A. Montesinos et J. Monterde

Abstract fluids as differential forms

A. Montesinos and J. Monterde
Departamento de Geometría y Topología
Universidad de Valencia
Burjasot (Valencia)
Spain

1. INTRODUCTION

This will be an expose of a previous work of the senior author [2], and of later joint work with J. Monterde. The starting problem was a question about the equation of continuity. It was the study of the possibility of changing the usual variational setup for fluids in such a manner that the equation of continuity could be obtained as a consequence of the variational principle instead of imposing it from the beginning as an a priori constraint.

So, let M be an oriented smooth n-dimensional manifold, which stands for spacetime. We shall consider fluids as cross sections of the fibre bundle of vector densities

$$E = TM \otimes \Lambda^n M \longrightarrow M.$$

This description of fluids as vector <u>densities</u> is certainly the usual thing since it amounts to take them as densities of mass (this would be the time component) and of momentum (the space component). Nevertheless, the practice is to convert them into <u>vector</u> fields by evaluating upon a unit volume once a metric has been fixed for the spacetime. For our approach, however, it is essential to keep densities as such.

We'll recall now some notations and concepts borrowed from Takens [4]. For the fibre bundle $\pi : E \longrightarrow M$ we denote by $\Gamma(E)$ the set of local smooth sections and put

$$\begin{array}{ccc} J(E) & \xrightarrow{p_0} & E \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \pi} \\ & & M, \end{array}$$

to fix the notation about the space of ∞-jets of local sections of π, $J(E)$. If $s \in \Gamma(E)$, then $j(s) : M \longrightarrow J(E)$ will stand for its ∞-jet. $V(M)$ will be the Lie algebra of vector fields on any manifold M.

For $X \in V(M)$ we put $X^H \in V(J(E))$ to denote its total vector field or horizontal lift. Let $V_\pi(E)$ be the Lie subalgebra of π-projectable vector fields on E. If $X \in V_\pi(E)$, then $\pi(X) \in V(M)$ will be its projection and $X^\infty \in V(J(E))$ its natural lift or integrable vector field.

The bundle TJ(E) splits as $TJ(E) = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V}$ is the p-vertical subbundle and $\mathcal{H}$ is given at each point, u, as the horizontal lift of $T_{p(u)}M$ to u. The algebra of differential forms on J(E) becomes thus bigraded. We put $H^r{}_s(E)$ to denote the module of (r+s)-forms on J(E) that are r times vertical and s times horizontal, and D, ∂ to denote the horizontal and vertical differentials. We have

$$D^2 = \partial^2 = D\partial + \partial D = 0.$$

A source form is an element $\omega \in H^1{}_n(E)$ such that $I_X\,\omega = 0$ for any p_0-vertical $X \in V(J(E))$. We have the fundamental result

<u>Theorem (Takens)</u>: Each $\omega \in H^1{}_n(E)$ can be written uniquely as $\omega = \omega_1 + \omega_2$ where ω_1 is a source form and $\omega_2 \in \operatorname{Im} D$.

2. VARIATIONAL EQUATIONS FOR FLUIDS.

Now we shall state the variational problem. Let A be a regular compact domain of M with boundary ∂A, and let $\lambda \in H^0{}_n(E)$, the Lagrangian, be given. Then we have a functional

$$s \in \Gamma(E|_A) \longrightarrow \int_A j(s)^* \lambda \,,$$

and the problem is to look for its critical points when certain class of variations are allowed.

A big class of variations is given by π-projectable vector fields. If $X \in V_\pi(E)$ we have another section $s_t = \tilde{\Phi}_t \cdot s \cdot \Phi_{-t}$ defined on $\Phi_t(A)$ for t small enough, where $\tilde{\Phi}_t$ and ϕ_t are respectively the flows of X and $\pi(X)$. Thus we must study the derivative

$$\delta_X = \frac{d}{dt}\Big|_{t=0} \int_{\phi_t(A)} j(s_t)^* \lambda = \int_A j(s)^* \mathcal{L}_{X^\infty} \lambda$$

whenever $X^\infty|_{\partial A} = 0$. Let $\alpha \in H^1{}_{n-1}(E)$ be such that $d\lambda = \omega + D\alpha$, ω being a source form (ω is thus uniquely defined, and in fact is determined by the classical Euler-Lagrange operator).

Thus, integration by parts gives

$$\delta_X = \int_A j(s)^* \, i_{X^\infty - \pi(X)^H} \, \omega$$

Now instead of taking X freely we make it lie in a subalgebra of $V_\pi(E)$, that is we impose a constraint upon the field s. This constraint makes all the difference between a fluid and a physical field (in the usual variational treatment).

