Lifting of Symplectic Structures to Graded Symplectic Structures

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Abstract. Given a symplectic structure on a differentiable manifold, M, we show, by applying a characterization of derivations, that there exists an associated even graded symplectic structure on any graded manifold whose underlying manifold is M. We also show a similar result for odd graded symplectic structures.

INTRODUCTION

The purpose of this work is to show that any graded manifold, whose underlying differentiable manifold has a symplectic structure, can itself be endowed with an associated graded symplectic structure. The association is given by the following condition: the graded symplectic structure induces on the differentiable manifold its initial symplectic structure.

For the case of an exact symplectic structure on the underlying manifold, this problem was solved in [3], but the general case was still open. The authors of this paper introduce the general problem and present some of the difficulties encountered to solve it. Following different methods, we rederive the same result in an intrinsic way and solve the general case.

By Batchelor's theorem [1], every graded manifold is (not canonically) isomorphic to a graded manifold of the kind $(M, \Gamma(\Lambda E))$, where E is a suitable vector bundle and $\Gamma(\Lambda E)$ is the sheaf of local sections of the exterior bundle. We study first such kind of graded manifolds because any object defined on them can be carried to the graded manifold by the isomorphism thus getting the same kind of object on the graded manifold.

Vector fields on a graded manifold are defined as derivations of the sheaf of algebras. Therefore we must study the derivations of $\Gamma(\Lambda E)$ that are characterized in [7].

Graded 1-forms are $\Gamma(\Lambda E)$-homomorphisms of the $\Gamma(\Lambda E)$-module of derivations on $\Gamma(\Lambda E)$. We characterize here such homomorphisms and so that we can define, in an intrinsic way, graded 1-forms.

Given an exact symplectic structure on M, $\bar{\omega} = d\bar{\lambda}$, and a metric on E^*, the dual vector bundle, we can define a graded 1-form such that its differential is an even exact nondegenerate graded 2-form, $\omega_{\bar{\lambda}}$. Moreover, if $d\bar{\lambda} = d\bar{\mu}$, then $\omega_{\bar{\lambda}} = \omega_{\bar{\mu}}$. Therefore, an usual argument for sheaves shows the general case.

In graded manifolds with the even dimension equal to the odd dimension there is room for odd symplectic structures. We show that an odd graded symplectic structure, ω defines a unique isomorphism $\Pi\omega : TM \to E$. And, as in the even case, given an isomorphism $\omega^{\#}$ between TM and E we can define an odd nondegenerate graded 2-form, ω such that $\Pi\omega = \omega^{\#}$.

1. DEFINITIONS

For a detailed study of graded manifolds we refer to [4]. Let $(M, \mathcal{A})$ be a graded manifold of graded dimension $dim(M, \mathcal{A}) = (m, n)$, in the following sense: M is a C^∞ real manifold of dimension m, and $\mathcal{A}$ is a sheaf of $\mathbb{Z}_2$ graded commutative algebras such that

(1) there exists a surjective sheaf morphism $\sim : \mathcal{A}(M) \to C^\infty(M)$ often called the natural morphism,

(2) there exists an open covering $\{U_i\}_{i\in I}$ of M and sheaf isomorphisms $\mathcal{A}(U_i) \to \Lambda \mathbb{R}^n(U_i)$.

The surjective sheaf morphism induces a map between the modules of differential forms $\kappa : \Omega(M, \mathcal{A}) \to \Omega(M)$ (See [4] page 257 for definition).

Let $\pi : E \to M^m$ be a real vector bundle where the dimension of M is m and the fibre dimension is n, and let $\pi : \Lambda E \to M$ be the exterior bundle of E.

Let $\Gamma(\Lambda E)$ be the sheaf of exterior $\mathbb{R}$-algebra of local smooth sections of ΛE (all objects are C^∞). The pair $(M, \Gamma(\Lambda E))$ is a graded manifold of graded dimension (m, n). Batchelor's theorem [1] asserts that any graded manifold is isomorphic to a graded manifold $(M, \Gamma(\Lambda E))$ for a suitable E. Thus we will focus our attention on such kind of graded manifolds.

