2] - (-1)^{(a+k_1)k_2}(\mathcal{L}_{K_2}\alpha) \wedge K_1 + (-1)^{a+k_1} d\alpha \wedge i(K_1)K_2.$$

2.- Generalized Hamiltonian Mechanics. (See [4] for details).

Let (M,ω) be a $2n$-dimensional symplectic manifold. ω induces a vector bundle isomorphism $\rho: T^*M \longrightarrow TM$, and, as a consequence, a linear homomorphism $A^1(M) \longrightarrow A^0(M;TM)$, which can be uniquely extended to an algebraic derivation of degree -1 $\rho: A(M) \longrightarrow A(M;TM)$

Let $H: A(M) \longrightarrow A(M;TM)$ be the generalized Hamiltonian mapping defined by $H_\alpha = \rho(d\alpha)$, where d denotes the exterior derivative.

Lemma 2.1:

1. $i(\rho\alpha)\omega = (-1)^{a-1}a\alpha$ for $\alpha \in A^a(M)$.
2. $i(H\alpha)\omega = (-1)^a(a+1)d\alpha$ for $\alpha \in A^a(M)$.
3. $\mathcal{L}_{H_\alpha}\omega = 0$.
4. $i([K,L])\omega = dA(K,L)\omega$ for $K \in A^k(M;TM)$, $L \in A^l(M;TM)$ such that $\mathcal{L}_K\omega = \mathcal{L}_L\omega = 0$, where

$$A(K,L) = (-1)^k i(K)i(L) - (-1)^{(k-1)l} i(i(L)K).$$

5. $i(i(K)H_\alpha)\omega = (-1)^\alpha a i(K)d\alpha$ for $\alpha \in A^a(M)$.

We will write $\rho^\#(\alpha) = (-1)^{a-1}\frac{1}{a}\rho(\alpha)$ for $\alpha \in A^a(M)$. $\rho^\#$ is a right inverse to $i(.)\omega$.

Among the several possibilities of definition of the generalized Poisson bracket presented in [4] we will choose the following one: $\{\alpha,\beta\} = \mathcal{L}_{H_\alpha}\beta$.

Lemma 2.2:

1.- $\{\alpha, \}$ is a derivation of degree a on the Cartan algebra of differential forms.
2.- $[H_\alpha, H_\beta] = H_{\{\alpha,\beta\}}$.
3.- $\{\ ,\ \}$ satisfies the graded Jacobi identity.
4.- $\{\ ,\ \}$ is not graded anticommutative. We have

$$\{\alpha,\beta\} + (-1)^{ab}\{\beta,\alpha\} = (-1)^a d(i(H_\alpha)\beta - (-1)^{(a-1)(b-1)} i(H_\beta)\alpha).$$

Thus $\{\ ,\ \}$ induces a structure of graded Lie algebra on $\frac{A(M)}{B(M)}$. Let $\Gamma(E_\omega)$ be the subespace of the elements $K \in A(M;TM)$ such that $i(K)\omega = 0$. We have the projector $P: A(M;TM) \longrightarrow \Gamma(E_\omega)$ given by $P(K) = K - \rho^\#(i(K)\omega)$.

The main result of [4] is the following:

Theorem 2.3: Let (M,ω) be a symplectic manifold.

1.- Let $B(M)$ be the space of exact forms, let $H^k(M)$ be the k-th singular homology group and let $V(M;TM)$ be the space of the elements $K \in A(M;TM)$ such that $\mathcal{L}_K\omega = 0$.

The following sequence is exact and consists of graded Lie algebras (the brackets are indicated below the spaces)

$$\begin{array}{ccccccc} 0 \longrightarrow & H^*(M) & \xrightarrow{i} & \dfrac{A(M)}{B(M)} & \xrightarrow{H} & V(M;TM). \\ & 0 & & \{\ ,\} & & [\ ,] \end{array}$$

All mappings are homomorphisms of graded Lie algebras.

2.- Let $\gamma(K)$ be the cohomology class of $i(K)\omega$. Then the following sequence is exact

$$\frac{A(M)}{B(M)} \xrightarrow{H} V(M;TM) \xrightarrow{\gamma+P} H^{*+1}(M) \oplus \Gamma(E_\omega) \longrightarrow 0.$$

3.- A Generalization of locally Hamiltonian fields.

