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CALCULUS OF VARIATIONS IN A SIMPLE SUPERDOMAIN WITHOUT THE BEREZINIAN INTEGRAL*

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Abstract. We study the setting for the calculus of variations in a simple superdomain from the point of view of a recently introduced integration formula for $\mathbb{Z}_2$-graded differential 1-forms along $(1,1)$-dimensional curves. The corresponding Euler-Lagrange equations are also obtained.

Key words: Supermanifolds, Berezin integral, Superdifferential Equations, Euler-Lagrange Equations

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Let $\mathbb{R}^{1|1}$ be the fundamental $(1,1)$-dimensional supermanifold $(\mathbb{R}, \mathcal{A}^{1|1})$, where $\mathcal{A}^{1|1}$ stands for the structure sheaf $C^\infty_{\mathbb{R}} \otimes \bigwedge \mathbb{R}$. Let $\mathbb{I}^{1|1}$ be the closed $(1,1)$-dimensional submanifold defined by restriction of $\mathcal{A}^{1|1}$ to the closed interval $[0,1]$. We shall consider the trivial bundle $\pi_1 \colon \mathbb{I}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{I}^{1|1}$, defined by the canonical projection of the product onto the first factor, together with its corresponding first jet of sections $J^1(\mathbb{I}^{1|1}, \mathbb{R}^{1|1})$. This is a bundle over $\mathbb{I}^{1|1} \times \mathbb{R}^{1|1}$ isomorphic to $\mathbb{I}^{1|1} \times \mathbb{T}\mathbb{R}^{1|1}$ (*cf*, [6]), where $\mathbb{T}\mathbb{R}^{1|1}$ denotes the $(3,3)$-dimensional supertangent bundle of $\mathbb{R}^{1|1}$ (*cf*, [2], [8] first and second factors, respectively, and $\pi: \mathbb{TR}^{1|1} \to \mathbb{R}^{1|1}$ the supertangent bundle projection.

A section of $\pi_1: \mathbb{I}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is of the form $\sigma = id \times \gamma$, with $\gamma: \mathbb{I}^{1|1} \to \mathbb{R}^{1|1}$ a $(1,1)$-dimensional supercurve. Its *first jet extension*, $j\sigma: \mathbb{R}^{1|1} \to \mathbb{I}^{1|1} \times \mathbb{TM}$, is also a $(1,1)$-dimensional supercurve of the form $j\sigma = id \times j\gamma$, where $j\gamma: \mathbb{R}^{1|1} \to \mathbb{TM}$ is the map obtained via the superjacobian of the section γ (*cf*, [2], [8], and [10-11]).

Since the differential equations deduced from a variational principle have a local nature, we shall fix some local coordinates to simplify the discussion. The reader is referred to [3] for the fundamental theory on supermanifolds, and [1-2], [7-8] for notation, conventions, and further definitions.

Let $\{x;\theta\}$ be some set of local coordinates on $\mathbb{R}^{1|1}$ and let $\{t;\tau\}$ be restriction to $\mathbb{I}^{1|1}$ of the usual abelian-supergroup coordinates on $\mathbb{R}^{1|1}$. We recall that there is a prefered (global) supercoordinate system $\{t,\tau\}$ on $\mathbb{R}^{1|1}$ which is particularly useful when $\mathbb{R}^{1|1}$ is regarded as an abelian Lie supergroup (*cf*, [1], and [7]); namely, the identity chart on the reals defines a linear functional t on the vector space $\mathbb{R}$, and its dual $\tau \in (\mathbb{R}^*)^*$, is the generator for the exterior factor of the structure sheaf $\mathcal{A}^{1|1}$ (see also [2-3]). Local coordinates on $\mathbb{TR}^{1|1}$ are then written in the form $\{(x;\theta), (x_t, \theta_\tau; x_\tau, \theta_t)\}$, where $(x;\theta)$ correspond to the coordinates on the base $\mathbb{R}^{1|1}$, and $(x_t, \theta_\tau; x_\tau, \theta_t)$ correspond to the coordinates on the superfibers. In the notation of [11], $x_\tau = \pi x$, and $\theta_\tau = \pi\theta$.

Following the analogy with the non-graded calculus of variations, a *Lagrangian* is a 1 form on the graded manifold $J^1(\mathbb{I}^{1|1}, \mathbb{R}^{1|1})$ of the special form

$$dt\, L + d\tau\, \lambda \tag{1}$$

where L and λ are superfunctions on $J^1(\mathbb{I}^{1|1}, \mathbb{R}^{1|1})$. In other words, a 1-form lying in the ideal generated by $d(\pi_1 \circ \rho)^* \mathcal{A}^{1|1}([0,1])$. (We also recall that the space of 1 forms on the graded manifold $\mathbb{I}^{1|1}$ defines a locally free sheaf of *right* modules over $C^\infty_{[0,1]} \otimes \bigwedge \mathbb{R}$ of rank $(1,1)$ locally generated by dt and $d\tau$; *cf*, [2-3]).