If $s \in \Gamma(\Lambda E)$ is homogeneous, say of degree p, then we will write $|s| = p$. If $s \in \Gamma(\Lambda E)$, we denote by $S_{(p)}$ the homogeneous component of s of degree p; thus $s = \sum_{p=0}^{n} s_{(p)}$. A linear endomorphism $D : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is said to be homogeneous of degree $|D|$ if $|D(s_{(p)})| = p + |D|$.

A homogeneous linear homomorphism $D : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is called a homogeneous derivation of degree $|D|$ if, for homogeneous $s_1, s_2 \in \Gamma(\Lambda E)$,

$$D(s_1 s_2) = (Ds_1)s_2 + (-1)^{|D||s_1|} s_1 (Ds_2).$$

We fix the following terminology: when we refer to the $\mathbb{Z}_2$-grading, we shall talk about *even* or *odd* objects if they are of degree 0 or 1, respectively. We shall reserve the expression *object of degree ...* for the $\mathbb{Z}$-grading.

Every homogeneous derivation is determined by its action on the elements of degree 0 and 1. Thus all derivations of degree less than -1 are zero. A linear endomorphism of $\Gamma(\Lambda E)$ is called a derivation if its homogeneous parts are derivations.

Let $\mathcal{M}, \mathcal{N}$ be graded $\Gamma(\Lambda E)$-modules. A morphism $\mathcal{S} : \mathcal{M} \to \mathcal{N}$ is called of degree $|\mathcal{S}|$ if $\mathcal{S}(x_{(p)})$ is of degree $|\mathcal{S}| + p$, and is called a $\Gamma(\Lambda E)$-endomorphism if $\mathcal{S}(sx) = (-1)^{||\mathcal{S}||s|} s\mathcal{S}(x), \qquad s \in \Gamma(\Lambda E), x \in \mathcal{M}.$

Two classes of derivations on $\Gamma(\Lambda E)$.

Let $TM \to M$ be the tangent bundle over M and $\Gamma(\Lambda E \otimes TM)$ the $\Gamma(\Lambda E)$-module of smooth sections of $\Lambda E \otimes TM$. We can define in $\Gamma(\Lambda E \otimes TM)$ a $\mathbb{Z}$ and a $\mathbb{Z}_2$-grading. Every $\Psi \in \Gamma(\Lambda E \otimes TM)$ can be expressed as a finite sum of decomposable homogeneous sections $s_{(p)} \otimes X$ where $s_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $X \in \Gamma(TM)$.

Let ∇ be a linear connection in E. If $\Psi = s_{(p)} \otimes X$, we define the endomorphism $\nabla_\Psi : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $\nabla_\Psi u = s_{(p)} \nabla_X u$, , where $u \in \Gamma(\Lambda E)$, and if $\Psi \in \Gamma(\Lambda E \otimes TM)$, we define ∇_Ψ by linear extension.

∇_Ψ is a derivation and we call it the *proper derivation associated to Ψ trough* ∇. If Ψ is a homogeneous element, in whatever grading, then ∇_Ψ is a derivation of degree $|\Psi|$.

Now, we shall define another type of derivations, the algebraic ones.

Let $\pi : E^* \to M$ be the dual bundle of E, and let $\Gamma(\Lambda E \otimes E^*)$ be the $\Gamma(\Lambda E)$-module of smooth sections of $\Lambda E \otimes E^*$. We define in $\Gamma(\Lambda E \otimes E^*)$ a $\mathbb{Z}$ and a $\mathbb{Z}_2$-grading.

Every $\Phi \in \Gamma(\Lambda E \otimes E^*)$ can be expressed as a finite sum of decomposable homogeneous sections $s_{(p)} \otimes \alpha$ where $s_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $\alpha \in \Gamma(E^*)$.

If $\Phi = s_{(p)} \otimes \alpha$, we define the endomorphism $i_\Phi : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $i_\Phi u = s_{(p)} i_\alpha u$, where $u \in \Gamma(\Lambda E)$, and where i_α is the interior multiplication; and if $\Phi \in \Gamma(\Lambda E \otimes E^*)$, we define i_Φ by linear extension.

i_Φ is a derivation that acts trivially on the smooth functions on M, and we call it the *algebraic derivation associated to Φ*. If Φ is a homogeneous element, in either grading, then i_Φ is a derivation of degree $|\Phi| - 1$ (modulo 2 if we are dealing with the $\mathbb{Z}_2$-grading).