If X is a locally Hamiltonian vector field different from zero then $i(X)\omega$ is always different from zero because ω is a symplectic form. However, in this generalization of Hamiltonian Mechanics, it is possible to have $i(K)\omega = 0$ with $K \neq 0$. In such case $\mathcal{L}_K\omega = 0$ trivially. Then, we must not consider all the TM-valued forms such that $\mathcal{L}_K\omega = 0$ as locally Hamiltonian. To get only the right ones, we project the space of the $K \in A(M;TM)$ such that $\mathcal{L}_K\omega = 0$ onto the complementary of $\Gamma(E_\omega)$.

Definition: $K \in A(M;TM)$ is called locally Hamiltonian if $\mathcal{L}_K\omega = 0$ and $P(K) = 0$, and is called globally Hamiltonian if there exist $\alpha \in A(M)$ such that $K = H_\alpha$.

We shall denote by $LHam(M)$ $(GHam(M))$ the space of all locally (globally) Hamiltonian $K \in A(M;TM)$. Obviously $GHam(M) = Im\ H$. We shall study the structure of both spaces with respect to the graded Lie bracket.

Lemma 3.1: LHam(M) is a graded Lie subalgebra for the Frölicher-Nijenhuis bracket.

Proof: If $K, L \in LHam(M)$ then, locally, there exist ϕ and ψ such that $K = H_\phi$ and $L = H_\psi$. Thus,

$$P([K,L]) = P([H_\phi, H_\psi]) = P(H_{\{\phi,\psi\}}) = 0. \blacksquare$$

Lemma 3.2: Let $K \in A(M;TM)$ be such that $L_K\omega = 0$. If $[K, Im\ H] \subset Im\ H$ then $[K, H_\phi] = H_{\mathcal{D}_K\phi}$ for all $\phi \in A^q(M)$, where

$$\mathcal{D}_K\phi = \frac{q}{q+k+1}\,\mathcal{L}_K\phi + \frac{k+1}{q+k+1}\,\mathcal{L}_{\rho^\#(i(K)\omega)}\phi.$$

Proof: In the following, $\rho^\#(i(K)\omega)$ will be denoted by $K^\#$. First, let us suppose that $K = K^\#$, i.e. that K is locally Hamiltonian. Then, the form $i(K)\omega$ is closed, and, if $p \in M$, there exists a homologically trivial neighbourhood U,of p, where $i(K)\omega$ is exact.

Since the sequence in theorem 2.3, with $M = U$, is exact, and since $K|_U$ is an element of the kernel of $\gamma + P$, then, there exists a $\psi \in A(U)$ such that $H_\psi = K|_U$.

Thus

$$[K, H_\phi](p) = [K|_U, H_\phi|_U](p) = [H_\psi|_U, H_\phi|_U](p) = H_{\{\psi|_U, \phi|_U\}}(p),$$

by lemma 2.2.2.

From the definition of Poisson bracket definition this is equal to

$$H_{\mathcal{L}_{H_{\psi|_U}}(\phi|_U)}(p) = H_{\mathcal{L}_K\phi}(p) = H_{\mathcal{D}_K\phi}(p).$$

Let us assume now that $[K, H_\phi] = H_{P(K,\phi)}$. Then

$$[K - K^\#, H_\phi] = H_{P(K,\phi) - \mathcal{L}_{K^\#}\phi}, \text{ where } i(K - K^\#)\omega = 0.$$

Moreover

$$i([K - K^\#, H_\phi])\omega = -(-1)^{kq} i([H_\phi, K - K^\#])\omega = \qquad \text{(by proposition 1.1.1)}$$

$$= (-1)^{kq+1}[\mathcal{L}_{H_\phi}, i_{K-K^\#}]\omega + (-1)^{kq+1+q(k-1)}\mathcal{L}_{i_{K-K^\#}H_\phi}\omega =$$

since

$$i(K - K^\#)\omega = 0 = \mathcal{L}_{H_\phi}\omega,$$

$$= (-1)^{q+k}\ di(i(K - K^\#)H_\phi)\omega =$$

by lemma 2.1.5

$$= (-1)^{q+k}\ q di(K - K^\#)d\phi = (-1)^{q+k} q\ d\mathcal{L}_{K-K^\#}\phi.$$

On the other hand, it follows from lemma 2.1.2, that

$$i(H_{P(K,\phi) - \mathcal{L}_{K^\#}\phi})\omega = (-1)^{q+k}(k+q+1)\ d(P(K,\phi) - \mathcal{L}_{K^\#}\phi).$$