What triggers the Calculus of Variations is an *action functional* $\mathcal{L}$ associated to the Lagrangian 1-form (1); *i.e.*, a linear map defined on the space of sections of $\pi_1: \mathbb{I}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{I}^{1|1}$. Now, in order to define such an action functional one only needs a notion of *line integral*, and the starting point of the Variational Calculus would be

$$\sigma \mapsto \mathcal{L}[\sigma] = \int_{j^1\sigma} (dt\, L + d\tau\, \lambda) \tag{2}$$

where $j^1\sigma: \mathbb{I}^{1|1} \to J^1(\mathbb{I}^{1|1}, \mathbb{R}^{1|1})$ is the first jet extension of the section $\sigma: \mathbb{I}^{1|1} \to \mathbb{R}^{1|1}$, as defined above. Naturality in the definition of the integral on the right requires that

$$\int_{j^1\sigma} (dt\, L + d\tau\, \lambda) = \int_{\mathbb{I}^{1|1}} (j^1\sigma)^* (dt\, L + d\tau\, \lambda). \tag{3}$$

Now, the right hand side can be explicitly computed, provided that an integration formula for 1-forms on $\mathbb{I}^{1|1}$ has been given. This was precisely the main goal of [12], and the purpose of this note is to deduce the differential equations associated to the Lagrangian (1) when the integral formula of [12] is used. The point is that, once the integral on the right is defined, the Variational Calculus calls for the *stationary values* of $\mathcal{L}$.

Definition: (*a*) A $(1,1)$-*variation* of the section $\sigma: \mathbb{I}^{1|1} \to \mathbb{I}^{1|1} \times \mathbb{R}^{1|1}$ is a morphism

$$\delta\sigma: \mathbb{I}^{1|1} \times \mathbb{E}^{1|1} \to \mathbb{I}^{1|1} \times \mathbb{R}^{1|1}$$

with

$$\delta\sigma = p_1 \times \delta\gamma, \qquad \delta\gamma: \mathbb{I}^{1|1} \times \mathbb{E}^{1|1} \to \mathbb{R}^{1|1}, \qquad \mathrm{ev}|_{s=0}(\delta\gamma)^* = \gamma^* \tag{4}$$

where $\mathbb{E}^{1|1}$ is a $(1,1)$-dimensional open subdomain of $\mathbb{R}^{1|1}$ containing $0 \in \mathbb{R}$—say, $((-\varepsilon,\varepsilon),\ C^\infty_{(-\varepsilon,\varepsilon)} \otimes \bigwedge \mathbb{R})$—with $\{s,\xi\}$ being the restriction to $\mathbb{E}^{1|1}$ of the natural abelian-supergroup coordinates above.

(*b*) The section $\sigma: \mathbb{I}^{1|1} \to \mathbb{I}^{1|1} \times \mathbb{R}^{1|1}$ produces a *stationary value* of the action functional (2) if *for any* $(1,1)$-*variation* $\delta\sigma$,

$$\mathrm{ev}|_{s=0} D\, \mathcal{L}[\delta\sigma] = 0 \tag{5}$$

where $D = \partial_s + \partial_\xi$ is the fundamental derivation of $\mathcal{A}^{1|1}|_{(-\varepsilon,\varepsilon)}$ (*cf*, [1], and [7]).

We can now follow the prescription given in [12] to explicitly compute the *superline integrals* of the type (3). According to the work there, and in terms of the coordinates $\{t,\tau\}$, the superline integral of the 1-superform $\omega = dt\,(f_0 + f_1\,\tau) + d\tau\,(g_0 + g_1\,\tau)$ on $\mathbb{I}^{1|1}$ is given by

$$\int_{\mathbb{I}^{1|1}} \omega = \int_0^1 \left(f_0 - \int_0^t g_1\right) + \tau \int_0^1 f_1. \tag{6}$$