Let $X \in V(M)$. Then its flow lifts, via its differential, to the frame bundle, and therefore X lifts to any tangent tensor bundle over M. In particular we denote by $\widetilde{X}$ its lift to E. So we have $\widetilde{V(M)} \subset V_\pi(E)$, that is a Lie subalgebra which will be our space of variations for fluids. Since ω is a source form, it is obvious that there is a unique 1-form ω_s on M such that $\omega_s \cdot u = j(s)^*(i_{\iota u}\omega)$ for any $u \in \Gamma(E)$, where the dot denotes full contraction, and $\iota : \Gamma(E) \longrightarrow V_\pi(E)$ is the vertical injection. Also it is true almost by definition that

$$(X^H - \widetilde{X}^\infty - (\iota \mathcal{L}_X s)^\infty) \cdot j(s) = 0.$$

After substitution and another integration by parts we get finally the variational equations for fluids:

$$d\omega_s \cdot s - \omega_s \cdot \delta s = 0 , \qquad (1)$$

where δ denotes divergence.

Corollary : If s satisfies (1), then $\delta s = 0$ in the open subset where $\omega_s \cdot s \neq 0$ (equation of continuity).

Proof : Contract (1) with any vector field parallel to the vector part of s.

3. EXAMPLES.

Example 1. Newtonian space time and conservative fluid.

Let $M = R \times R^3$ be the Newtonian space-time with coordinates (t,x^a). Let $\tau = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$, and let

$$s = \left(s^t \frac{\partial}{\partial t} + s^a \frac{\partial}{\partial x^a} \right) \otimes \tau$$

be a fluid . We put $\rho = s^t$, $\rho v^a = s^a$ and fix the lagrangian

$$\lambda = (\rho v^2/2 - (V + e)\rho) \tau \quad .$$

where V and e are the potential and internal energy per unit of mass, and where the latter is supposed to depend only on the density.

Then, equation (1) splits into

$$\rho \frac{dv}{dt} + \rho \operatorname{grad} V + \operatorname{grad} P = \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v)\right) \tag{2}$$

$$\frac{1}{2} \rho \frac{dv^2}{dt} + v \cdot (\rho \operatorname{grad} V + \operatorname{grad} P) = \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v)\right)\left(\frac{v^2}{2} + V + e + \frac{P}{\rho}\right) \tag{3}$$

where $P = \rho^2 \, {}^{de}/_{d\rho}$ is the pressure and ${}^{d}/_{dt}$ is the Eulerian derivative . If we multiply (2) by v and subtract from (3) we get the following equation of compatibility (cf. Corollary):

$$\left(\frac{1}{2} \rho v^2 - P - \rho(V+e)\right)\left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v)\right) = 0 \tag{4}$$

Thus, the equation of continuity holds in the points where

$$\rho v^2/2 \neq P + (V + e)\rho. \tag{5}$$

Otherwise, the fluid may not satisfy it.

We wish to remark two things. First, that the zero level of the potential or internal energy is not indifferent to our equations. And second, that (5) is the classical condition of the no cavitation. We perhaps have thus in (4) a sort of classical foreboding of a creation and annihilation mechanism. Of course, if the equation of continuity holds good, we have the usual equations for the momentum and energy.

<u>Example 2. Schrödinger equation</u>:

There is another important equation of continuity in Physics, the conservation of probability in Schrödinger equation. After a convenient choice of units, Schrödinger equation reads

$$i \frac{\partial \Psi}{\partial t} = -\Delta \Psi + \frac{1}{2} V \Psi. \tag{6}$$

If we put

$$\Psi = (\rho e^{i\theta})^{\frac{1}{2}},$$

so that ρ is the density of probability, then (6) decomposes into

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho \operatorname{grad}\theta) = 0 \tag{7}$$

$$\Delta\rho = \frac{1}{2}\frac{(\operatorname{grad}\rho)^2}{\rho} + \frac{1}{2}\rho(\operatorname{grad}\theta)^2 + V\rho + \frac{\partial\theta}{\partial t}. \tag{8}$$

Now this is fantastic! The modulus of the square root of a complex function on R^4 combines with the gradient of its argument to give a vector density with zero divergence! This is so queer that should deserve a look from another perspective. Let us take the Lagrangian density

$$\lambda = \left(\frac{1}{2}\rho v^2 - V\rho - \frac{1}{2}\frac{(\operatorname{grad}\rho)^2}{\rho}\right)\tau .$$