Therefore

$$dP(K,\phi) = \frac{q}{q+k+1}\, d\mathcal{L}_K\phi + \frac{k+1}{q+k+1}\, d\mathcal{L}_{K^\#}\phi = d\mathcal{D}_K\phi.$$

Then

$$H_{P(K,\phi)} = H_{D_K\phi}. \blacksquare$$

Lemma 3.3:Let K be a TM-valued form with no part of degree $n =$ dimension of M.

$\mathcal{L}_K\phi$ is closed for every $\phi \in A(M)$ if and only if $K = \alpha \wedge Id$ with $d\alpha = 0$.

Proof: We can assume that K is homogeneous of degree $n > k \geq 0$ and that ϕ is homogeneous of degree p. First, if $K = \alpha \wedge Id$ with $d\alpha = 0$ then

$$d\mathcal{L}_K\phi = d\mathcal{L}_{\alpha\wedge Id}\phi = di(\alpha \wedge Id)d\phi = pd(\alpha \wedge d\phi) = 0.$$

Assume now that $\mathcal{L}_K\phi$ is closed for every $\phi \in A(M)$. Let us write $K = K^l \otimes \frac{\partial}{\partial x^l}$ locally, where $K^l \in A^{(k)}(M)$. For $\phi = x^l$ we have $dK^l = 0$, and for $\phi = x^l x^j$ we have $dx^j \wedge K^l + dx^l \wedge K^j = 0$. Evaluating this expression on $\frac{\partial}{\partial x^j}$ and adding from $j = 1$ to n we get

$$(n-k+1)K^l = dx^l \wedge \sum_{j=1}^{n} K^j(\frac{\partial}{\partial x^j}, \,) = 0.$$

Then, there exits α such that $K^l = \alpha \wedge dx^l$. $d\alpha = 0$ is a consequence of $dK^l = 0$ and degree of $\alpha < n-1$. ■

Proposition 3.4: $[K, H\phi] = H_{\mathcal{L}_K\phi}$ for every $\phi \in A(M)$ iff K is locally Hamiltonian.

Proof: We can assume that the degree of K is k and that the degree os ϕ is q. By lemma 3.1, if K is locally Hamiltonian, then $[K, Im\ H] \subset Im\ H$. Now, it follows from lemma 3.2 that $[K, H_\phi] = H_{\mathcal{L}_K\phi}$.

Assume now that $[K, H_\phi] = H_{\mathcal{L}_K\phi}$ for every $\phi \in A(M)$. First, we shall see that $\mathcal{L}_K\omega = 0$. We can suppose that, locally, the symplectic form is the canonical form $\omega = dp_l \wedge dq^l$, $\{p_l, q^l\}$ being the Darboux coordinates. Let $\phi = p_i dq^i$ be the Liouville form so that $d\phi = \omega$. Thus, we have $\mathcal{L}_{H_\phi} = -d$, the exterior derivative. Then,

$$\mathcal{L}_{H_{\mathcal{L}_K\phi}} = \mathcal{L}_{[K,H_\phi]} = \mathcal{L}_K\mathcal{L}_{H_\phi} - (-1)^k\mathcal{L}_{H_\phi}\mathcal{L}_K = -\mathcal{L}_K d + (-1)^k d\mathcal{L}_K = 0.$$

Hence $H_{\mathcal{L}_K\phi} = 0$. This means that $\mathcal{L}_K\phi$ must be a closed form, and thus

$$0 = d(\mathcal{L}_K\phi) = \mathcal{L}_K d\phi = \mathcal{L}_K\omega.$$

We shall see now that $P(K) = 0$. From lemma 3.2 $[K, H_\phi] = H_{D_K\phi}$, and then, $H_{\mathcal{L}_K\phi} = H_{\mathcal{D}_K\phi}$. Then $\frac{q}{q+k+1}\, d\mathcal{L}_K\phi + \frac{k+1}{q+k+1}\, d\mathcal{L}_{K^\#}\phi = d\mathcal{L}_K\phi$.