Now, a direct computation shows that the condition $\mathrm{ev}|_{s=0} D\, \mathcal{L}[\delta\sigma] = 0$ implies that

$$\int_{\mathbb{I}^{1|1}} \omega_{A,B} = 0 \tag{7}$$

where

$$\begin{aligned} \omega_{A,B} \;=\; & dt \left(A\frac{\partial L}{\partial x} + B\frac{\partial L}{\partial \theta} + \frac{dA}{dt}\frac{\partial L}{\partial x_t} + \frac{dB}{dt}\frac{\partial L}{\partial \theta_t} + \frac{dA}{d\tau}\frac{\partial L}{\partial x_\tau} + \frac{dB}{d\tau}\frac{\partial L}{\partial \theta_\tau} \right) \\ & + d\tau \left(A\frac{\partial \lambda}{\partial x} + B\frac{\partial \lambda}{\partial \theta} + \frac{dA}{dt}\frac{\partial \lambda}{\partial x_t} + \frac{dB}{dt}\frac{\partial \lambda}{\partial \theta_t} + \frac{dA}{d\tau}\frac{\partial \lambda}{\partial x_\tau} + \frac{dB}{d\tau}\frac{\partial \lambda}{\partial \theta_\tau} \right) \end{aligned} \tag{8}$$

and

$$A = \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial \xi}\right)_{s=0} (\delta\gamma)^{*} x = A_0 + A_1 \tau,$$

$$B = \left(\frac{\partial}{\partial s} + \frac{\partial}{\partial \xi}\right)_{s=0} (\delta\gamma)^{*} \theta = B_0 + B_1 \tau. \qquad (9)$$

The resulting Euler-Lagrange equations that we have found from (7) are:

$$\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x_t} + \frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right) = 0$$

$$\left(\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \theta_t} + \frac{d}{d\tau}\frac{\partial L}{\partial \theta_\tau}\right) = 0$$

$$(1-t)\left(\frac{\partial \lambda}{\partial x} - \frac{d}{dt}\frac{\partial \lambda}{\partial x_t} + \frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right) = \left(\frac{\partial L}{\partial x_\tau} - \frac{\partial \lambda}{\partial x_t}\right) \qquad (10)$$

$$(1-t)\left(\frac{\partial \lambda}{\partial \theta} - \frac{d}{dt}\frac{\partial \lambda}{\partial \theta_t} + \frac{d}{d\tau}\frac{\partial \lambda}{\partial \theta_\tau}\right) = \left(\frac{\partial L}{\partial \theta_\tau} - \frac{\partial \lambda}{\partial \theta_t}\right).$$

We remark that these equations are not equivalent to those deduced from the Berezinian density approach when a Lagrangian of the form $dt\, L$ is used (*cf.* [4-5]). In fact, the equations obtained here imply that L does not depend on x_τ nor on θ_τ when $\lambda = 0$. There are also some new cases to be explored that arise from the *super-interaction* symmetries,

$$\frac{\partial L}{\partial x_\tau} - \frac{\partial \lambda}{\partial x_t} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \theta_\tau} - \frac{\partial \lambda}{\partial \theta_t} = 0. \qquad (11)$$

We shall close this report with a brief outline of how the Euler-Lagrange equations above are deduced. Complete details will be given elsewhere (*cf.* [9]).

First, one notes that there are some derivatives with respect to t and τ appearing in $\omega_{A,B}$. We have found, however, that the common technique in the calculus of variations of integrating by parts to produce a total derivative contributing only with a boundary term, does not have an exact counterpart in this setting. What we have instead is the following lemma, whose proof is a straightforward consequence of the definition (6):

1. Lemma: *Let* $\omega = dt\,(f_0+f_1\tau)+d\tau\,(g_0+g_1\tau)$. *Let* $B = B_0+B_1\tau \in \mathcal{A}^{1|1}([0,1])$. *Then,*

$$(1) \quad \int_{\mathbb{I}^{1|1}} \omega = \int_{\mathbb{I}^{1|1}} \left(\omega + d\tau\, \partial_\tau(B)\right)$$

$$(2) \quad \int_{\mathbb{I}^{1|1}} \omega = \int_{\mathbb{I}^{1|1}} \left(\omega + d\tau\, \partial_t(B)\right) + \int_0^1 B_1 - B_1(0)$$

$$(3) \quad \int_{\mathbb{I}^{1|1}} \omega = \int_{\mathbb{I}^{1|1}} \left(\omega + dt\, \partial_\tau(B)\right) - \int_0^1 B_1$$

$$(4) \quad \int_{\mathbb{I}^{1|1}} \omega = \int_{\mathbb{I}^{1|1}} \left(\omega + dt\, \partial_t(B)\right) - \left(B_0(1) + B_1(1)\tau - (B_0(0) + B_1(0)\tau)\right).$$

We have also found that when the integral of the product of a 1-superform with an arbitrary smooth function on $[0,1] \subset \mathbb{R}$ is identically zero, it does not necessarily follow that the superform itself is equal to zero. The precise statement is given in the following lemma whose proof requires only a little analysis on $C^\infty([0,1])$.