Then

$$\omega_s = \left(\frac{\Delta\rho}{\rho} - \frac{1}{2}\frac{(\operatorname{grad}\rho)^2}{\rho} - \frac{1}{2}v^2 - V\right)dt + v^a dx^a$$

We can recover (7) and (8) by a modification of our variational principle. Let $Z = \{ u \in \Gamma(E) \mid \delta u = 0 \}$. Then $\widetilde{V(M)} \oplus \iota Z$ is a subalgebra of $V_\pi(E)$, and to take it as the space of variations amounts to impose that, though the variation of s may not be that of a fluid, the induced variation of its divergence is. So, we call then s a <u>prefluid</u>.

A computation shows easily that then the variational equations are

$$\omega_s \cdot \delta s = 0$$

$$\omega_s \text{ is exact,} \tag{9}$$

and this is (7) and (8) in the points where $\omega_s \neq 0$.

So, one can adopt the unorthodox attitude of thinking that the possibility of a complex linear description of quantum waves is a mere coincidence for the case of the Newtonian space-time. In other words, the complex wave function would be only a clever construction for simplifying the true image which would be that of an abstract prefluid.

4 . ABSTRACT FLUIDS AS DIFFERENTIAL FORMS.

Our first attempt to push forward this idea was to try a relativistic generalization. We can put, for a Minkowski metric g and a prefluid $s \otimes \tau : \rho = g(s,s)^{1/2}$ and

$$\lambda = \left(\rho - \frac{g(\operatorname{grad}\rho, \operatorname{grad}\rho)}{\rho} \right) \tau$$

as suggested by the Lagrangian for the Schrödinger equation.

In few words it can be said that this approach leads to a failure. Certainly one can get plane wave solutions and also it is true that the solutions of our equation approach Klein-Gordon's when $g(\operatorname{grad}\rho, \operatorname{grad}\rho)$ is almost constant. But in the search for solutions of the hydrogen atom problem one is bound to hit against singularities.

The first idea that comes to the mind at this point is to ascertain whether a similar variational treatment can be made upon spinors. However, the senior author [3] showed that they are not well fit for it due to the lack of a natural Lie derivative. With the precedent of [1] we took general differential forms, that is, forms of all degrees, as the basic description of our would-be alternative approach to wave mechanics.

Of course, since there is a natural isomorphism between vector densities and (n-1)-forms, it is easy to translate the preceding picture in terms of differential forms. Classical fluids become 3-forms, which is a most natural thing to do. In fact they can be integrated over 3-dimensional spacelike surfaces, so giving the matter contents on them. Also, the integration over a (1,2) hypersurface (1 time and 2 space dimensions) gives the amount of fluid that crossed that space surface during that lapse of time. In this picture the preceding variational equations are obtained in the same way and give, for each homogeneous component s of the fluid, the following equation:

$$ds \cdot \omega_S - s \cdot \delta\omega_S = 0.$$

Then, for any homogeneous component of s, the number of equations is n, the dimension of M; that is too little a number in general. The need for more equations leads to the use of deformations more than simply variations on the variational principle. A vector field $X \in V(J(E))$ is (essentially) called a deformation (cfr. [4]) if for any given $s \in \Gamma(E)$, the field $X \cdot j(s)$ can be locally put as $Y^{\infty} \cdot j(s)$, for some vector field $Y \in V_{\pi}(E)$.

Now, in the preceding process of variation we have

$$\int_A j(s)^* \mathcal{L}_{\tilde{X}^{\infty}} \lambda = \int_A j(s)^* \mathcal{L}_{\tilde{X}^{\infty} - X^H} \lambda =$$

$$\int_A j(s)^* \mathcal{L}_{(\iota \mathcal{L}_X s)^{\infty}} \lambda = \int_A j(s)^* \mathcal{L}_K \lambda$$

where $K \in V(J(E))$ is the deformation defined by

$$K \cdot j(s) = (\iota \mathcal{L}_X s)^{\infty} \cdot j(s).$$

In other words, we have a variation given essentially through the Lie derivative. The obvious generalization of this is to take the derivations of the algebra of differential forms, A(M), that commute with the differential, i.e. Nijenhuis operators.