By computation we get $d\mathcal{L}_{K-K^\#}\phi = 0$. From lemma 3.3, $K - K^\# = \alpha \wedge Id$ with $d\alpha = 0$. But

$$0 = i(K - K^\#)\omega = i(\alpha \wedge Id)\omega = 2\alpha \wedge \omega.$$

Using Lepage's divisibility theorem ([3], p. 49) we get that if the degree of $\alpha = n-r$ with $r \geq 1$ then condition $\alpha \wedge \omega = 0$ implies $\alpha = 0$; and that if the degree of $\alpha = n+r$

with $r \geq 0$ then α may be written uniquely as $\alpha = \beta \wedge \omega^r$ where $\beta \in A^{n-r}(M)$. In such case, condition $\alpha \wedge \omega = 0$ implies that β is effective,([3], proposition 15.11, p. 46), i.e.,

$$\sum_{j=1}^{n} \beta(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^j}, \ ) = 0.$$

On the other hand, $[K - K^{\#}, H_\phi] = 0$ for every $\phi \in A^q(M)$. From proposition 1.1.2, we get

$$0 = [\alpha \wedge Id, H_\phi] = (-1)^{kq} \mathcal{L}_{H_\phi} \alpha \wedge Id.$$

Thus, $0 = \mathcal{L}_{H_f} \alpha = \mathcal{L}_{H_f} \beta \wedge \omega^r$ for every $f \in A^0(M)$. Using Lepage's divisibility theorem we get that $\mathcal{L}_{H_f} \beta = 0$ for every $f \in A^0(M)$. Finally, using the same techniques as in lemma 3.3 we get $\beta = 0$.

In all cases we get $\alpha = 0$ and then $K - K^{\#} = 0$, so that $P(K) = 0$. ∎

Corollary 3.5: $GHam(M)$ is an ideal of $LHam(M)$.

Theorem 3.6: Let (M, ω) be a symplectic manifold. The following sequence is exact and consists of graded Lie algebras, whose brackets are written under the spaces.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(M) & \xrightarrow{i} & \dfrac{A(M)}{B(M)} & \xrightarrow{H} & Lham(M) & \xrightarrow{\gamma} & H^{*+1}(M) \longrightarrow 0 \\ & & 0 & & \{\ ,\ \} & & [\ ,\ ] & & 0 \end{array}$$

All mappings are homomorphisms of graded Lie algebras. $\{\ ,\ \}$ is the induced graded Poisson bracket.

Proof: If $K \in Ker\ \gamma$, as we have, by definition, that $P(K) = 0$, $K \in Ker\ (\gamma + P)$. By 2.3.2, we also have that $K \in Im\ H$. And, finally, by 2.1.4, $\gamma([K, L]) = 0$. The rest is proved in [4]. ∎

As in the classic case ([1], p. 194, or also [3], p. 98) we have

Corollary 3.7: $[LHam(M), LHam(M)] \subset GHam(M)$.

Proof: We know that if $K, L \in LHam(M)$ then $[K, L] \in LHam(M)$. On the other hand $i([K, L])\omega = dA(K, L)\omega$, an exact form, whence $[K, L] \in Ker\ \gamma$. By the previous theorem, $[K, L] \in Im\ H = GHam(M)$. ∎

Corollary 3.8: $dim \frac{LHam^k(M)}{GHam^k(M)} = b_{k+1}(M)$, the $(k+1)$-th Betti number.

Proof:

$$\frac{LHam^k(M)}{GHam^k(M)} = \frac{LHam^k(M)}{Im\ H^k(M)} = \frac{LHam^k(M)}{Ker\ \gamma} \cong Im\ \gamma = Ker\ 0 = H^{k+1}(M). \ ∎$$

4.- Symplectomorphisms and Poisson automorphisms.

The characterization of Poisson automorphisms is a very hard problem, even for zero degree. But the aim of the present work is not to study the automorphisms completely, but to find the orbits associated to the generating forms.

Definition: An $\mathbb{R}$-linear automorphism $S: A(M) \longrightarrow A(M)$ is called a symplectomorphism if $S\omega = \omega$.