2. Lemma: *Let* $\omega = dt\,(f_0 + f_1\tau) + d\tau\,(g_0 + g_1\tau)$. *Let* $A = A_0 \in C^\infty(\mathbb{R})$. *Then,*

$$\int_{\mathbb{I}^{1|1}} \omega\, A_0 = 0, \quad \text{for all } A_0 \qquad \Longrightarrow \qquad f_0(t) = g_1(t)(1-t) \quad \text{and} \quad f_1 = 0.$$

Using these two results one can now look at the equation (7) by studying separately the cases $B_0 = 0 = B_1$, and $A_0 = 0 = A_1$. Let us here consider in some detail the case $B_0 = B_1 = 0$. It is easy to check that

$$\omega_A = dt\left(A\frac{\partial L}{\partial x} - A\frac{d}{dt}\frac{\partial L}{\partial x_t} - \bar{A}\frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right) + dt\left(\frac{d}{dt}\left(A\frac{\partial L}{\partial x_t}\right) + \frac{d}{d\tau}\left(A\frac{\partial L}{\partial x_\tau}\right)\right)$$

$$+ d\tau\left(A\frac{\partial \lambda}{\partial x} - A\frac{d}{dt}\frac{\partial \lambda}{\partial x_t} - \bar{A}\frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right) + dt\left(\frac{d}{dt}\left(A\frac{\partial \lambda}{\partial x_t}\right) + \frac{d}{d\tau}\left(A\frac{\partial \lambda}{\partial x_\tau}\right)\right) \qquad (12)$$

where $\bar{A} = A_0 - A_1\,\tau$, whenever $A = A_0 + A_1\,\tau$. We then apply the first lemma to obtain,

$$\int_{\mathbb{I}^{1|1}} \omega_A = \int_{\mathbb{I}^{1|1}} \left\{ dt\left(A\frac{\partial L}{\partial x} - A\frac{d}{dt}\frac{\partial L}{\partial x_t} - \bar{A}\frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right)\right.$$

$$\left. + d\tau\left(A\frac{\partial \lambda}{\partial x} - A\frac{d}{dt}\frac{\partial \lambda}{\partial x_t} - \bar{A}\frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right)\right\} + \int_0^1 \left\{\left(A\frac{\partial L}{\partial x_\tau} - A\frac{\partial \lambda}{\partial x_t}\right)_1\right\} \qquad (13)$$

We can look first at the subcase $A_0 = 0$; *i.e.*,

$$0 = \int_{\mathbb{I}^{1|1}} \omega_{A_1} = \int_0^1 \left\{ A_1\left(\frac{\partial L}{\partial x_\tau} - \frac{\partial \lambda}{\partial x_t}\right)_0 - \int_0^{(\cdot)} A_1\left(\frac{\partial \lambda}{\partial x} - \frac{d}{dt}\frac{\partial \lambda}{\partial x_t} + \frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right)_0 \right\}$$

$$+ \tau \int_0^1 A_1\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x_t} + \frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right)_0 \qquad (14)$$

Using the second lemma we conclude that

$$\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x_t} + \frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right)_0 = 0$$

$$(1-t)\left(\frac{\partial \lambda}{\partial x} - \frac{d}{dt}\frac{\partial \lambda}{\partial x_t} + \frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right)_0 = \left(\frac{\partial L}{\partial x_\tau} - \frac{\partial \lambda}{\partial x_t}\right)_0 \qquad (15)$$

To investigate the condition $A_1 = 0$, and $A_0 \neq 0$ one proceeds in an entirely similar manner, but using the fact that $f_0 = 0$ already. The resutls are then

$$\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial x_t} + \frac{d}{d\tau}\frac{\partial L}{\partial x_\tau}\right)_1 = 0$$

$$(1-t)\left(\frac{\partial \lambda}{\partial x} - \frac{d}{dt}\frac{\partial \lambda}{\partial x_t} + \frac{d}{d\tau}\frac{\partial \lambda}{\partial x_\tau}\right)_1 = \left(\frac{\partial L}{\partial x_\tau} - \frac{\partial \lambda}{\partial x_t}\right)_1 \tag{16}$$

Finally, the case when $A_0 = A_1 = 0$, and $B \neq 0$ is handled similarly. The results are the same except for the fact that the coordinate x gets replaced by the coordinate θ.

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