2. Characterization of the derivations on $\Gamma(\Lambda E)$

The following characterization can be found in [7]. We will include here the proof because it shows how a graded vector field of the graded manifold $(M, \Gamma(\Lambda E))$ admits a brief representation by means of the two previous kinds of derivations.

The characterization is analogous to that of Frölicher-Nijenhuis [2].

Proposition 1. *Let D be a derivation on $\Gamma(\Lambda E)$, and let ∇ be a connection in E. Then, there are unique fields $\Psi \in \Gamma(\Lambda E \otimes TM)$ and $\Phi \in \Gamma(\Lambda E \otimes E^*)$ such that*

$$D = i_\Phi + \nabla_\Psi .$$

Proof: Suppose that D is a homogeneous derivation of degree $k \geq 0$, and let $\alpha^1, \ldots, \alpha^k \in \Gamma(E^*)$ be smooth sections. The map

$$f \to (Df)(\alpha^1, \ldots, \alpha^k), \qquad f \in C^\infty(M) \subset \Gamma(\Lambda E),$$

is a derivation and then it defines a vector field on M, which we denote by $\Psi(\alpha^1, \ldots, \alpha^k)$. The map from $\Gamma(E^*) \times \cdots \times \Gamma(E^*)$ to $\Gamma(TM)$ defined by $(\alpha^1, \ldots, \alpha^k) \to \Psi(\alpha^1, \ldots, \alpha^k)$ is $C^\infty(M)$-linear and skewsymmetric, whence it de fines a section $\Psi \in \Gamma(\Lambda E \otimes TM)$ that satisfies $\nabla_\Psi f = Df$ for every $f \in C^\infty(M)$.

Then, the operator $D - \nabla_\Psi$ is a derivation of degree k that acts trivially o $C^\infty(M)$; therefore it is a $C^\infty(M)$-linear endomorphism which is determined by it action on the sections of degree 1. Then, if $s \in \Gamma(E)$ the map $s \to (D - \nabla_\Psi)$ defines a homomorphism from $\Gamma(E)$ into $\Gamma(\Lambda^{k+1} E)$ and therefore there is a sectio $\Phi \in \Gamma(\Lambda E \otimes E^*)$ such that $(D - \nabla_\Psi)s = i_\Phi s$. The operator i_Φ is a derivation o degree k that acts trivially on $C^\infty(M)$ and as $D - \nabla_\Psi$ on the sections of E. The $D = \nabla_\Psi + i_\Phi$.

The case of k = -1 is trivial. ∎

Let $U \subset M$ be a coordinate open sibset with coordinate functions $\{x_i\}_{i=1}^m$ and le $\{s_j\}_{j=1}^n$ be a local basis for $\Gamma(E(U))$. To avoid confussion, let us denote by $\{r_i\}_{i=1}^m$ the same functions $\{x_i\}_{i=1}^m$ but now considered as elements of $\Gamma(\Lambda(E(U)))$, i.e., a even graded coordinates. Then $\{r_i, s_j\}$ are graded coordinates on $(U, \Gamma(\Lambda(E(U))))$ By theorem 2.8 of [4] there exist unique derivations $\{\frac{\partial}{\partial r_i}, \frac{\partial}{\partial s_j}\}$ with the followin properties:

$$\frac{\partial r_k}{\partial r_i} = \delta_{ik} 1_U, \quad \frac{\partial s_\ell}{\partial r_i} = 0,$$

$$\frac{\partial r_k}{\partial s_j} = 0, \quad \frac{\partial s_\ell}{\partial s_j} = \delta_{j\ell} 1_U.$$

Furthermore, $\frac{\partial}{\partial r_i}$ is an even derivation and $\frac{\partial}{\partial s_j}$ is an odd derivation and any derivation D on $\Gamma(\Lambda(E(U)))$ can be uniquely written as

$$D = \sum_{i=1}^m a_i \frac{\partial}{\partial r_i} + \sum_{j=1}^n b_j \frac{\partial}{\partial s_j},$$

where $a_i, b_j \in \Gamma(\Lambda(E(U)))$.