Let $V^k(M) = \Gamma(TM \otimes \Lambda^k M)$, $K \in V^k(M)$. If we put

$$\mathcal{L}_K = i_K \cdot d + (-1)^k d \cdot i_K ,$$

then $\mathcal{L}_K$ is a derivation of degree k in A(M) that commutes with the differential in the following sense:

$$\mathcal{L}_K \cdot d = (-1)^k d \cdot \mathcal{L}_K .$$

Let $E = \Lambda(M) \rightarrow M$ be the bundle of forms of any degree on M; if, as before, $K \in V^k(M)$, we define the vector field $\tilde{K} \in V(J(E))$ by $\tilde{K} \cdot j(s) = (\iota L_K s)^{\infty} \cdot j(s)$.

Then K is well defined and it is obviously a deformation. Let $\lambda \in H^0{}_n(E)$ be a given Lagranglan and $s \in \Gamma(E)$. We define the variation δ_K of the functional by

$$\delta_K = \int_A j(s)^* L_{\tilde{K}} \lambda$$

Note that here $L_{\tilde{K}}$ is an ordinary Lie derivative.

After a similar (but longer) process of integration by parts, we get the desired variational equations for abstract fluids as differential forms

$$\sum_{p=0}^{n} \left(ds_{(p)} \overset{\cdot}{(p)} \omega_{s_{(p+k)}} - s_{(p)} \overset{\cdot}{(p-1)} \delta \omega_{s_{(p+k)}} \right) = 0 \qquad , \qquad 0 \leq k \leq n \qquad (10)$$

where the letter p under the dot means a p-fold contraction and ω_s is obtained as before but now $\omega_{s_{(p)}} \in \Gamma(\Lambda_p M \otimes \Lambda^n M)$.

We have, after a lot of algebraic work :

<u>Theorem</u> : Let $\mathcal{U} = \{ m \in M \mid s_{(1)}(m) \neq 0 \,,\ \omega_{s(n)} \neq 0 \}$. Then $ds = \delta\omega_s = 0$ in $\mathcal{U}$ if s satisfies (10).

Notice that $ds = \delta\omega_s = 0$ is always a sufficient condition for solving (10); in $\mathcal{U}$ it is also necessary. The condition $ds = 0$ is the equation of continuity; and the condition $\delta\omega_s = 0$ is almost the condition for having a prefluid (of course, in R^4, where every closed form is exact, both conditions coincide). So, we have a clean and nice generalization of Schrödinger equation as far as its formal aspects are concerned.

We have begun to study the system $ds = \delta\omega_s = 0$ for the simplest case, that is $n = 1 + 1$. The first thing to do is to choose the Lagrangian. For the better known field, the electromagnetic one, $F \in \Gamma(\Lambda^2(M))$,we have that the energy is given by $g(F,F)$; and it must also be remarked that the equations $ds = \delta\omega_s = 0$ are then Maxwell's equations for empty space. So, we have chosen, for a form $s = \Sigma^n{}_{p=0}\, s_{(p)}$ the Lagrangian

$$\lambda = \left(\rho - \frac{g(d\rho, d\rho)}{\rho}\right)\tau$$

where $\rho = g(s,s) = \Sigma_p\, g(s_{(p)}, s_{(p)})$ is calculated as usual. Then, we have a classical term, ρ, and the quantum one $-\, g(d\rho,d\rho)/_\rho$, which is suggested naturally by the Schrödinger equation. This last term would be very small in everyday units.

So, let $n = 2$, and take $g = dt^2 - dx^2$. We put

$$s = A + (Bdt + Edx) + Fdt \wedge dx$$
$$+\text{-}\mu^2 = A^2 + B^2 - E^2 - F^2$$
$$Q = 1 + (\mu_{tt} - \mu_{xx})/_\mu .$$

The equations become

$$A = k_0 \in R \qquad QF = k_2 \in R$$

$$B = u_t \qquad QE = v_t$$

$$\underbrace{E = u_x}_{ds = 0} \qquad \underbrace{QB = v_x}_{\delta\omega_S = 0,}$$

where u and v stand for two unknown functions on R^2.