An $\mathbb{R}$-linear automorphism $S: A(M) \longrightarrow A(M)$ is called a $\mathbb{Z}_2$-graded algebra automorphism if $S(\alpha \wedge \beta) = S(\alpha) \wedge S(\beta)$ for all $\alpha, \beta \in A(M)$.

An $\mathbb{R}$-linear automorphism $S: A(M) \longrightarrow A(M)$ is called a Poisson automorphism if $S\{\alpha, \beta\} = \{S(\alpha), S(\beta)\}$ for all $\alpha, \beta \in A(M)$.

Any linear endomorphism T of $A(M)$ can be uniquely written in the form $T = T_0 + T_1$, where T_0 preserves the $\mathbb{Z}_2$-grading and T_1 reverses it.

Definition: We shall say that T commutes with the exterior derivative if $d \circ T_i = (-1)^i T_i \circ d$ for $i = 0, 1$.

From now on, we shall work with endomorphisms that commute with the exterior derivative.

Let G_s (resp. G_a, G_p) be the group of all the symplectomorphisms (resp. $\mathbb{Z}_2$-graded algebra automorphisms, Poisson automorphisms).

Remark: $S \in G_p$, iff $S \circ \mathcal{L}_{H_\alpha} \circ S^{-1} = \mathcal{L}_{H_{S\alpha}}$ for all $\alpha \in A(M)$.

Let us recall the following result that characterizes the $\mathbb{Z}_2$-graded algebra automorphisms of $A(M)$ that commute with the exterior derivative:

Theorem 4.1: ([5],[6]). If $S \in G_a$, there exists:

a) a unique diffeomorphism $\psi: M \longrightarrow M$, and

b) a unique section $K_{(2k)} \in A^{2k}(M; TM)$, for all $k > 0$, such that $S = B_{(r)} \circ \ldots \circ B_{(2)} \circ B_{(1)} \circ (\psi^*)^{-1}$, where ψ^* is the pull-back of ψ, and $B_{(k)} = \exp(\mathcal{L}_{K_{(2k)}})$, and r is the integral part of $\frac{n}{2}$.

The following theorem characterizes the elements of G_p among those of G_a:

Theorem 4.2: If $S \in G_a$, then $S \in G_p$ iff $K_{(2i)} \in LHam(M)$ for all $i > 0$ and $\psi^* \in G_s$.

Proof: If $S \in G_a$, then S can be written in the form $S = B_{(r)} \circ \ldots \circ B_{(2)} \circ B_{(1)} \circ (\psi^*)^{-1}$.

As it is well known ([1], proposition 3.3.20), ψ is a symplectomorphism of M, if and only if $\{\psi^*\alpha, \psi^*\beta\} = \psi^*\{\alpha, \beta\}$ for all $\alpha, \beta \in A(M)$. Then we can suppress ψ^* without loss of generality.

We will see now that an element $K \in LHam(M)$ of degree $2k$ satisfies

$$\exp(\mathcal{L}_K)\{\alpha, \beta\} = \{\exp(\mathcal{L}_K)\alpha, \exp(\mathcal{L}_K)\beta\} \text{ for all } \alpha, \beta \in A(M) \tag{4}$$

if and only if

$$[K, H_\alpha] = H_{\mathcal{L}_K \alpha} \text{ for all } \alpha \in A(M). \tag{5}$$

Let a (resp. b) be the degree of α (resp. β). The term of degree $a + b$ of both sides of (4) is $\{\alpha, \beta\}$. Equating the terms of degree $a + b + 2k$ of (4) we have

$$\mathcal{L}_K \mathcal{L}_{H_\alpha} \beta = \mathcal{L}_{H_{\mathcal{L}_K \alpha}} \beta + \mathcal{L}_{H_\alpha} \mathcal{L}_K \beta$$

which is (5).

Conversely, this last equality implies

$$(\mathcal{L}_K)^n \mathcal{L}_{H_\alpha}\beta = \sum_{p=0}^{n} \binom{n}{p} \mathcal{L}_{H_{(\mathcal{L}_K)^p\alpha}} (\mathcal{L}_K)^{n-p}\beta,$$

but this expression is the term of degree $a + b + 2kn$ of $\{\exp(\mathcal{L}_K)\alpha, \exp(\mathcal{L}_K)\beta\}$.