Let us compare this with the two kinds of derivations of proposition 1. It is clear that, if $\{\theta^j\}_{j=1}^n$ is the dual local basis of $\{s_j\}_{j=1}^n$, then $\frac{\partial}{\partial s_j} = i_{\theta^j}$. On the other hand, $\frac{\partial}{\partial r_i}$ and $\nabla_{\frac{\partial}{\partial x_i}}$ are in general different derivations and the relation is

$$\nabla_{\frac{\partial}{\partial x_i}} = \frac{\partial}{\partial r_i} + s_\ell \Gamma^j_{i\ell} \frac{\partial}{\partial s_j},$$

where $\Gamma^j_{i\ell}$ are the Christoffel symbols of ∇.

3. Characterization of the graded 1-forms on $(M, \Gamma(\Lambda E))$

Graded 1-forms are $\Gamma(\Lambda E)$-linear homomorphisms from the module of graded vector fields into $\Gamma(\Lambda E)$. By Proposition 1 we have that a graded vector field, i.e., a derivation D, is uniquely determined by two objects: $\Psi \in \Gamma(\Lambda E \otimes TM)$ and $\Phi \in \Gamma(\Lambda E \otimes E^*)$. Moreover, the maps from $Der(\Gamma(\Lambda E))$ into $\Gamma(\Lambda E \otimes TM)$ and $\Gamma(\Lambda E \otimes E^*)$ defined by $D \to \Psi$ and $D \to \Phi$, respectively, are $\Gamma(\Lambda E)$-homomorphisms of $\Gamma(\Lambda E)$-modules. Indeed, if $s \in \Gamma(\Lambda E)$, $s\nabla_\Psi = \nabla_{s\Psi}$ and $si_\Phi = i_{s\Phi}$.

Then, to study graded 1-forms is equivalent to study two sets of $\Gamma(\Lambda E)$-homomorphisms, the first from $\Gamma(\Lambda E \otimes TM)$ into $\Gamma(\Lambda E)$ and the second from $\Gamma(\Lambda E \otimes E^*)$ into $\Gamma(\Lambda E)$. By definition of $\Gamma(\Lambda E)$-homomorphism, this is equivalent to study $C^\infty(M)$-linear homomorphisms from $\Gamma(TM)$ into $\Gamma(\Lambda E)$ and from $\Gamma(E^*)$ into $\Gamma(\Lambda E)$, respectively.

Let S be a $C^\infty(M)$-linear homomorphism from $\Gamma(TM)$ into $\Gamma(\Lambda E)$. If $X \in \Gamma(TM)$ let us suppose that $S(X)$ is an element of degree k and let $\alpha^1, \ldots, \alpha^k \in \Gamma(E^*)$ be smooth sections. Then, $S(X)(\alpha^1, \ldots, \alpha^k) \in C^\infty(M)$, thus it defines a differential form of degree 1, that we denote by $K(\alpha^1, \ldots, \alpha^k)$. The map $(\alpha^1, \ldots, \alpha^k) \to K(\alpha^1, \ldots, \alpha^k)$ is $C^\infty(M)$-linear and skewsymmetric, whence it defines a unique section $K \in \Gamma(\Lambda E \otimes T^*M)$.

Let us denote by $\rho(K)$ the $\Gamma(\Lambda E)$-homomorphism from $\Gamma(\Lambda E \otimes TM)$ into $\Gamma(\Lambda E)$ defined by $K \in \Gamma(\Lambda E \otimes T^*M)$. If $K = s \otimes \beta$, where $s \in \Gamma(\Lambda E)$ and $\beta \in A^1(M)$ and if $\Psi = u \otimes X \in \Gamma(\Lambda E \otimes TM)$ then

$$\rho(s \otimes \beta)(u \otimes X) = su\beta(X).$$

It is easy to check that, for homogeneous elements,

$$\rho(s \otimes \beta)(u \wedge \Psi) = (-1)^{|s||u|} u \rho(s \otimes \beta)(\Psi).$$