We look first for a static solution. Then $u = f(x) + at$, $v = h(x) + bt$. Therefore

$$\pm \mu^2 = k_0^2 + a^2 - \frac{k_2^2 + b^2}{Q^2} .$$

Putting

$$0 \neq m^2 = \frac{k_2^2 + b^2}{k_0^2 + a^2} , \qquad p = \sqrt{k_0^2 + a^2} , \qquad z = \frac{\mu}{p} ,$$

we get the equation

$$z'' = z \left(1 - \frac{m}{\sqrt{1 \pm z^2}} \right) .$$

There is one non-zero solution that tends rapidly to zero at infinity (lump solution). Its exact expression varies heavily with the value of the parameter m. The formula for the simplest case , m=3/4,(fig. 1) is

$$z = \left(1 - \left(\frac{2q}{3+q}\right)^2\right)^{1/2} , \text{ where } \qquad q = \sqrt{1 + 8 \tanh^2 \frac{x}{2}} ,$$

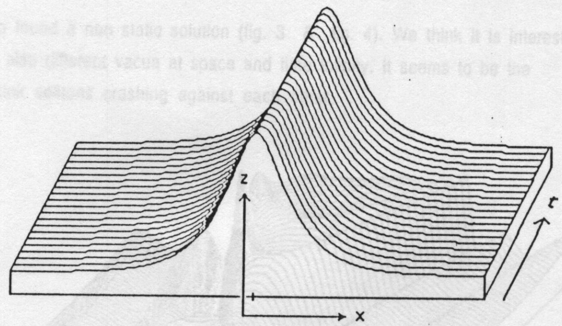

Figure 1

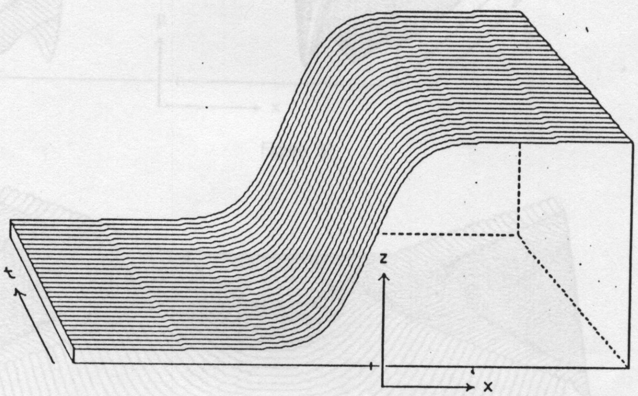

Figure 2

There is also one solution of the kink type (fig. 2). It connects two different vacua at infinity.

We have also found a non static solution (fig. 3 & fig. 4). We think it is interesting because it connects also different vacua at space and time infinity. It seems to be the description of two kink solitons crashing against each other.

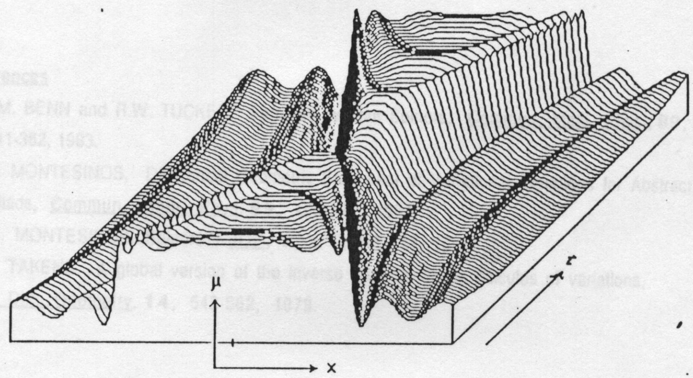

Figure 3′

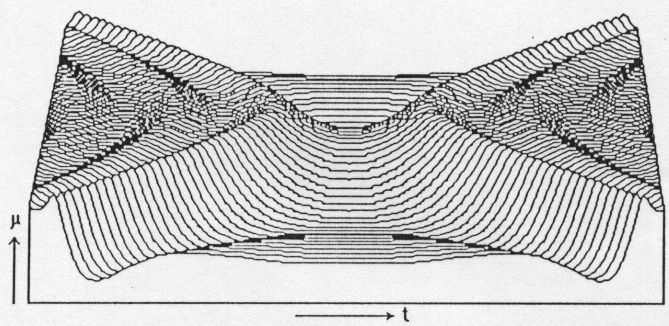

Figure 4

Of course, since we have relativistic invariance, all these solutions can be boosted into a uniform movement.

For real spacetime (n = 1 + 3), we know that at least there is a solution of the lump type.

References

1. I.M. BENN and R.W. TUCKER, Fermions without Spinors, Commun. Math. Phys., **89**, 341-362, 1983.
2. A. MONTESINOS, Differential Geometric Statement of Variational Equations for Abstract Fluids, Commun. Math. Phys, **95**, 185-197, 1984.
3. A. MONTESINOS, Spinors?, JGP, **2**, n. 2, 145-153, 1985.
4. F. TAKENS, A global version of the inverse problem of the calculus of variations, J. Diff. Geometry, **14**, 543-562, 1979.