By proposition 3.3, (5) holds if and only if $K \in LHam(M)$. Thus we can suppress the factors $\exp(\mathcal{L}_K)$ until the dimension of M is reached.■

Definition: A Poisson automorphism is said to be regular if there exists a diffeomorphism $\psi: M \longrightarrow M$ such that, for any $m \in M$, $(S\alpha)_m$ only depends on α in a neighbourhood of $\psi(m)$.

We have the following relation between Poisson automorphisms and symplectomorphisms.

Theorem 4.3: Every regular Poisson automorphism that commutes with the exterior derivative is a symplectomorphism.

Proof: Let S be a regular Poisson automorphism. For any $p \in M$ let V a contractible open neighbourhood of p, and let ψ be the diffeomorphism of M defined by S. Let ϕ be a differential form such that $d\phi = \omega$, in $\phi^{-1}(V)$. Then $\mathcal{L}_{H_\phi} = -d$ in $\phi^{-1}(V)$.

The condition that S be a Poisson automorphism is equivalent to $S \circ \{\phi, \ \} \circ S^{-1} = \{S\phi, \}$. In V we have that $S \circ \{\phi, \} \circ S^{-1} = -d = \mathcal{L}_{H_\phi} = \mathcal{L}_{H_{S\phi}}$. Thus $\omega = d\phi = dS\phi$. Since S commutes with d, then $\omega = S_0\omega - S_1\omega$. However, $S_1\omega$ is of even degree, hence it must be zero. Thus $S\omega = S_0\omega = \omega$. ■

5.- Orbits in the generalized Hamiltonian Mechanics.

In Hamiltonian Mechanics the orbits of the dynamics associated to a generating function f are given by the flow, ψ_t, of the field H_f. The pull-backs ψ_t^* are a family of linear automorphisms of degree zero, that integrate $\mathcal{L}_{H_f}$ in the following sense:

$$\frac{d}{dt}\psi_t^*(\beta) = \mathcal{L}_{H_f} \circ \psi_t^*(\beta) \quad \text{for all } \beta \in A(M).$$

Or, in Poisson bracket form

$$\frac{d}{dt}\psi_t^*(\beta) = \{f, \psi_t^*(\beta)\} \quad \text{for all } \beta \in A(M),$$

which are the Hamilton equations.

Furthermore, $\psi_t^*(\beta)$ is, at the same time, a family of $\mathbb{Z}_2$-algebra automorphisms and of Poisson automorphisms, since $\mathcal{L}_{H_f}$ is a derivation on the Cartan algebra and a derivation on the Poisson algebra, and a family of symplectomorphisms, because $\mathcal{L}_{H_f}\omega = \omega$.

This example gives rise to the following definition: the orbit associated to a generating form α is the integral curve of the derivation $\mathcal{L}_{H_\alpha}$. Let us recall some definitions and results of [5].

Definition: A curve of linear automorphisms of $A(M)$ is a one-parameter family of $\mathbb{R}$-linear isomorphisms $T_t: A(\mathcal{D}_{-t}) \longrightarrow A(\mathcal{D}_t)$, $t \in \mathbb{R}$, where $\mathcal{D}_t$ is an open subset of M, with the following conditions:

a) $\mathcal{D}_0 = M$, $T_0 = Id$, $\bigcup_{t>0} \mathcal{D}_t = \bigcup_{t<0} \mathcal{D}_t = M$, $\mathcal{D}_{t_1} \subset \mathcal{D}_{t_2}$ if $t_1 \geq t_2 \geq 0$, or if $t_1 \leq t_2 \leq 0$, and $\{t \in \mathbb{R} \mid m \in \mathcal{D}_t\}$ is an open interval for every $m \in M$;

b) if $\alpha \in A(M)$, let us write $T_t(\alpha) = T_t(\alpha|_{\mathcal{D}_{-t}})$; then, if $m \in M$, the curve $t \longrightarrow (T_t\alpha)(m) \in \Lambda_m T^*M$, for t such that $m \in \mathcal{D}_t$, is smooth; and the map $m \in \mathcal{D}_{t_0} \longrightarrow (\frac{\partial}{\partial t})|_{t_0}(T_t\alpha)(m)$, which we will denote by $T'_{t_0}\alpha$, belongs to $A(\mathcal{D}_t)$.