Analogously, a $C^\infty(M)$-linear homomorphism from $\Gamma(E^*)$ into $\Gamma(\Lambda E)$ is uniquely determined by a section $L \in \Gamma(\Lambda E \otimes E)$. Let us denote by $\mu(L)$ the $\Gamma(\Lambda E)$-homomorphism from $\Gamma(\Lambda E \otimes E^*)$ into $\Gamma(\Lambda E)$ defined by $L \in \Gamma(\Lambda E \otimes E)$. If $L = s \otimes Y$, where $s \in \Gamma(\Lambda E)$ and $Y \in \Gamma(E)$ and if $\Phi = u \otimes \alpha \in \Gamma(\Lambda E \otimes E^*)$ then

$$\mu(s \otimes Y)(u \otimes \alpha) = su\alpha(Y).$$

It is easy to check that, for homogeneous elements,

$$\mu(s \otimes Y)(u\Phi) = (-1)^{|s||u|} u \wedge \mu(s \otimes Y)(\Phi).$$

We have proved the following

Proposition 2. *Let ∇ be a fixed connection in E. Let Ω be a graded 1-form on the graded manifold $(M, \Gamma(\Lambda E))$ then there are unique fields $K \in \Gamma(\Lambda E \otimes T^*M)$ and $L \in \Gamma(\Lambda E \otimes E)$ such that*

$$\Omega(\nabla_\Psi + i_\Phi) = \rho(K)(\Psi) + \mu(L)(\Phi).$$

Remark: We will denote by $\Omega(D)$ the action of the graded form on the derivation, but having in mind that the action is by the left, i.e., in the notation of [4], it is $< D, \Omega >$.

4. Graded symplectic forms on $(M, \Gamma(\Lambda E))$

The module of derivations has a $\mathbb{Z}$-graduation, then the space of graded p-forms, for each p, has also an induced $\mathbb{Z}$-graduation, not only a $\mathbb{Z}_2$-graduation.

Definition. *A graded p-form ω is said to be of degree k if*

$$\omega(D_1, \ldots, D_p) \in \Gamma(\Lambda^{|D_1|+\cdots+|D_p|+k} E),$$

for all $D_1, \ldots, D_p$ graded vector fields of degree $|D_1|, \ldots, |D_p|$.

If k is even (odd) then ω is an even (odd) form.

Let us fix a connection ∇ in E.

Even symplectic structures. Let $\bar{\omega} = d\bar{\lambda}$ be an exact 2-form on M, and let $g : E^* \times_M E^* \to \mathbb{R}$ be a symmetric bilinear form on E^*. We can consider the graded 1-form defined, as in proposition 2, by $\bar{\lambda} \in \Gamma(T^*M) \subset \Gamma(\Lambda E \otimes T^*M)$ and by $g \in \Gamma(E \otimes E) \subset \Gamma(\Lambda E \otimes E)$.

The homomorphism $\rho(\bar{\lambda})$ is of degree 0 and the homomorphism $\mu(g)$ is of degree 1.

Let us recall that, if $X \in \Gamma(TM)$ and $\alpha \in \Gamma(E^*)$, then

$$\lambda(\nabla_X) = \rho(\bar{\lambda})(X) = \bar{\lambda}(X) \in C^\infty(M), \quad \lambda(i_\alpha) = \mu(g)(\alpha) = g(\alpha, \ ) \in \Gamma(E).$$

Thus λ sends even derivations to even sections and odd derivations to odd sections, i.e., λ is an even graded 1-form.

Let us put $\omega = d\lambda$, the graded 2-form differential of λ. ω is determined by its action on the following pairs of derivations $(\nabla_X, \nabla_Y), (i_\alpha, \nabla_X)$ and (i_α, i_β) where $X, Y \in \Gamma(TM)$ and $\alpha, \beta \in \Gamma(E^*)$. But first, we need to compute the brackets of each pair of derivations.

By proposition 1, the derivation $[\nabla_X, \nabla_Y]$ can be expressed as a sum of a proper and an algebraic derivation. The proper derivation is determined by the action of $[\nabla_X, \nabla_Y]$ on functions. It is easy to chek that $[\nabla_X, \nabla_Y]f = \nabla_{[X,Y]}f$. Therefore the algebraic derivation is $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and we have $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]} + i_{R(X,Y)}$, where $[X, Y]$ is the Lie bracket of the vector fields X, Y and where the curvature of ∇, $R(X, Y) \in \Gamma(E \otimes E^*)$, is given by the usual formula $R(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \in \Gamma(E)$, for every $s \in \Gamma(E)$.

$[i_\alpha, \nabla_X]$ is a derivation of degree -1, then, it is determined by its action on $s \in \Gamma(E)$.

$$[i_\alpha, \nabla_X]s = i_\alpha \nabla_X s - X(i_\alpha s) = \alpha(\nabla_X s) - X(\alpha(s)) = -(\nabla_X \alpha)s = -i_{\nabla_X \alpha} s.$$

Finally, $[i_\alpha, i_\beta] = 0$ because it is a derivation of degree -2.

Let us compute now such actions. By application of proposition 4.3.6 of [4], we get

$$\begin{aligned}\omega(\nabla_X,\nabla_Y) &= \nabla_X\lambda(\nabla_Y)-\nabla_Y\lambda(\nabla_X)-\lambda([\nabla_X,\nabla_Y])\\ &= X(\bar\lambda(Y)-Y(\bar\lambda(X)-\bar\lambda([X,Y])-\mu(g)(R(X,Y))\\ &= \bar\omega(X,Y)-\mu(g)(R(X,Y)),\\ \omega(i_\alpha,\nabla_X) &= i_\alpha\lambda(\nabla_X)-\nabla_X\lambda(i_\alpha)-\lambda([i_\alpha,\nabla_X])\\ &= -\nabla_X\mu(g)\alpha+\mu(g)\nabla_X\alpha\\ &= -(\nabla_X g)(\alpha,\ )\\ \omega(i_\alpha,i_\beta) &= i_\alpha\lambda(i_\beta)+i_\beta\lambda(i_\alpha)-\lambda([i_\alpha,i_\beta])\\ &= 2g(\alpha,\beta).\end{aligned}$$

Then, we can state the following lemma, first proved by different methods in [3]

Lemma 1. *Let M be a differentiable manifold supporting a symplectic structure defined by an exact 2-form $\bar\omega = d\bar\lambda$ (dimension of $M = m = 2\ell$) and let $(M,\mathcal{A})$ be a graded manifold over M. Then there exists an even exact nondegenerate graded 2-form ω such that $\kappa\omega = \bar\omega$.*

Proof: Let us suppose first that the graded manifold is $(M,\Gamma(\Lambda E))$. Let us choose a metric g on E^* and let ∇ be a connection in E.

Let ω be the graded form defined as before by $\bar\omega$, g and ∇. Then ω is an exact 2-form because it is the differential of a graded 1-form. It is nondegenerate by proposition 4.5.2 of [4]. Indeed, let us note that the part of degree 0 of $\omega(\nabla_X,\nabla_Y)$ is $\bar\omega(X,Y)$, and the part of degree 0 of $\omega(i_\alpha,i_\beta)$ is $2g(\alpha,\beta)$. As $\bar\omega$ and g are non-singular then ω is non-singular too. Let us denote this graded symplectic form by $\omega_{(g,\nabla)}$.

By Batchelor's theorem [1], there is a vector bundle $E\to M$ such that the graded manifolds $(M,\mathcal{A})$ and $(M,\Gamma(\Lambda E))$ are isomorphic. Let ω be the graded 2-form on $(M,\mathcal{A})$ image by the isomorphism of $\omega_{(g,\nabla)}$. Then ω is a symplectic structure on $(M,\mathcal{A})$ that satisfies the desired condition. ∎

Note that the graded 2-form thus defined depends on $\bar\omega = d\lambda$ but not directly on λ. This fact will allow us to generalize the construction to closed 2-forms.

Theorem 1. *Let M be a differentiable manifold supporting a symplectic structure defined by a 2-form $\bar\omega$ and let $(M,\mathcal{A})$ be a graded manifold over M. Then there exists an even nondegenerate graded 2-form ω such that $\kappa\omega = \bar\omega$.*

Proof: Let $\{U_i\}_{i\in I}$ be an open cover of M by simply connected coordinates sets. First, let us recall that the module of graded differential forms, $\Omega(M,\mathcal{A})$, has a sheaf structure (see [4] page 247). For $V\subset U$, let $\rho_{U,V}:\Omega(U,\mathcal{A}(U))\to\Omega(V,\mathcal{A}(V))$ be the restriction map