A curve of linear automorphisms of $A(M)$ is called a curve of symplectomorphisms (resp. $\mathbb{Z}_2$-graded algebra automorphisms, Poisson automorphisms) if , for every $t \in \mathbb{R}$, $T_t \in G_s$ (resp. G_a, G_p).

Let T_t be a curve of linear automorphisms of $A(M)$ and put $D_t = T_t{}^{-1}T'_t$. Then D_t is a linear endomorphism of $A(\mathcal{D}_{-t})$). It is easy to check that when T_t is a curve of $\mathbb{Z}_2$-graded algebra automorphisms (resp. of Poisson automorphisms), then D_t is an even derivation of the Cartan algebra (resp. of the Poisson algebra), i.e. $D_t(\alpha \wedge \beta) = (D_t\alpha) \wedge \beta + \alpha \wedge (D_t\beta)$ (resp. $D_t\{\alpha, \beta\} = \{D_t\alpha, \beta\} + \{\alpha, D_t\beta\}$). Therefore, we cannot obtain the odd derivations (both Cartan and Poisson) as derivatives of curves of $\mathbb{Z}_2$-graded algebra automorphisms or Poisson automorphisms. Since our wish is to include even and odd derivations, we have chosen curves of symplectomorphisms instead.

We need the following

Definition: A linear endomorphism D is said to be localizable if there is a map $\psi: M \longrightarrow M$ such that, for each $\alpha \in A(M)$ and $m \in M$, $(D\alpha)(m)$ is determined by the infinite jet of α at $\psi(m)$.

The following theorem assures the existence and uniqueness of integral curves:

Theorem 5.1: ([5]). Let D be a localizable linear endomorphism of A(M) such that

$$D = D_{(0)} + ... + D_{(p)} + ..., \quad (D_{(p)} \text{ of degree } p)$$

with $D_{(0)}$ a Cartan derivation. Then there exists a unique curve T_t of linear automorphisms of $A(M)$ such that $T'_t = T_t \circ D = D \circ T_t$, and $T_0 = Id$. If D is an even Cartan derivation (resp. even Poisson derivation), then T_t is a curve of $\mathbb{Z}_2$-graded algebra automorphisms (resp. Poisson automorphisms).

For the Poisson case, derive the expression $T_t\{\alpha, \beta\} = \{T_t\alpha, T_t\beta\}$.

Definition: We shall call <u>orbit generated by</u> $\alpha \in A(M)$ the curve of linear automorphisms T_t integral of $\mathcal{L}_{H_\alpha}$, i.e., such that in Poisson bracket form

$$T'_t(\beta) = \{\alpha, T_t(\beta)\} \text{ and } T_0(\beta) = \beta, \text{ for all } \beta \in A(M).$$

Or by definition of the Poisson bracket

$$T'_t = \mathcal{L}_{H_\alpha} \circ T_t, \text{ and } T_0 = Id.$$

If α is even then $\mathcal{L}_{H_\alpha}$ is, at the same time, a derivation of $(A(M), \wedge)$ and of $(A(M), \{\ ,\ \})$. Thus $T_t \in G_a \cap G_p$. But if α is odd then, in general, it is neither a $\mathbb{Z}_2$-algebra automorphism nor a Poisson automorphism. However it is a symplectomorphism: $T_t \in G_s$, i.e. $T_t\omega = \omega$, because $\mathcal{L}_{H_\alpha}\omega = 0$.

We can say that $T_t' = \mathcal{L}_{H_\alpha} \circ T_t$ is a generalization of Hamilton equations. In [5] is shown how they can be integrated in a constructive way.

Proposition 5.2: Let T_t be the orbit generated by $\alpha \in A(M)$.

a) The flow T_t preserves β iff $\{\alpha, \beta\} = 0$.

b) If $[\alpha] \in \frac{A(M)}{B(M)}$ is of even degree, then $T_t[\alpha]$ is well defined and is constant. ("conservation of the energy").

Proof: a) Is a consequence of $\frac{d}{dt}|_{t=0} T_t \beta = T_0 \circ \mathcal{L}_{H_\alpha} \beta = \{\alpha, \beta\}$.

b) Is a consequence of $\{[\alpha], [\alpha]\} = [\{\alpha, \alpha\}] = 0$ if α is of even degree. (See 4.6.3 [4]). ∎

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