For each $i\in I$ we have that there exists a 1-form $\bar\lambda_i$ such that $\bar\omega = d\bar\lambda_i$ in U_i. Let ω_i be the graded symplectic form of the graded manifold $(U_i,\mathcal{A}(U_i))$ defined by λ_i, a metric on E, g, and a fixed linear connection on E, ∇, as follows from lemma 1. It is clear, by the proof of lemma 1, that for every $i,j\in I$,

$$\rho_{U_i,U_i\cap U_j}(\omega_i) = \rho_{U_j,U_i\cap U_j}(\omega_j).$$

Then, by the definition of sheaf, we have that there exists a unique $\omega\in\Omega(M,\mathcal{A})$ such that $\rho_{X,U_i}(\omega) = \omega_i$. ∎

The graded symplectic structure thus defined depends on the choice of a metric and on a linear connection; there are many degrees of freedom. But there are situations where this choice can be made canonically. For instance, in general relativity we have a fixed metric and a distinguished connection, the metric connection. Therefore if the graded manifold has as underlying differentiable manifold a Lorentz manifold of even dimension, M, and the sheaf is the Cartan algebra of the manifold, i.e., the algebra of differential forms ($\Gamma(\Lambda E) = \Gamma(\Lambda T^*M)$), then there is a canonical lifting of symplectic structures on M to graded symplectic structures on $(M,\Gamma(\Lambda T^*M))$.

Odd symplectic structures. In some cases, it is also possible to construct, following this method, odd graded symplectic forms. [6]

If ω is an even graded symplectic structure then $\kappa\omega$ is a symplectic structure. But this is not true for odd graded symplectic structures. If ω is an odd graded symplectic structure then $\kappa\omega = 0$. Anyway, we can obtain a similar condition:

Let us suppose that the fiber dimension $n = m$, thus the graded manifold $(M,\Gamma(\Lambda E))$ has the same even and odd dimension.

PROPOSITION 3. *Let ω be an odd graded symplectic structure on $(M,\Gamma(\Lambda E))$. Then it defines an isomorphism $\Pi\omega : TM\to E$.*

PROOF: Let ∇ be a connection in E. For a fixed $X\in\Gamma(TM)$ the map from $\Gamma(E^*)$ into $C^\infty(M)$ given by $\alpha\to(\omega(\nabla_X,i_\alpha))^\sim$ is $C^\infty(M)$-linear, thus it defines a local section of E that we shall call $\Pi(X)$.

The map from $\Gamma(TM)$ into $\Gamma(E)$ given by $X\to\Pi(X)$ is $C^\infty(M)$-linear, thus it defines a homomorphism $\Pi_\omega : TM\to E$. Since ω is non-singular, then $\Pi\omega$ is an isomorphism.

Let us see now that the isomorphism, $\Pi\omega$ does not depend on the choice of the connection. If $\bar\nabla$ is another connection, then $\bar\nabla_X = \nabla_X + i_{A(X)}$, where $A(X)\in\Gamma(E\otimes E^*)$.

If $\alpha,\beta\in\Gamma(E^*)$ then i_α, i_β are odd derivations, then, as ω is an odd graded form, $\omega(i_\alpha,i_\beta)$ is an odd element of $\Gamma(\Lambda E)$. Thus, $(\omega(i_\alpha,i_\beta))^\sim = 0$, and therefore, $(\omega(i_{A(X)},i_\alpha))^\sim = 0$.

We get then that $(\omega(\bar\nabla_X,i_\alpha))^\sim = (\omega(\nabla_X,i_\alpha))^\sim$, thus the isomorphism does not depend on ∇. ∎

Note that every odd graded symplectic structure is exact. Indeed, as κ induces an algebra isomorphism of the de Rham cohomologies (see [4]) and since $\kappa\omega = 0$, then ω must be exact.

Now, we shall construct odd symplectic structures as we have done before for even symplectic structures.

Let $\omega^\# : TM \to E$ be an isomorphism. It can be seen as an element of $\Gamma(E \otimes T^*M)$. Let ∇ be a connection on E and denote by d_∇ the exteriour covariant derivative $d_\nabla : \Gamma(E \otimes \Lambda^k(T^*M)) \to \Gamma(E \otimes \Lambda^{k+1}(T^*M))$, given by

$$(d_\nabla \Phi)(X_0, \ldots, X_k) = \sum_{i=0}^{p} (-1)^i (\nabla_{X_i} \Phi)(X_0, \ldots, \widehat{X}_i, \ldots, X_k) +$$

$$+ \sum_{i<j} (-1)^{i+j} \Phi([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k),$$

where $\Phi \in \Gamma(E \otimes \Lambda^k(T^*M))$.

Let λ be the graded 1-form defined, following proposition 2, by $\omega^\#$, i.e. $\lambda = \rho(\omega^\#)$. $\rho(\omega^\#)$ is a homomorphism of degree 1. Its action on derivations is given by

$$\lambda(\nabla_\Psi + i_\Phi) = \rho(\omega^\#)\Psi.$$

Then λ is an odd graded 1-form.

As before, let $\omega = d\lambda$. We have that

$$\omega(\nabla_X, \nabla_Y) = (d_\nabla \omega^\#)(X, Y) \in \Gamma(E),$$
$$\omega(i_\alpha, \nabla_X) = \alpha(\omega^\#(X)) \in C^\infty(M),$$
$$\omega(i_\alpha, i_\beta) = 0.$$

Since $\omega^\#$ is an isomomorphism, then ω is non-singular. It is also clear that $\Pi\omega = \omega^\#$.

We have then proved the following

Theorem 2. *Let $(M, \Gamma(\Lambda E))$ be a graded manifold of graded dimension (m, m). Let us suppose that there exist an isomorphism $\omega^\# : TM \to E$. Then there exists an odd graded symplectic structure ω such that $\Pi\omega = \omega^\#$.*

Local expression. Let us choose $E = T^*M$ and let $\omega^\# : TM \to T^*M$ be an isomorphism, and let ∇ be a linear connection on M such that $\nabla\omega^\# = 0$. Let $U \subset M$ be a coordinate open subset with coordinate functions $\{x_k\}_{k=1}^m$, then, $\{s_k = \omega^\#(\frac{\partial}{\partial x_k})\}_{k=1}^m$ is a local basis in T^*M, let $\{\theta^k\}_{k=1}^m$ be its dual.

$$\omega(i_{\theta^k}, \nabla_{\frac{\partial}{\partial x_l}}) = \theta^k(\omega^\#(\frac{\partial}{\partial x_l})) = \delta_{kl}.$$

Note that $\nabla\omega^\# = 0$. Then, locally, $\omega = \sum_{k=1}^m ds_k \wedge dr_k$.

Note that a symplectic form $\bar{\omega}$ on M defines an isomorphism $\omega^\# : TM \to T^*M$ and with this homomorphism we can define an odd graded symplectic form.

Example. The Schouten-Nijenhuis bracket

As a final example, we shall see that the the Schoten-Nijuenhuis bracket can be expressed as the Poisson bracket of an odd symplectic structure. This was first observed in [6]. A dual construction on differential forms was constructed in [5].

Let us consider as vector bundle E the tangent bundle, TM. Thus, the graded manifold is $(M, \Gamma(\Lambda TM))$. The elements of $\Gamma(\Lambda TM)$ are called multivectors. Let us choice as isomorphism from TM into itself the identity map, and let ∇ be a linear connection in TM.

Let λ be the graded 1-form defined by the identity isomorphism and let $\omega = d\lambda$ be the associated odd symplectic form.

It it easy to check that

$$\omega(\nabla_X, \nabla_Y) = T(X, Y), \quad \omega(i_\alpha, \nabla_X) = \alpha(X), \quad \omega(i_\alpha, i_\beta) = 0,$$

where T is the torsion of ∇.

Theorem 3. *The Poisson bracket on multivectors induced by ω is the Schouten-Nijenhuis bracket.*

Proof: Let $X \in \Gamma(TM)$. The Hamiltonian graded vector field defined by X is the derivation $H_X = \nabla_X + i_{\nabla X}$.

Thus $\{X, Y\} = -\omega(H_X, H_Y) = -\omega(\nabla_X, \nabla_Y) - \omega(\nabla_X, i_{\nabla Y}) - \omega(i_{\nabla X}, \nabla_Y) = -T(X, Y) + \nabla_X Y - \nabla_Y X = [X, Y]$. Then, the Poisson bracket and the Schouten-Nijenhuis bracket agree on vector fields. By the properties of both brackets, they agree on multivectors.

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